Upper and lower almost $\beta(\Lambda, sp)$-continuous multifunctions

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Abstract: This paper deals with the concepts of upper and lower almost $\beta(\Lambda, sp)$-continuous multifunctions. Moreover, several characterizations concerning upper and lower almost $\beta(\Lambda, sp)$-continuous multifunctions are investigated.

Key–Words: $\beta(\Lambda, sp)$-open set, upper almost $\beta(\Lambda, sp)$-continuous multifunction, lower almost $\beta(\Lambda, sp)$-continuous multifunction

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1 Introduction

In 1983, Abd El-Monsef et al., [1], introduced the concept of β-continuous functions as a generalization of semi-continuity in the sense of Levive, [14], and percontinuity due to Mashhour et al., [16]. Borsík and Doboš, [9], introduced the notion of almost quasi-continuity which is weaker than that of quasi-continuity in the sense of Marcus, [15], and obtained a decomposition theorem of quasi-continuity. Moreover, Popa and Noiri, [23], studied some characterizations of β-continuity and showed that almost quasi-continuity is equivalent to β-continuity. The equivalence of almost quasi-continuity and β-continuity is also shown by Borsík, [8], and Ewert, [10]. In 1997, Nasef and Noiri, [17], introduced and investigated the concept of almost β-continuity. Several different forms of continuous multifunctions have been introduced and studied over the years. Many authors have researched and studied several stronger and weaker forms of continuous functions and multifunctions. The notions of upper and lower β-continuous multifunctions were studied by Popa and Noiri, [22]. In 1999, Noiri and Popa, [20], obtained some characterizations and several properties concerning upper and lower almost β-continuous multifunctions. In 2008, Noiri and Popa, [18], introduced and studied the notions of upper and lower C-m-continuous multifunctions as multifunctions defined on a set satisfying some minimal conditions. In 2018, Boonpok et al., [7], introduced and investigated the concepts of upper and lower almost (τ_1, τ_2)-precontinuous multifunctions. In 2020, Laprom et al., [13], introduced and studied the notions of upper and lower almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. The notion of (Λ, sp)-continuous multifunctions was introduced in [6]. In [5], the present authors introduced and studied the concepts of upper and lower slightly (Λ, sp)-continuous multifunctions. Quite recently, Khampakdee and Boonpok, [12], introduced and investigated the notions of upper and lower $\beta(\Lambda, sp)$-continuous multifunctions. The purpose of the present paper is to introduce the concepts of upper and lower almost $\beta(\Lambda, sp)$-continuous multifunctions. Moreover, several characterizations and some basic properties of upper and lower almost $\beta(\Lambda, sp)$-continuous multifunctions are discussed.

2 Preliminaries

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of

A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be β-*open* [1] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set is called β-*closed*. The family of all β-open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$. Let A be a subset of a topological space (X, τ). A subset $\Lambda_{sp}(A)$ [19] is defined as follows: $\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$.

Lemma 1. [19] *For subsets A, B and $A_\alpha(\alpha \in \nabla)$ of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq \Lambda_{sp}(A)$.

(2) If $A \subseteq B$, *then* $\Lambda_{sp}(A) \subseteq \Lambda_{sp}(B)$.

(3) $\Lambda_{sp}(\Lambda_{sp}(A)) = \Lambda_{sp}(A)$.

(4) If $U \in \beta(X, \tau)$, *then* $\Lambda_{sp}(U) = U$.

(5) $\Lambda_{sp}(\cap\{A_\alpha|\alpha \in \nabla\}) \subseteq \cap\{\Lambda_{sp}(A_\alpha)|\alpha \in \nabla\}$.

(6) $\Lambda_{sp}(\cup\{A_\alpha|\alpha \in \nabla\}) = \cup\{\Lambda_{sp}(A_\alpha)|\alpha \in \nabla\}$.

Recall that a subset A of a topological space (X, τ) is called a Λ_{sp}-*set* [19] if $A = \Lambda_{sp}(A)$.

Lemma 2. [19] *For subsets A and $A_\alpha(\alpha \in \nabla)$ of a topological space (X, τ), the following properties hold:*

(1) $\Lambda_{sp}(A)$ *is a* Λ_{sp}*-set.*

(2) If A is β*-open, then A is a* Λ_{sp}*-set.*

(3) If A_α *is a* Λ_{sp}*-set for each* $\alpha \in \nabla$, *then* $\cap_{\alpha\in\nabla} A_\alpha$ *is a* Λ_{sp}*-set.*

(4) If A_α *is a* Λ_{sp}*-set for each* $\alpha \in \nabla$, *then* $\cup_{\alpha\in\nabla} A_\alpha$ *is a* Λ_{sp}*-set.*

Definition 3. [6] *A subset A of a topological space (X, τ) is called (Λ, sp)-closed if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called (Λ, sp)-open.*

Definition 4. [6] *Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a (Λ, sp)-cluster point of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the (Λ, sp)-closure of A and is denoted by $A^{(\Lambda,sp)}$.*

Lemma 5. [6] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-closure, the following properties hold:*

(1) $A \subseteq A^{(\Lambda,sp)}$ *and* $[A^{(\Lambda,sp)}]^{(\Lambda,sp)} = A^{(\Lambda,sp)}$.

(2) If $A \subseteq B$, *then* $A^{(\Lambda,sp)} \subseteq B^{(\Lambda,sp)}$.

(3) $A^{(\Lambda,sp)}$ *is* (Λ, sp)*-closed.*

(4) A is (Λ, sp)*-closed if and only if* $A = A^{(\Lambda,sp)}$.

Definition 6. [6] *Let A be a subset of a topological space (X, τ). The union of all (Λ, sp)-open sets contained in A is called the (Λ, sp)-interior of A and is denoted by $A_{(\Lambda,sp)}$.*

Lemma 7. [6] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-interior, the following properties hold:*

(1) $A_{(\Lambda,sp)} \subseteq A$ *and* $[A_{(\Lambda,sp)}]_{(\Lambda,sp)} = A_{(\Lambda,sp)}$.

(2) If $A \subseteq B$, *then* $A_{(\Lambda,sp)} \subseteq B_{(\Lambda,sp)}$.

(3) $A_{(\Lambda,sp)}$ *is* (Λ, sp)*-open.*

(4) A is (Λ, sp)*-open if and only if* $A_{(\Lambda,sp)} = A$.

(5) $[X - A]^{(\Lambda,sp)} = X - A_{(\Lambda,sp)}$.

(6) $[X - A]_{(\Lambda,sp)} = X - A^{(\Lambda,sp)}$.

Definition 8. [6] *A subset A of a topological space (X, τ) is said to be:*

(i) $r(\Lambda, sp)$*-open if* $A = [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$;

(ii) $s(\Lambda, sp)$*-open if* $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$;

(iii) $p(\Lambda, sp)$*-open if* $A \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$;

(iv) $\alpha(\Lambda, sp)$*-open if* $A \subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$;

(v) $\beta(\Lambda, sp)$*-open if* $A \subseteq [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

The family of all $r(\Lambda, sp)$-open (resp. $s(\Lambda, sp)$-open, $p(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open, $\alpha(\Lambda, sp)$-open) sets in a topological space (X, τ) is denoted by $r\Lambda_{sp}O(X, \tau)$ (resp. $s\Lambda_{sp}O(X, \tau)$, $p\Lambda_{sp}O(X, \tau)$, $\beta\Lambda_{sp}O(X, \tau)$, $\alpha\Lambda_{sp}O(X, \tau)$). The complement of a $r(\Lambda, sp)$-open (resp. $s(\Lambda, sp)$-open, $p(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open, $\alpha(\Lambda, sp)$-open) set is called $r(\Lambda, sp)$-*closed* (resp. $s(\Lambda, sp)$-*closed*, $p(\Lambda, sp)$-*closed*, $\beta(\Lambda, sp)$-*closed*, $\alpha(\Lambda, sp)$-*closed*).

Let A be a subset of a topological space (X, τ). The intersection of all $s(\Lambda, sp)$-closed (resp. $p(\Lambda, sp)$-closed, $\beta(\Lambda, sp)$-closed, $\alpha(\Lambda, sp)$-closed) sets of X containing A is called the $s(\Lambda, sp)$-*closure* [24] (resp. $p(\Lambda, sp)$-*closure* [3], $\beta(\Lambda, sp)$-*closure* [12], $\alpha(\Lambda, sp)$-*closure* [4, 25]) of A and is denoted by $A^{s(\Lambda,sp)}$ (resp. $A^{p(\Lambda,sp)}$, $A^{\beta(\Lambda,sp)}$, $A^{\alpha(\Lambda,sp)}$). The union of all $s(\Lambda, sp)$-open (resp. $p(\Lambda, sp)$-open, $\beta(\Lambda, sp)$-open, $\alpha(\Lambda, sp)$-open) sets of X contained in A is called the *$s(\Lambda, sp)$-interior* (resp. *$p(\Lambda, sp)$-interior*, *$\beta(\Lambda, sp)$-interior*, *$\alpha(\Lambda, sp)$-interior*) of A and is denoted by $A_{s(\Lambda,sp)}$ (resp. $A_{p(\Lambda,sp)}$, $A_{\beta(\Lambda,sp)}$, $A_{\alpha(\Lambda,sp)}$).

Definition 9. [11] *A subset N_x of a topological space (X, τ) is said to be (Λ, sp)-neighbourhood of a point $x \in X$ if there exists a (Λ, sp)-open set U such that $x \in U \subseteq N_x$.*

Lemma 10. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) $A^{\beta(\Lambda,sp)}$ is $\beta(\Lambda, sp)$-closed in X;

(2) A is $\beta(\Lambda, sp)$-closed in X if and only if $A = A^{\beta(\Lambda,sp)}$;

(3) $[X - A]^{\beta(\Lambda,sp)} = X - A_{\beta(\Lambda,sp)}$.

Lemma 11. *Let A be a subset of a topological space (X, τ). Then, $x \in A^{\beta(\Lambda,sp)}$ if and only if $A \cap U \neq \emptyset$ for each $\beta(\Lambda, sp)$-open set U containing x.*

Lemma 12. *Let A be a subset of a topological space (X, τ). Then, $A^{s(\Lambda,sp)} = A \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$.*

Proof. Since $A^{s(\Lambda,sp)}$ is $s(\Lambda, sp)$-closed, we have $[[A^{s(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A^{s(\Lambda,sp)}$. Therefore, $[A^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A^{s(\Lambda,sp)}$ and hence

$$A \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A^{s(\Lambda,sp)}.$$

To establish the opposite inclusion we observe that

$$\begin{aligned}
&[[A \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \\
&= [A^{(\Lambda,sp)} \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \\
&\subseteq A^{(\Lambda,sp)} \cup [[[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \\
&= A^{(\Lambda,sp)} \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)} \\
&= A^{(\Lambda,sp)}.
\end{aligned}$$

Thus, $[[A \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq [A^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and so $A \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is $s(\Lambda, sp)$-closed. This shows that $A^{s(\Lambda,sp)} \subseteq A \cup [A^{(\Lambda,sp)}]_{(\Lambda,sp)}$. □

Lemma 13. *Let (X, τ) be a topological space. Then, $V^{s(\Lambda,sp)} = [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ for every $p(\Lambda, sp)$-open set V of X.*

Proof. Let $V \in p\Lambda_{sp}O(X, \tau)$. Then, $V \subseteq [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and by Lemma 12, we have $V^{s(\Lambda,sp)} = V \cup [V^{(\Lambda,sp)}]_{(\Lambda,sp)} = [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$. □

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [2] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X \mid F(x) \subseteq B\}$$

and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Then, F is said to be a *surjection* if $F(X) = Y$, or equivalently, if for each $y \in Y$, there exists an $x \in X$ such that $y \in F(x)$. Moreover, $F : X \to Y$ is called *upper semi-continuous* (resp. *lower semi-continuous*) if $F^+(V)$ (resp. $F^-(V)$) is open in X for every open set V of Y [21].

3 Characterizations of upper and lower almost $\beta(\Lambda, sp)$-continuous multifunctions

We begin this section by introducing the notions of upper and lower almost $\beta(\Lambda, sp)$-continuous multifunctions. Furthermore, several characterizations of upper and lower almost $\beta(\Lambda, sp)$-continuous multifunctions are discussed.

Definition 14. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:*

(i) upper almost $\beta(\Lambda, sp)$-continuous at $x \in X$ if, for each (Λ, sp)-open set V of Y containing $F(x)$, there exists a $\beta(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$;

(ii) lower almost $\beta(\Lambda, sp)$-continuous at $x \in X$ if, for each (Λ, sp)-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a $\beta(\Lambda, sp)$-open set U of X containing x such that

$$F(z) \cap [V^{(\Lambda,sp)}]_{(\Lambda,sp)} \neq \emptyset$$

for every $z \in U$;

(iii) upper (lower) almost $\beta(\Lambda, sp)$-continuous if F has this property at each point of X.

Lemma 15. *Let A be a subset of a topological space (X, τ). Then, $A^{\beta(\Lambda,sp)} = A \cup [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$.*

Proof. We observe that

$$\begin{aligned}&[[[A\cup[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}\\&\subseteq[[[A\cup[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}\\&\subseteq[[A_{(\Lambda,sp)}\cup[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}\\&=[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}\\&\subseteq A\cup[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}.\end{aligned}$$

Thus, $A\cup[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is $\beta(\Lambda,sp)$-closed and hence $A^{\beta(\Lambda,sp)}\subseteq A\cup[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$. On the other hand, since $A^{\beta(\Lambda,sp)}$ is $\beta(\Lambda,sp)$-closed, we have $[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}\subseteq[[[A^{\beta(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}\subseteq A^{\beta(\Lambda,sp)}$ and hence $A\cup[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}\subseteq A^{\beta(\Lambda,sp)}$. □

Theorem 16. *For a multifunction $F:(X,\tau)\to(Y,\sigma)$, the following properties are equivalent:*

(1) F is upper almost $\beta(\Lambda,sp)$-continuous at $x\in X$;

(2) $x\in[[[F^{+}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ for any (Λ,sp)-open set V of Y containing $F(x)$;

(3) for each (Λ,sp)-open neighbourhood U of x and each (Λ,sp)-open set V of Y containing $F(x)$, there exists a (Λ,sp)-open set G of X such that $\emptyset\neq G\subseteq U$ and $G\subseteq[F^{+}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}$;

(4) for each (Λ,sp)-open set V of Y containing $F(x)$, there exists a $s(\Lambda,sp)$-open set U of X containing x such that $U\subseteq[F^{+}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}$.

Proof. (1) ⇒ (2): Let V be any (Λ,sp)-open set of Y containing $F(x)$. Then, there exists $U\in\beta\Lambda_{sp}O(X,\tau)$ containing x such that $F(U)\subseteq V^{s(\Lambda,sp)}=[V^{(\Lambda,sp)}]_{(\Lambda,sp)}$. Then, $U\subseteq F^{+}(V^{s(\Lambda,sp)})$. Since U is $\beta(\Lambda,sp)$-open, we have $x\in U\subseteq[[U^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}\subseteq[[[F^{+}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

(2) ⇒ (3): Let V be any (Λ,sp)-open set of Y containing $F(x)$ and let U be a (Λ,sp)-open set of X containing x. Since $x\in[[[F^{+}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$, we have $U\cap[[F^{+}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}\neq\emptyset$. Put $G=U\cap[[F^{+}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}$, then G is a nonempty (Λ,sp)-open set, $G\subseteq U$ and $G\subseteq[[F^{+}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}\subseteq[F^{+}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}$.

(3) ⇒ (4): Let V be any (Λ,sp)-open set of Y containing $F(x)$. By $\mathcal{U}(x)$, we denote the family of all (Λ,sp)-open neighbourhood of x. For each $U\in\mathcal{U}(x)$, there exists a (Λ,sp)-open set G_U of X such that $\emptyset\neq G_U\subseteq U$ and $G_U\subseteq[F^{+}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}$. Put $W=\cup\{G_U\mid U\in\mathcal{U}(x)\}$, then W is (Λ,sp)-open set of X, $x\in W^{(\Lambda,sp)}$ and $W\subseteq[F^{+}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}$. Moreover, if we put $U_0=W\cup\{x\}$, then we obtain U_0 is a $s(\Lambda,sp)$-open set of X containing x and $U_0\subseteq[F^{+}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}$.

(4) ⇒ (1): Let V be any (Λ,sp)-open set of Y containing $F(x)$. There exists a $s(\Lambda,sp)$-open set G of X containing x such that $G\subseteq[F^{+}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}$. Thus,

$$\begin{aligned}&x\in G\cap F^{+}(V)\\&\subseteq F^{+}(V^{s(\Lambda,sp)})\cap[G_{(\Lambda,sp)}]^{(\Lambda,sp)}\\&\subseteq F^{+}(V^{s(\Lambda,sp)})\cap[[[F^{+}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}\\&=[F^{+}(V^{s(\Lambda,sp)})]_{\beta(\Lambda,sp)}.\end{aligned}$$

Put $U=[F^{+}(V^{s(\Lambda,sp)})]_{\beta(\Lambda,sp)}$, then U is a $\beta(\Lambda,sp)$-open set of X containing x such that $F(U)\subseteq[V^{(\Lambda,sp)}]_{(\Lambda,sp)}$. This shows that F is upper almost $\beta(\Lambda,sp)$-continuous at x. □

Theorem 17. *For a multifunction $F:(X,\tau)\to(Y,\sigma)$, the following properties are equivalent:*

(1) F is lower almost $\beta(\Lambda,sp)$-continuous at a point x of X;

(2) $x\in[[[F^{-}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ for any (Λ,sp)-open set V of Y such that $F(x)\cap V\neq\emptyset$;

(3) for each (Λ,sp)-open neighbourhood U of x and each (Λ,sp)-open set V of Y such that $F(x)\cap V\neq\emptyset$, there exists a nonempty (Λ,sp)-open set G of X such that $G\subseteq U$ and $G\subseteq[F^{-}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}$;

(4) for each (Λ,sp)-open set V of Y such that $F(x)\cap V\neq\emptyset$, there exists a $s(\Lambda,sp)$-open set U of X containing x such that $U\subseteq[F^{-}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}$.

Proof. The proof is similar to that of Theorem 16 and is thus omitted. □

Definition 18. *A function $f:(X,\tau)\to(Y,\sigma)$ is called almost $\beta(\Lambda,sp)$-continuous at a point $x\in X$ if, for each (Λ,sp)-open set V of Y containing $f(x)$, there exists a $\beta(\Lambda,sp)$-open set U of X containing x such that $f(U)\subseteq[V^{(\Lambda,sp)}]_{(\Lambda,sp)}$. If f has this property at each point of X, then f is called almost $\beta(\Lambda,sp)$-continuous.*

Corollary 19. *For a function $f:(X,\tau)\to(Y,\sigma)$, the following properties are equivalent:*

(1) f is almost $\beta(\Lambda, sp)$-continuous at a point x of X;

(2) $x \in [[[f^{-1}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ for any (Λ, sp)-open set V of Y containing $f(x)$;

(3) for each (Λ, sp)-open neighbourhood U of x and each (Λ, sp)-open set V of Y containing $f(x)$, there exists a nonempty (Λ, sp)-open set G of X such that $G \subseteq U$ and $G \subseteq [f^{-1}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}$;

(4) for each (Λ, sp)-open set V of Y containing $f(x)$, there exists a $s(\Lambda, sp)$-open set U of X containing x such that $U \subseteq [f^{-1}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}$.

Theorem 20. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost $\beta(\Lambda, sp)$-continuous;

(2) for each $x \in X$ and each (Λ, sp)-open set V of Y containing $F(x)$, there exists a $\beta(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq V^{s(\Lambda,sp)}$;

(3) for each $x \in X$ and each $r(\Lambda, sp)$-open set V of Y containing $F(x)$, there exists a $\beta(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq V$;

(4) $F^+(V)$ is $\beta(\Lambda, sp)$-open in X for every $r(\Lambda, sp)$-open set V of Y;

(5) $F^-(K)$ is $\beta(\Lambda, sp)$-closed in X for every $r(\Lambda, sp)$-closed set K of Y;

(6) $F^+(V) \subseteq [F^+(V^{s(\Lambda,sp)})]_{\beta(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y;

(7) $[F^-(K_{s(\Lambda,sp)})]^{\beta(\Lambda,sp)} \subseteq F^-(K)$ for every (Λ, sp)-closed set K of Y;

(8) $[F^-([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\beta(\Lambda,sp)} \subseteq F^-(K)$ for every (Λ, sp)-closed set K of Y;

(9) $[F^-([[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\beta(\Lambda,sp)} \subseteq F^-(B^{(\Lambda,sp)})$ for every subset B of Y;

(10) $[[[F^-([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq F^-(K)$ for every (Λ, sp)-closed set K of Y;

(11) $[[[F^-(K_{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq F^-(K)$ for every (Λ, sp)-closed set K of Y;

(12) $F^+(V) \subseteq [[[F^+(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ for every (Λ, sp)-open set V of Y.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3): The proofs are obvious.

(3) $\Rightarrow$ (4): Let V be any $r(\Lambda, sp)$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$ and there exists a $\beta(\Lambda, sp)$-open set U_x of X containing x such that $F(U_x) \subseteq V$. Thus, $x \in U_x \subseteq F^+(V)$ and hence $F^+(V) \in \beta\Lambda_{sp}O(X, \tau)$.

(4) $\Rightarrow$ (5): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(5) $\Rightarrow$ (6): Let V be any (Λ, sp)-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V \subseteq V^{s(\Lambda,sp)}$ and hence

$$x \in F^+(V^{s(\Lambda,sp)}) = X - F^-(Y - V^{s(\Lambda,sp)}).$$

Since $Y - V^{s(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed in Y, $F^-(Y - V^{s(\Lambda,sp)})$ is $\beta(\Lambda, sp)$-closed in X. Thus, $F^+(V^{s(\Lambda,sp)})$ is a $\beta(\Lambda, sp)$-open set of X containing x and hence $x \in [F^+(V^{s(\Lambda,sp)})]_{\beta(\Lambda,sp)}$. This shows that $F^+(V) \subseteq [F^+(V^{s(\Lambda,sp)})]_{\beta(\Lambda,sp)}$.

(6) $\Rightarrow$ (7): Let K be any (Λ, sp)-closed set of Y. Then, $Y - K$ is (Λ, sp)-open in Y and by (6),

$$\begin{aligned} X - F^-(K) &= F^+(Y - K) \\ &\subseteq [F^+([Y - K]^{s(\Lambda,sp)})]_{\beta(\Lambda,sp)} \\ &= [F^+(Y - K_{s(\Lambda,sp)})]_{\beta(\Lambda,sp)} \\ &= [X - F^-(K_{s(\Lambda,sp)})]_{\beta(\Lambda,sp)} \\ &= X - [F^-(K_{s(\Lambda,sp)})]^{\beta(\Lambda,sp)}. \end{aligned}$$

Thus, $[F^-(K_{s(\Lambda,sp)})]^{\beta(\Lambda,sp)} \subseteq F^-(K)$.

(7) $\Rightarrow$ (8): The proof is obvious since $K_{s(\Lambda,sp)} = [K_{(\Lambda,sp)}]^{(\Lambda,sp)}$ for every (Λ, sp)-closed set K.

(8) $\Rightarrow$ (9): The proof is obvious.

(9) $\Rightarrow$ (10): It follows from Lemma 15 that $[[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \subseteq A^{\beta(\Lambda,sp)}$ for every subset A. Thus, for every (Λ, sp)-closed set K of Y, we have

$$\begin{aligned} &[[[F^-([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \\ &\subseteq [F^-([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\beta(\Lambda,sp)} \\ &= [F^-([[K^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\beta(\Lambda,sp)} \\ &\subseteq F^-(K^{(\Lambda,sp)}) = F^-(K). \end{aligned}$$

(10) $\Rightarrow$ (11): The proof is obvious since $K_{s(\Lambda,sp)} = [K_{(\Lambda,sp)}]^{(\Lambda,sp)}$ for every (Λ, sp)-closed set K.

(11) $\Rightarrow$ (12): Let V be any (Λ, sp)-open set of Y. Then, $Y - V$ is (Λ, sp)-closed in Y and we have

$$\begin{aligned} &[[[F^-([Y - V]_{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)} \\ &\subseteq F^-(Y - V) = X - F^+(V). \end{aligned}$$

Moreover, we have

$$\begin{aligned}&[[[F^-([Y-V]_{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}\\&=[[[F^-(Y-V^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}\\&=[[[X-F^+(V^{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}\\&=X-[[[F^+(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}.\end{aligned}$$

Thus, $F^+(V)\subseteq[[[F^+(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

(12) $\Rightarrow$ (1): Let x be any point of X and V any (Λ,sp)-open set of Y containing $F(x)$. Then, $x\in F^+(V)\subseteq[[[F^+(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ and hence F is upper almost $\beta(\Lambda,sp)$-continuous at x by Theorem 16. This shows that F is upper almost $\beta(\Lambda,sp)$-continuous. □

Theorem 21. *For a multifunction $F:(X,\tau)\to(Y,\sigma)$, the following properties are equivalent:*

(1) F is lower almost $\beta(\Lambda,sp)$-continuous;

(2) for each $x\in X$ and each (Λ,sp)-open set V of Y such that $F(x)\cap V\neq\emptyset$, there exists a $\beta(\Lambda,sp)$-open set U of X containing x such that $U\subseteq F^-(V^{s(\Lambda,sp)})$;

(3) for each $x\in X$ and each $r(\Lambda,sp)$-open set V of Y such that $F(x)\cap V\neq\emptyset$, there exists a $\beta(\Lambda,sp)$-open set U of X containing x such that $U\subseteq F^-(V)$;

(4) $F^-(V)$ is $\beta(\Lambda,sp)$-open in X for every $r(\Lambda,sp)$-open set V of Y;

(5) $F^+(K)$ is $\beta(\Lambda,sp)$-closed in X for every $r(\Lambda,sp)$-closed set K of Y;

(6) $F^-(V)\subseteq[F^-(V^{s(\Lambda,sp)})]_{\beta(\Lambda,sp)}$ for every (Λ,sp)-open set V of Y;

(7) $[F^+(K_{s(\Lambda,sp)})]^{\beta(\Lambda,sp)}\subseteq F^+(K)$ for every (Λ,sp)-closed set K of Y;

(8) $[F^+([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\beta(\Lambda,sp)}\subseteq F^+(K)$ for every (Λ,sp)-closed set K of Y;

(9) $[F^+([[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\beta(\Lambda,sp)}\subseteq F^+(B^{(\Lambda,sp)})$ for every subset B of Y;

(10) $[[[F^+([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}\subseteq F^+(K)$ for every (Λ,sp)-closed set K of Y;

(11) $[[[F^+(K_{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}\subseteq F^+(K)$ for every (Λ,sp)-closed set K of Y;

(12) $F^-(V)\subseteq[[[F^-(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ for every (Λ,sp)-open set V of Y.

Proof. The proof is similar to that of Theorem 20 and is thus omitted. □

Corollary 22. *For a function $f:(X,\tau)\to(Y,\sigma)$, the following properties are equivalent:*

(1) f is almost $\beta(\Lambda,sp)$-continuous;

(2) for each $x\in X$ and each (Λ,sp)-open set V of Y containing $f(x)$, there exists a $\beta(\Lambda,sp)$-open set U of X containing x such that $f(U)\subseteq V^{s(\Lambda,sp)}$;

(3) for each $x\in X$ and each $r(\Lambda,sp)$-open set V of Y containing $f(x)$, there exists a $\beta(\Lambda,sp)$-open set U of X containing x such that $f(U)\subseteq V$;

(4) $f^{-1}(V)$ is $\beta(\Lambda,sp)$-open in X for every $r(\Lambda,sp)$-open set V of Y;

(5) $f^{-1}(K)$ is $\beta(\Lambda,sp)$-closed in X for every $r(\Lambda,sp)$-closed set K of Y;

(6) $f^{-1}(V)\subseteq[f^{-1}(V^{s(\Lambda,sp)})]_{\beta(\Lambda,sp)}$ for every (Λ,sp)-open set V of Y;

(7) $[f^{-1}(K_{s(\Lambda,sp)})]^{\beta(\Lambda,sp)}\subseteq f^{-1}(K)$ for every (Λ,sp)-closed set K of Y;

(8) $[f^{-1}([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\beta(\Lambda,sp)}\subseteq f^{-1}(K)$ for every (Λ,sp)-closed set K of Y;

(9) $[f^{-1}([[B^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)})]^{\beta(\Lambda,sp)}\subseteq f^{-1}(B^{(\Lambda,sp)})$ for every subset B of Y;

(10) $[[[f^{-1}([K_{(\Lambda,sp)}]^{(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}\subseteq f^{-1}(K)$ for every (Λ,sp)-closed set K of Y;

(11) $[[[f^{-1}(K_{s(\Lambda,sp)})]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}\subseteq f^{-1}(K)$ for every (Λ,sp)-closed set K of Y;

(12) $f^{-1}(V)\subseteq[[[f^{-1}(V^{s(\Lambda,sp)})]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$ for every (Λ,sp)-open set V of Y.

Theorem 23. *For a multifunction $F:(X,\tau)\to(Y,\sigma)$, the following properties are equivalent:*

(1) F is upper almost $\beta(\Lambda,sp)$-continuous;

(2) $[F^-(V)]^{\beta(\Lambda,sp)}\subseteq F^-(V^{(\Lambda,sp)})$ for every $\beta(\Lambda,sp)$-open set V of Y;

(3) $[F^-(V)]^{\beta(\Lambda,sp)}\subseteq F^-(V^{(\Lambda,sp)})$ for every $s(\Lambda,sp)$-open set V of Y;

(4) $F^+(V)\subseteq[F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{\beta(\Lambda,sp)}$ for every $p(\Lambda,sp)$-open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be any $\beta(\Lambda, sp)$-open set of Y. Since $V^{(\Lambda,sp)}$ is $r(\Lambda, sp)$-closed in Y and by Theorem 20, $F^-[V^{(\Lambda,sp)}]$ is $\beta(\Lambda, sp)$-closed in X. Thus, $[F^-(V)]^{\beta(\Lambda,sp)} \subseteq F^-[V^{(\Lambda,sp)}]$.

(2) $\Rightarrow$ (3): This is obvious since $s\Lambda_{sp}O(Y,\sigma) \subseteq \beta\Lambda_{sp}O(Y,\sigma)$.

(3) $\Rightarrow$ (4): Let $V \in p\Lambda_{sp}O(Y,\sigma)$. Then, we have $V \subseteq [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and $Y-V \supseteq [[Y-V]_{(\Lambda,sp)}]^{(\Lambda,sp)}$. Since $[[Y-V]_{(\Lambda,sp)}]^{(\Lambda,sp)} \in s\Lambda_{sp}O(Y,\sigma)$, we have

$$\begin{aligned} X - F^+(V) &= F^-(Y-V) \\ &\supseteq F^-([[Y-V]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \\ &\supseteq [F^-([[Y-V]_{(\Lambda,sp)}]^{(\Lambda,sp)}])^{\beta(\Lambda,sp)} \\ &= [F^-(Y-[V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{\beta(\Lambda,sp)} \\ &= [X - F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]^{\beta(\Lambda,sp)} \\ &= X - [F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{\beta(\Lambda,sp)} \end{aligned}$$

and hence $F^+(V) \subseteq [F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{\beta(\Lambda,sp)}$.

(4) $\Rightarrow$ (1): Let V be any $r(\Lambda, sp)$-open set of Y. Since $V \in p\Lambda_{sp}O(Y,\sigma)$, we have $F^+(V) \subseteq [F^+([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{\beta(\Lambda,sp)} = [F^+(V)]_{\beta(\Lambda,sp)}$ and hence $F^+(V) \in \beta\Lambda_{sp}O(X,\tau)$. It follows from Theorem 20 that F is upper almost $\beta(\Lambda, sp)$-continuous. □

Theorem 24. *For a multifunction $F : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) F is lower almost $\beta(\Lambda, sp)$-continuous;

(2) $[F^+(V)]^{\beta(\Lambda,sp)} \subseteq F^+(V^{(\Lambda,sp)})$ for every $\beta(\Lambda, sp)$-open set V of Y;

(3) $[F^+(V)]^{\beta(\Lambda,sp)} \subseteq F^+(V^{(\Lambda,sp)})$ for every $s(\Lambda, sp)$-open set V of Y;

(4) $F^-(V) \subseteq [F^-([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{\beta(\Lambda,sp)}$ for every $p(\Lambda, sp)$-open set V of Y.

Proof. The proof is similar to that of Theorem 23 and is thus omitted. □

Corollary 25. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) f is almost $\beta(\Lambda, sp)$-continuous;

(2) $[f^{-1}(V)]^{\beta(\Lambda,sp)} \subseteq f^{-1}(V^{(\Lambda,sp)})$ for every $\beta(\Lambda, sp)$-open set V of Y;

(3) $[f^{-1}(V)]^{\beta(\Lambda,sp)} \subseteq f^{-1}(V^{(\Lambda,sp)})$ for every $s(\Lambda, sp)$-open set V of Y;

(4) $f^{-1}(V) \subseteq [f^{-1}([V^{(\Lambda,sp)}]_{(\Lambda,sp)})]_{\beta(\Lambda,sp)}$ for every $p(\Lambda, sp)$-open set V of Y.

For a multifunction $F : (X,\tau) \to (Y,\sigma)$, by $F^{(\Lambda,sp)} : (X,\tau) \to (Y,\sigma)$ we shall denote a multifunction defined as follows: $F^{(\Lambda,sp)}(x) = [F(x)]^{(\Lambda,sp)}$ for each $x \in X$. Similarly, we can define $F^{s(\Lambda,sp)}$, $F^{p(\Lambda,sp)}$, $F^{\alpha(\Lambda,sp)}$ and $F^{\beta(\Lambda,sp)}$.

Theorem 26. *A multifunction $F : (X,\tau) \to (Y,\sigma)$ is upper almost $\beta(\Lambda, sp)$-continuous if and only if $F^{s(\Lambda,sp)} : (X,\tau) \to (Y,\sigma)$ is upper almost $\beta(\Lambda, sp)$-continuous.*

Proof. Suppose that F is upper almost $\beta(\Lambda, sp)$-continuous. Let $x \in X$ and V be any (Λ, sp)-open set of Y such that $F^{s(\Lambda,sp)}(x) \subseteq V$. Then, $F(x) \subseteq V$ and by Theorem 20, there exists a $\beta(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq V^{\beta(\Lambda,sp)}$. For each $z \in U$, $F(z) \subseteq V^{s(\Lambda,sp)}$ and hence $[F(U)]^{s(\Lambda,sp)} \subseteq V^{s(\Lambda,sp)}$. Therefore, we have $F^{s(\Lambda,sp)}(U) \subseteq V^{s(\Lambda,sp)}$ and by Theorem 20, $F^{s(\Lambda,sp)}$ is upper almost $\beta(\Lambda, sp)$-continuous.

Conversely, suppose that $F^{s(\Lambda,sp)}$ is upper almost $\beta(\Lambda, sp)$-continuous. Let $x \in X$ and V be any (Λ, sp)-open set of Y containing $F(x)$. Then, $F(x) \subseteq V$ and $[F(x)]^{s(\Lambda,sp)} \subseteq V^{s(\Lambda,sp)}$. Since $V^{s(\Lambda,sp)} = [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ is (Λ, sp)-open, there exists a $\beta(\Lambda, sp)$-open set U of X containing x such that $F^{s(\Lambda,sp)}(U) \subseteq [V^{s(\Lambda,sp)}]^{s(\Lambda,sp)} = V^{s(\Lambda,sp)}$. Thus, $F(U) \subseteq V^{s(\Lambda,sp)}$ and hence F is upper almost $\beta(\Lambda, sp)$-continuous. □

Definition 27. [11] *A subset A of a topological space (X,τ) is said to be:*

(i) (Λ, sp)-paracompact if every cover of A by (Λ, sp)-open sets of X is refined by a cover of A which consists of (Λ, sp)-open sets of X and is locally finite in X;

(ii) (Λ, sp)-regular if, for each $x \in A$ and each (Λ, sp)-open set U of X containing x, there exists a (Λ, sp)-open set V of X such that $x \in V \subseteq V^{(\Lambda,sp)} \subseteq U$.

Lemma 28. [11] *If A is a (Λ, sp)-regular (Λ, sp)-paracompact set of a topological space (X,τ) and U is a (Λ, sp)-open neighbourhood of A, then there exists a (Λ, sp)-open set V of X such that $A \subseteq V \subseteq V^{(\Lambda,sp)} \subseteq U$.*

Lemma 29. *If $F : (X,\tau) \to (Y,\sigma)$ is a multifunction such that $F(x)$ is (Λ, sp)-regular (Λ, sp)-paracompact for each $x \in X$, then for each (Λ, sp)-open set V of Y, $G^+(V) = F^+(V)$, where G denotes $F^{\beta(\Lambda,sp)}$, $F^{p(\Lambda,sp)}$, $F^{\alpha(\Lambda,sp)}$, or $F^{(\Lambda,sp)}$.*

Theorem 30. *Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is (Λ, sp)-paracompact and (Λ, sp)-regular for each $x \in X$. Then, the following properties are equivalent:*

(1) F is upper almost $\beta(\Lambda, sp)$-continuous;

(2) $F^{\beta(\Lambda,sp)}$ is upper almost $\beta(\Lambda, sp)$-continuous;

(3) $F^{p(\Lambda,sp)}$ is upper almost $\beta(\Lambda, sp)$-continuous;

(4) $F^{\alpha(\Lambda,sp)}$ is upper almost $\beta(\Lambda, sp)$-continuous;

(5) $F^{(\Lambda,sp)}$ is upper almost $\beta(\Lambda, sp)$-continuous.

Proof. Similarly to Lemma 29, we put $G = F^{\beta(\Lambda,sp)}$, $F^{p(\Lambda,sp)}$, $F^{\alpha(\Lambda,sp)}$ or $F^{(\Lambda,sp)}$. First, suppose that F is upper almost $\beta(\Lambda, sp)$-continuous. Let $x \in X$ and V be any (Λ, sp)-open set of Y containing $G(x)$. By Lemma 29, $x \in G^+(V) = F^+(V)$ and there exists a $\beta(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq V^{s(\Lambda,sp)}$. Since $F(z)$ is (Λ, sp)-paracompact and (Λ, sp)-regular for each $z \in U$, by Lemma 28 there exists a (Λ, sp)-open set H such that $F(z) \subseteq H \subseteq H^{(\Lambda,sp)} \subseteq V^{s(\Lambda,sp)}$; hence $G(z) \subseteq H^{(\Lambda,sp)} \subseteq V^{s(\Lambda,sp)}$ for each $z \in U$. This shows that G is upper almost $\beta(\Lambda, sp)$-continuous.

Conversely, suppose that G is upper almost $\beta(\Lambda, sp)$-continuous. Let $x \in X$ and V be any (Λ, sp)-open set of Y containing $F(x)$. By Lemma 29, we have $x \in F^+(V) = G^+(V)$. Therefore, $G(x) \subseteq V$. Then, there exists a $\beta(\Lambda, sp)$-open set U of X containing x such that $G(U) \subseteq V^{s(\Lambda,sp)}$. Thus, $F(U) \subseteq V^{s(\Lambda,sp)}$ and hence F is upper almost $\beta(\Lambda, sp)$-continuous. □

Lemma 31. *If $F : (X, \tau) \to (Y, \sigma)$ is a multifunction, then for each (Λ, sp)-open set V of Y, $G^-(V) = F^-(V)$, where G denotes $F^{\beta(\Lambda,sp)}$, $F^{s(\Lambda,sp)}$, $F^{p(\Lambda,sp)}$, $F^{\alpha(\Lambda,sp)}$, or $F^{(\Lambda,sp)}$.*

Theorem 32. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost $\beta(\Lambda, sp)$-continuous;

(2) $F^{\beta(\Lambda,sp)}$ is lower almost $\beta(\Lambda, sp)$-continuous;

(3) $F^{s(\Lambda,sp)}$ is lower almost $\beta(\Lambda, sp)$-continuous;

(4) $F^{p(\Lambda,sp)}$ is lower almost $\beta(\Lambda, sp)$-continuous;

(5) $F^{\alpha(\Lambda,sp)}$ is lower almost $\beta(\Lambda, sp)$-continuous;

(6) $F^{(\Lambda,sp)}$ is lower almost $\beta(\Lambda, sp)$-continuous.

Proof. Similarly to Lemma 31, we put $G = F^{\beta(\Lambda,sp)}$, $F^{s(\Lambda,sp)}$, $F^{p(\Lambda,sp)}$, $F^{\alpha(\Lambda,sp)}$, or $F^{(\Lambda,sp)}$. First, suppose that F is lower almost $\beta(\Lambda, sp)$-continuous. Let $x \in X$ and V be any (Λ, sp)-open set of Y such that $G(x) \cap V \neq \emptyset$. Since V is (Λ, sp)-open, $F(x) \cap V \neq \emptyset$ and there exists a $\beta(\Lambda, sp)$-open set U of X containing x such that $F(z) \cap V^{s(\Lambda,sp)} \neq \emptyset$ for each $z \in U$. Thus, $G(z) \cap V^{s(\Lambda,sp)} \neq \emptyset$ for each $z \in U$. This shows that G is lower almost $\beta(\Lambda, sp)$-continuous.

Conversely, suppose that G is lower almost $\beta(\Lambda, sp)$-continuous. Let $x \in X$ and V be any (Λ, sp)-open set of Y such that $F(x) \cap V \neq \emptyset$. Since $F(x) \subseteq G(x)$, $G(x) \cap V \neq \emptyset$ and there exists a $\beta(\Lambda, sp)$-open set U of X containing x such that $G(z) \cap V^{s(\Lambda,sp)} \neq \emptyset$ for each $z \in U$. By Lemma 13, $V^{s(\Lambda,sp)} = [V^{(\Lambda,sp)}]_{(\Lambda,sp)}$ and $F(z) \cap V^{s(\Lambda,sp)} \neq \emptyset$ for each $z \in U$. Thus, by Theorem 21, F is lower almost $\beta(\Lambda, sp)$-continuous. □

Definition 33. *Let A be a subset of a topological space (X, τ). The $\beta(\Lambda, sp)$-frontier of A, denoted by $\beta\Lambda_{sp}Fr(A)$, is defined by*

$$\begin{aligned}\beta\Lambda_{sp}Fr(A) &= A^{\beta(\Lambda,sp)} \cap [X - A]^{\beta(\Lambda,sp)} \\ &= A^{\beta(\Lambda,sp)} - A_{\beta(\Lambda,sp)}.\end{aligned}$$

Theorem 34. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is not upper almost $\beta(\Lambda, sp)$-continuous at $x \in X$ if and only if x is in the union of the $\beta(\Lambda, sp)$-frontier of the upper inverse images of $r(\Lambda, sp)$-open sets containing $F(x)$.*

Proof. Let x be a point of X at which F is not upper almost $\beta(\Lambda, sp)$-continuous. Then, there exists a $r(\Lambda, sp)$-open set V of Y containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $\beta(\Lambda, sp)$-open set U of X containing x. By Lemma 11, we have

$$x \in [X - F^+(V)]^{\beta(\Lambda,sp)}.$$

Since $x \in F^+(V)$, we obtain $x \in [F^+(V)]^{\beta(\Lambda,sp)}$ and hence $x \in \beta\Lambda_{sp}Fr(F^+(V))$.

Conversely, suppose that V is a $r(\Lambda, sp)$-open set of Y containing $F(x)$ such that

$$x \in \beta\Lambda_{sp}Fr(F^+(V)).$$

If F is upper almost $\beta(\Lambda, sp)$-continuous at x, then there exists a $\beta(\Lambda, sp)$-open set of X containing x such that $F(U) \subseteq V$. Therefore, $x \in U \subseteq [F^+(V)]_{\beta(\Lambda,sp)}$. This is a contradiction to

$$x \in \beta\Lambda_{sp}Fr(F^+(V)).$$

Thus, F is not upper almost $\beta(\Lambda, sp)$-continuous at x. □

Theorem 35. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is not lower almost $\beta(\Lambda, sp)$-continuous at $x \in X$ if and only if x is in the union of the $\beta(\Lambda, sp)$-frontier of the lower inverse images of $r(\Lambda, sp)$-open sets meeting $F(x)$.*

Proof. The proof is similar to that of Theorem 34 and is thus omitted. □

4 Conclusion

Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the researching of generalizations of continuity in topological spaces. Using different forms of open sets, many authors have introduced and studied various types of weak forms of continuity for functions and multifunctions. This work is concerned with the notions of of upper and lower almost $\beta(\Lambda, sp)$-continuous multifunctions. Moreover, some characterizations of upper and lower almost $\beta(\Lambda, sp)$-continuous multifunctions are established. The ideas and results of this work may motivate further research.

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