Logarithmic wave equation involving variable-exponent nonlinearities:Well-posedness and blow-up

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Abstract: In this paper, we focus on a class of existence, uniqueness, and explosion in a finite time of solving a logarithmic wave equation model with nonlinearities with variable exponents and nonlinear source terms under homogeneous Dirichlet boundary conditions.

$$u_{tt} - \Delta u + |u_t|^{m(.)-2} u_t = |u|^{p(.)-2} u \ln |u|$$

We applied the Faedo-Galerkin method in combination with the Banach fixed point theorem to determine the existence and uniqueness of a local solution in time. Various inequality techniques were used under appropriate conditions to obtain the blow-up of a solution. This type of equation is related to fluid dynamics, electrorheological fluids, quantum mechanics theory, nuclear physics, optics, and geophysics.

Key-Words: Wave equations; Logarithmic nonlinearity; variable exponents spaces; Existence; Finite time blow-up.

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1 Introduction

In recent years, many authors have paid attention to the study of nonlocal logarithmic differential equations. This is partly due to the wide use of this species to model various phenomena such as fluid dynamics, electrorheological fluids, nuclear physics, optics, geophysics, quantum mechanics theory. In this work we treat the following semilinear wave equation with logarithmic nonlinear source term under homogeneous Dirichlet boundary condition

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m(.)-2} u_t = |u|^{p(.)-2} u \ln |u|, \\ \text{in } \Omega \times (0,T) \\ u(x,t) = 0, \\ \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \\ \text{in } \Omega, \end{cases} \tag{1.1}$$

In (1.1), Ω be a bounded domain in $\mathbb{R}^n (n \geq 1)$ with a smooth boundary $\partial\Omega$, for all $m(.)$, $p(.) : \Omega \to \mathbb{R}$ measurable functions satisfying

$$\begin{cases} 2 \leq q_1 \leq q(x) \leq q_2 \leq \overline{\frac{2n}{n-2}},\ n \geq 3, \\ 2 \leq q_1 \leq q(x) \leq q_2 < \infty,\ n \leq 2, \end{cases} \tag{1.2}$$

with

$$q_1 := \operatorname{ess}\inf_{x\in\Omega} q(x), \quad q_2 := \operatorname{ess}\sup_{x\in\Omega} q(x)$$

and the log–Hölder continuity condition:

$$|q(x) - q(y)| \leq \overline{\frac{-A}{\log|x-y|}}, \text{ for a.e. } x, y \in \Omega, \\ \text{with } 0 < |x-y| < \delta,\ A > 0,\ \delta < 1 \tag{1.3}$$

In case m, p are constants, local, global existence and long-time behavior have been considered by many authors. For example, the logarithmic nonlinearity term $|u|^{p-2} u \ln(|u|)$ in the absence of the damping term $|u_t|^{m-2} u_t$ causes an infinite time blow -up of solutions with negative initial energy [4, 1, 11, 12], in contrast to the power source term $|u|^{p-2}u$, which causes a finite time blow-up of solutions [5, 6], it is known that the damping term $|u_t|^{m-2} u_t$ for any initial data [7, 8, 13] ensures global existence. We also refer to [9, 10] and its references for logarithmic nonlinearity problems. These semilinear wave equations arise when studying various problems and can be used as models for viscoelastic liquids, processes of filtration through a porous medium and liquids with temperature-dependent viscosity, filtration theory, etc. (see [36, 35]). We also refer to [14, 15] and its references for other issues in this direction.

In recent years, some partial differential equations with logarithmic nonlinearity term have attracted much attention due to their wide application in physics and other applied sciences, such as heat conduction with two temperature systems [17], seepage of homogeneous fluids through a fissured rock [16], unidirectional propagation of nonlinear, dispersive, long waves [17, 18], fluid flow in fissured porous media [19], two-phase flow in porous media with dynamic capillary pressure [20, 21] and the aggregation of populations [22]. Pseudo-parabolic equations can also be viewed as Sobolev-type or Sobolev-Galpern-type equations, see [23, 24] and many articles have been devoted to the study of well-posedness and qualitative properties of solutions to these partial differential equations with constant exponents. It is important to point out that the calculation of blow-up time and rate on nonlinear evolution equations is an important topic (see [25, 26]), and such evaluations be able conclusively characterize the blow-up phenomenon. The terminology variable exponents comes from the fact that $m(.)$ and $p(.)$ are functions and not real numbers. This term $|u_t|^{m(.)-2}u_t - |u|^{p(.)-2}u\ln|u|$ is then a generalization of $|u_t|^{m-2}u_t - |u|^{p-2}u$, which corresponds to $m(.)$, $p(.) > 1$ and $\ln|u|$. In fact, (1.1) can be cast as an extension of the variable case of the second-order viscoelastic wave equation with variable growth conditions

$$u_{tt} - \Delta u + |u_t|^{m(.)-2}u_t = |u|^{p(.)-2}u, \text{ in } \Omega\times(0,T) \tag{1.4}$$

what one gets when $|u_t|^{m(.)-2}u_t - |u|^{p(.)-2}u\ln|u|$ considered. Equation (1.4) is a well-known model for electrorheological fluids [32] that occurs in the treatment of fluid dynamics. On the other hand, results for the viscoelastic wave equation with logarithmic damping and variable growth conditions are limited and rare, and the literature on these equations is much less extensive, see [37, 39, 38].

The interest in the mathematical analysis of partial differential equations in recent years has been driven by inhomogeneous differential operators with variable exponents (see eg [29, 28, 27]). The study of these systems is based on the use of Lebesgue and Sobolev spaces with variable exponents. Note that the problems of differential equations with non-standard $p(x)$ growth are an unfamiliar and interesting topic. These are nonlinear theory of elasticity, electrorheological fluids, etc. These fluids retain the motivating property that their viscosity depends on the electric field in the fluid. For general accounts of the underlying physics see [31] and for the mathematical visions see [30]. A number of papers on problems in so-called rheological and electrorheological fluids that indicate spaces with variable exponents have recently been published by Diening and Rŭzicka [32, 33]. The results of this work were summarized in the books [32, 33]. Numerous mathematical models in fluid mechanics, elasticity theory (recently in image processing), see eg [34], etc. have been shown which are obviously related to the non-standard local growth problem. In this article we consider (1.1) and establish a local existence result. We also show that the solution explodes in finite time T for suitable initial dates.

2 Preliminaries

Let $p:\Omega\to[1,\infty]$ be a measurable function. $L^{p(\cdot)}(\Omega)$ denotes the set of the real measurable functions u on Ω such that

$$\int_\Omega |\lambda u(x)|^{p(x)}\,\mathrm{d}x < \infty \text{ for some } \lambda > 0.$$

The variable-exponent space $L^{p(\cdot)}(\Omega)$ equipped with the Luxemburg–type norm

$$\|u\|_{p(.)} = \inf\left\{\lambda > 0,\ \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)}\mathrm{d}x \le 1\right\},$$

is a Banach space. Throughout the paper, we use $\|.\|_q$ to indicate the L^q-norm for $1 \le q \le +\infty$. $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the following norm:

$$\|u\|_{H_0^1(\Omega)} = \left(\|\nabla u\|_2^2 + \|u\|_2^2\right)^{\frac{1}{2}}.$$

It is known that for the elements of $H_0^1(\Omega)$ the Poincaré inequality holds,

$$\|u\|_2 \le C^*\|\nabla u\|_2, \text{ for all } u\in H_0^1(\Omega).$$

and an equivalent norm of $H_0^1(\Omega)$ can be defined by

$$\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_2 = \left(\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

Lemma 2.1 [28, 29]. If $p:\Omega\to[1,\infty)$ is a measurable function and

$$2 \le p_1 \le p(x) \le p_2 < \frac{2n}{n-2},\ n\ge 3. \tag{2.1}$$

Then, the embedding $H_0^1(\Omega)\hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous and compact.

3 Existence of weak solutions

In this section we present the local existence and uniqueness of solutions for the system (1.1). Our proof method is based on Banach's fixed point theorem.

Theorem 3.1 Let $m(.)$, and $p(.)$ satisfies (1.2), (1.3), and in addition $p(.)$ satisfy

$$2 < p_1 \leq p(x) \leq p_2 < 2\frac{n-1}{n-2},\ n \geq 3. \quad (3.1)$$

Then, for any given $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ it exists $T > 0$ and a unique solution u of the problem (1.1) on $(0, T)$ such that

$$u \in C\left((0,T), H_0^1(\Omega)\right) \cap C^1\left((0,T), L^2(\Omega)\right) \quad (3.2)$$
$$\cap L^{m(.)}(\Omega \times (0,T)),$$
$$u_{tt} \in L^2\left((0,T), H^{-1}(\Omega)\right).$$

To prove the main theorem we need the local existence and uniqueness of the solution of a related problem. Then, given v, consider the following initial boundary value problem:

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m(.)-2} u_t = v(x,t), & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \quad (3.3)$$

where the exponent $m(.)$ is a given measurable function satisfying (1.2) and (1.3). We now have the following existence result of the local solution of the problem (3.3) for $v \in L^2(\Omega \times (0,T))$, and suitable initial value $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, which we created using the Galerkin method as in [2], or in [3, Theorem 3.1, Chapter 1].

Lemma 3.2 Suppose that $m(.)$ satisfies (1.2), and (1.3). Then, for all $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $v \in L^2(\Omega \times (0,T))$, there is a unique local solution u of the problem (3.3),

$$\begin{gathered} u \in L^\infty\left((0,T), H_0^1(\Omega)\right), \\ u_t \in L^\infty\left((0,T), L^2(\Omega)\right) \cap L^{m(.)}(\Omega \times (0,T)) \\ u_{tt} \in L^2\left((0,T), H^{-1}(\Omega)\right). \end{gathered} \quad (3.4)$$

proof.

1. Uniqueness: If the problem (3.3) has two solutions u and v. Then, $w = u - v$ must verify

$$\begin{cases} w_{tt} - \Delta w + u_t |u_t|^{m(.)-2} - v_t |v_t|^{m(.)-2} = 0, \\ \text{in } \Omega \times (0,T), \\ w(x,t) = 0, \\ \text{on } \partial\Omega \times (0,T), \\ w(x,0) = w_t(x,0) = 0, \\ \text{in } \Omega. \end{cases}$$

Formally, multiplying by u_t and integrate over $\Omega\times (0,t)$, gives

$$\begin{gathered} \int_\Omega \left(w_t^2 + |\nabla w|^2\right) \\ +2\int_0^t \int_\Omega \left(u_t |u_t|^{m(x)-2} - v_t |v_t|^{m(x)-2}\right)(u_t \\ -v_t)\,\mathrm{d}x\mathrm{d}s = 0. \end{gathered}$$

By using the inequality

$$\left(|\mathbf{a}|^{m(x)-2}\mathbf{a} - |\,\mathbf{b}^{m(x)-2}\mathbf{b}\right).(\mathbf{a} - \mathbf{b}) \geq 0 \quad (3.5)$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and a.e $x \in \Omega$, we get

$$\int_\Omega \left(w_t^2 + |\nabla w|^2\right) = 0$$

which means that $w = 0$, since $w = 0$ on $\partial\Omega$. Therefore, the uniqueness follows.

2. Existence. Let $\left\{(v_j)_{j=1}^\infty\right\}$ be an orthonormal basis of $H_0^1(\Omega)$, with

$$-\Delta v_j = \lambda_j v_j \ \text{ in } \Omega,\ v_j = 0,\ \text{on } \partial\Omega,$$

let determine the finite-dimensional subspace $V_k = \mathrm{span}\{v_1, \ldots, v_k\}$, without loss of generality we may take $\|v_j\|_2 = 1$. We will construct a convergent sequence $\left\{u^k(x,t)\right\}$,

$$u^k(x,t) = \sum_{j=1}^k a_{kj}(t) v_j,$$

where $u^k(x,t)$ satisfy the system of linear differential equations

$$\begin{gathered} \int_\Omega u_{tt}^k(x,t) v_j(x)\mathrm{d}x + \int_\Omega \nabla u^k(x,t)\nabla v_j(x)\mathrm{d}x \\ + \int_\Omega \left|u_t^k(x,t)\right|^{m(x)-2} u_t^k(x,t) v_j(x)\mathrm{d}x = \int_\Omega v(t) v_j(x)\mathrm{d}x \\ u^k(x,0) = u_0^k,\ u_t^k(x,0) = u_1^k \quad \forall j = 1.2\ldots\ldots k, \end{gathered} \quad (3.6)$$

where

$$u_0^k = \sum_{i=1}^k (u_0, v_i)\, v \to u_0 \text{ in } H_0^1(\Omega),$$

$$u_1^k = \sum_{i=1}^k (u_1, v_i)\, v_i \to u_1 \text{ in } L^2(\Omega).$$

Note that (3.6) is a system of ordinary differential equations for $a_{kj}(t)$. The local existence of solutions of the system (3.6) is guaranteed by the Picard-Lindelŏf Theorem on functional analysis concepts, which is known to have a local solution in an interval $[0, T_k)$ with $0 < T_k \leq T_{\max} < +\infty$. The extension of the solution to the entire interval $[0,+\infty)$ is a consequence of the following estimates.

Multiplying (3.6) by $a'_{kj}(t)$ and sum over j to find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_\Omega\left(\left|u_t^k(x,t)\right|^2\mathrm{d}x+\left|\nabla u^k(x,t)\right|^2\right)\mathrm{d}x\right] + \int_\Omega\left|u_t^k(x,t)\right|^{m(x)}\mathrm{d}x=\int_\Omega v(x,t)u_t^k(x,t)\mathrm{d}x$$

A simple integration on $(0,t)$ yields

$$\begin{aligned}&\frac{1}{2}\int_\Omega\left(\left|u_t^k(x,t)\right|^2\mathrm{d}x+\left|\nabla u^k(x,t)\right|^2\right)\mathrm{d}x\\&+\int_0^t\int_\Omega\left|u_t^k(x,s)\right|^{m(x)}\mathrm{d}x\mathrm{d}s\\&=\frac{1}{2}\int_\Omega\left(\left|u_1^k\right|^2+\left|\nabla u_0^k\right|^2\right)\mathrm{d}x\\&+\int_0^t\int_\Omega v(x,s)u_t^k(x,s)\mathrm{d}x\mathrm{d}s\\&\le\frac{1}{2}\int_\Omega\left(u_1^2+\left|\nabla u_0\right|^2\right)\mathrm{d}x\\&+\varepsilon\int_0^t\int_\Omega\left|u_t^k\right|^2\mathrm{d}x\mathrm{d}s+c_\varepsilon\int_0^T\int_\Omega v^2\mathrm{d}x\mathrm{d}s\\&\le C_\varepsilon+\varepsilon\sup_{(0,t_k)}\int_\Omega\left|u_t^k(x,t)\right|^2\mathrm{d}x,\\&\forall t\in[0,t_k)\end{aligned}\tag{3.7}$$

Hence

$$\begin{aligned}&\frac{1}{2}\sup_{(0,t_k)}\int_\Omega\left|u_t^k(x,t)\right|^2\mathrm{d}x\\&+\frac{1}{2}\sup_{(0,t_k)}\int_\Omega\left|\nabla u^k(x,t)\right|^2\mathrm{d}x\\&+\int_0^{t_k}\int_\Omega\left|u_t^k(x,s)^{m(x)}\right|\mathrm{d}x\mathrm{d}s\le C_\varepsilon\\&+\varepsilon\sup_{(0,t_k)}\int_\Omega\left|u_t^k(x,t)\right|^2\mathrm{d}x\end{aligned}$$

Taking $\varepsilon=\frac{1}{4}$, we arrive at

$$\begin{aligned}&\sup_{(0,t_k)}\int_\Omega\left|u_t^k(x,t)\right|^2\mathrm{d}x\\&+\sup_{(0,t_k)}\int_\Omega\left|\nabla u^k(x,t)\right|^2\mathrm{d}x\\&+\int_0^{t_k}\int_\Omega\left|u_t^k(x,s)\right|^{m(x)}\mathrm{d}x\mathrm{d}s\le C\end{aligned}$$

Therefore, the solution can be prolonged to $[0,T)$ and, besides, we have

$$\left(u^k\right)\text{ is a bounded sequence in } L^\infty\left((0,T),H_0^1(\Omega)\right),$$

$$\left(u_t^k\right)\text{ is a bounded sequence in } L^\infty\left((0,T),L^2(\Omega)\right)\cap L^{m(.)}(\Omega\times(0,T)),$$

$$\left|u_t^k\right|^{m(.)-2}u_t^k \text{ is a bounded sequence in } L^{\frac{m(.)}{m(.)-1}}(\Omega\times(0,T)).$$

From Dunford–Pettis theorem, we can extract from $\left\{\left(u^k\right)\right\}$ a subsequence still denoted by $\left\{\left(u^k\right)\right\}$ such that

$$u^k\to u\text{ weakly }*\text{ in }L^\infty\left((0,T),H_0^1(\Omega)\right),\tag{3.8}$$

$$u_t^k\to u_t\text{ weakly}*\text{in }L^\infty\left((0,T),L^2(\Omega)\right)\text{ and weakly in }L^{m(.)}(\Omega\times(0,T)),\tag{3.9}$$

$$\left|u_t^k\right|^{m(.)-2}u_t^k\to\psi\text{ weakly in }L^{\frac{m(.)}{m(.)-1}}(\Omega\times(0,T)).\tag{3.10}$$

Limits (3.8)–(3.10) allow us to pass to the limit in the approximate equation so that we can deduce that

$u\in C\left([0,T],L^2(\Omega)\right)$, and therefore $u(x,0)$ has a sense.

Now we show that $u\in C\left([0,T],L^2(\Omega)\right)$ is a solution to the system (3.3). First we try to prove that $\psi=|u_t|^{m(.)-2}u_t$, for all $v\in L^\infty\left((0,T),L^2(\Omega)\right)$, in (3.6), integrate over $(0,t)$, and make $k\to\infty$ in the results, we can derive for a.e $t\in[0,T]$ that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_t\varphi+\int_\Omega(\nabla u.\nabla\varphi+\psi\varphi)\mathrm{d}x=\int_\Omega v\varphi\mathrm{d}x,\forall\varphi\in H_0^1(\Omega).\tag{3.11}$$

For simplicity let $A(\varphi)=|\varphi|^{m(x)-2}\varphi$ and define (see [2, Proposition 2.5.]),

$$X^k=\int_0^t\int_\Omega\left(A\left(u_t^k\right)-A(\varphi)\right)\left(u_t^k-\varphi\right)\mathrm{d}t\ge0,$$

$$\forall\varphi\in L^{m(.)}\left((0,T);H_0^1(\Omega)\right)$$

So if we using (3.7) we get

$$\begin{aligned}X^k=&\int_0^t\int_\Omega vu_t^k\mathrm{d}x\mathrm{d}s+{}^1{}_2\int_\Omega\left(\left|u_1^k\right|^2+\left|\nabla u_0^k\right|^2\right)\mathrm{d}x\mathrm{d}s\\&-\frac{1}{2}\int_\Omega\left|u_t^k(x,t)\right|^2\mathrm{d}x\\&-\frac{1}{2}\int_\Omega\left|\nabla u^k(x,t)\right|^2\mathrm{d}x-\int_0^t\int_\Omega A\left(u_t^k\right)\varphi\mathrm{d}x\mathrm{d}s\\&-\int_0^t\int_\Omega A(\varphi)\left(u_t^k-\varphi\right)\mathrm{d}x\mathrm{d}s\end{aligned}$$

Taking $k\to\infty$ we get

$$\begin{aligned}0\le\limsup_k X^k\le&\int_0^t\int_\Omega vu_t\mathrm{d}x\mathrm{d}s\\&+\frac{1}{2}\int_\Omega\left(u_1^2+|\nabla u_0|^2\right)\mathrm{d}x\mathrm{d}s-\frac{1}{2}\int_\Omega|u_t(t)|^2\mathrm{d}x\\&-\frac{1}{2}\int_\Omega|\nabla u(x,t)|^2\mathrm{d}x-\int_0^t\int_\Omega\psi\varphi\mathrm{d}x\mathrm{d}s\\&-\int_0^t\int_\Omega A(\varphi)\left(u_t-\varphi\right)\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{3.12}$$

If we put $\varphi = u_t$ in (3.11) and integrate over $(0,T)$, we get

$$\begin{aligned}\int_0^t \int_\Omega v u_t \mathrm{d}x\mathrm{d}s &= \tfrac{1}{2}\int_\Omega |u_t(x,t)|^2 \mathrm{d}x\mathrm{d}s \\ -\tfrac{1}{2}\int_\Omega u_1^2 \mathrm{d}x\mathrm{d}s &+ \tfrac{1}{2}\int_\Omega |\nabla u(x,t)|^2 \mathrm{d}x \\ -\tfrac{1}{2}\int_\Omega |\nabla u_0|^2 \mathrm{d}x &+ \int_0^t \int_\Omega \psi u_t \mathrm{d}x\mathrm{d}s.\end{aligned} \tag{3.13}$$

Combine (3.12) and (3.13) gives

$$0 \le \limsup_k X^k \le \int_0^t \int_\Omega \psi u_t \mathrm{d}x\mathrm{d}s$$

$$-\int_0^t \int_\Omega \psi\varphi \mathrm{d}x\mathrm{d}s - \int_0^t \int_\Omega A(\varphi)\,(u_t-\varphi)\,\mathrm{d}x\mathrm{d}s.$$

That is

$$\int_0^t \int_\Omega (\psi - A(\varphi))\,(u_t-\varphi)\,\mathrm{d}x\mathrm{d}s \ge 0,$$

$$\forall \varphi \in L^{m(.)}\left((0,T);H_0^1(\Omega)\right).$$

Consequently

$$\int_0^t \int_\Omega (\psi - A(\varphi))\,(u_t-\varphi)\,\mathrm{d}x\mathrm{d}s \ge 0,$$

$$\forall \varphi \in L^{m(.)}(\Omega\times(0,T)),$$

by density of $H_0^1(\Omega)$ in $L^{m(.)}(\Omega)$.

Now, let $\varphi = \lambda w + u_t$, $w \in L^{m(.)}(\Omega\times(0,T))$. Hence, we know

$$-\lambda \int_0^t \int_\Omega (\psi - A(\lambda w + u_t))\, w \mathrm{d}x\mathrm{d}s \ge 0,$$

$$\forall w \in L^{m(.)}(\Omega\times(0,T)),$$

for $\lambda > 0$, we have

$$\int_0^t \int_\Omega (\psi - A(\lambda w + u_t))\, w \mathrm{d}x\mathrm{d}s \le 0,$$

$$\forall w \in L^{m(.)}(\Omega\times(0,T)).$$

If we take $\lambda \to 0$ and using the hemi-continuity of A, we get

$$\int_0^t \int_\Omega (\psi - A(u_t))\, w \mathrm{d}x\mathrm{d}s \le 0,$$

$$\forall w \in L^{m(.)}(\Omega\times(0,T)) \tag{3.14}$$

Similarly we find for $\lambda < 0$

$$\int_0^t \int_\Omega (\psi - A(u_t))\, w \mathrm{d}x\mathrm{d}s \ge 0,$$

$$\forall w \in L^{m(.)}(\Omega\times(0,T)) \tag{3.15}$$

From (3.14) and (3.15), for $k \to +\infty$ we get $\psi = A(u_t)$ and

$$\left|u_t^k\right|^{m(.)-2} u_t^k \to |u_t|^{m(.)-2} u_t$$

weakly in $L^{\frac{m(.)}{m(.)-1}}(\Omega\times(0,T))$.

Therefore, from the above result and (3.8)–(3.10), we deduce that there is $u \in C\left([0,T],L^2(\Omega)\right)$ that satisfies the following equation

$$\left(u_{tt} - \Delta u + |u_t|^{m(.)-2} u_t - v, \varphi\right) = 0$$

for all $\varphi \in H_0^1(\Omega)$ and the initial conditions

$$u(0) = u_0,\ u_t(0) = u_1,$$

which completes the existence proof in Lemma (3.2).

■

The following lemma crucial for the proof of our main result

Lemma 3.3 For a.e $x \in \Omega$ and $p(.)$ that satisfy (3.1), the function $\mathrm{F}(s) = |s|^{p(x)-2}s\,(\ln|s|)$ is differentiable and

$$\begin{aligned}|\mathrm{F}'(s)| &\le (p_2-1)\,|s|^{p(x)-2}\,|\ln|s|| \\ &\quad + |s|^{p(x)-2} \\ &\le \tfrac{2(p_2-1)}{e((p_1-2)-k_1)}|s|^{k_1} + \tfrac{2(p_2-1)}{e(k_2-(p_2-2))}|s|^{k_2} \\ &\quad + \left(|s|^{p_1-2} + |s|^{p_2-2}\right),\ s \ne 0,\end{aligned} \tag{3.16}$$

where

$$\begin{aligned} p_1 - 2 \le p_2 - 2 < k_2 \le \tfrac{2}{n-2}, \\ \text{for } n \ge 3, \\ 0 < p_1 - 2 \le p_2 - 2 < k_2, \\ \text{for } n = 1,2, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} 0 < k_1 < p_1 - 2 \le p_2 - 2 \le \tfrac{2}{n-2}, \\ \text{for } n \ge 3, \\ 0 < k_1 < p_1 - 2 \le p_2 - 2, \\ \text{for } n = 1,2. \end{aligned} \tag{3.18}$$

proof. Obviously we have for $k \ne 0$ since $\ln\zeta \le \frac{1}{ek}\zeta^k$ for every $\zeta \ge 1$ and $\ln\zeta \ge -\frac{1}{ek}\zeta^{-k}$, $\zeta < 1$

then for every $k > 0$

$$\begin{aligned}|\mathrm{F}'(s)| &= \left|(p(x)-1)\,|s|^{p(x)-2}(\ln|s|) + |s|^{p(x)-2}\right| \\ &\leq \tfrac{p_2-1}{ek}\left(|s|^{p_1+k-2}+|s|^{p_2+k-2}\right) \\ &\quad + \tfrac{p_2-1}{ek}\left(|s|^{p_1-k-2}+|s|^{p_2-k-2}\right) \\ &\quad + \left(|s|^{p_1-2}+|s|^{p_2-2}\right) \\ &\leq 2\tfrac{p_2-1}{ek}|s|^{p_2+k-2} + 2\tfrac{p_2-1}{ek}|s|^{p_1-k-2} \\ &\quad + \left(|s|^{p_1-2}+|s|^{p_2-2}\right) \\ &= \tfrac{2(p_2-1)}{e((p_1-2)-k_1)}|s|^{k_1} + \tfrac{2(p_2-1)}{e(k_2-(p_2-2))}|s|^{k_2} \\ &\quad + \left(|s|^{p_1-2}+|s|^{p_2-2}\right),\end{aligned}$$

with k_1, and k_2 are in (3.17)-(3.18). ■

Proof of Theorem (3.1).

1. Existence. Let $v \in L^\infty\left((0,T), H_0^1(\Omega)\backslash\{0\}\right)$. Then

$$\begin{aligned}&\left\||v|^{p(.)-2}v\ln|v|\right\|_2^2 \\ &\leq \int_\Omega |v|^{2p_1-2}(\ln|v|)^2\,\mathrm{d}x \\ &\quad + \int_\Omega |v|^{2p_2-2}(\ln|v|)^2\,\mathrm{d}x \\ &= \int_{\{x\in\Omega:|v(t)|<1\}} |v|^{2p_1-2}(\ln|v|)^2\,\mathrm{d}x \\ &\quad + \int_{\{x\in\Omega:|v(t)|<1\}} |v|^{2p_2-2}(\ln|v|)^2\,\mathrm{d}x \\ &\quad + \int_{\{x\in\Omega:|v(t)|\geq1\}} |v|^{2p_1-2}(\ln|v|)^2\,\mathrm{d}x \\ &\quad + \int_{\{x\in\Omega:|v(t)|\geq1\}} |v|^{2p_2-2}(\ln|v|)^2\,\mathrm{d}x.\end{aligned}$$

Choosing σ such that

$$\begin{gathered}2 \leq 2(p_1-1) \leq 2(p_2-1) < \sigma \leq \frac{2n}{n-2}, \\ \text{for } n \geq 3, \\ 2 \leq 2(p_1-1) \leq 2(p_2-1) < \sigma, \\ \text{for } n = 1,2,\end{gathered}$$

and by $\ln\zeta \leq \frac{1}{es}\zeta^s$ for any $\zeta \geq 1$, $s > 0$, we have

$$\begin{aligned}&\int_{\{x\in\Omega:|v(t)|<1\}} |v|^{2p_1-2}(\ln|v|)^2\,\mathrm{d}x \\ &+ \int_{\{x\in\Omega:|v(t)|\geq1\}} |v|^{2p_1-2}(\ln|v|)^2\,\mathrm{d}x \\ &\leq \tfrac{|\Omega|}{e^2} + \tfrac{1}{e^2}\left(\tfrac{2}{\sigma+2-2p_1}\right)^2 \int_\Omega |v|^\sigma\,\mathrm{d}x \\ &\leq \tfrac{|\Omega|}{e^2} + \tfrac{1}{e^2}C_s^\sigma\left(\tfrac{2}{\sigma+2-2p_1}\right)^2 \|\nabla v\|_2^\sigma < \infty,\end{aligned} \tag{3.19}$$

similarly

$$\begin{aligned}&\int_{\{x\in\Omega:|v(t)|<1\}} |v|^{2p_2-2}(\ln|v|)^2\,\mathrm{d}x \\ &+ \int_{\{x\in\Omega:|v(t)|\geq1\}} |v|^{2p_2-2}(\ln|v|)^2\,\mathrm{d}x \\ &\leq \tfrac{|\Omega|}{e^2} + \tfrac{1}{e^2}C_s^\sigma\left(\tfrac{2}{\sigma+2-2p_2}\right)^2 \|\nabla v\|_2^\sigma < \infty,\end{aligned} \tag{3.20}$$

where C_s is the optimal constant of Sobolev embedding $H_0^1(\Omega) \to L^\sigma(\Omega)$. So, in this case.

$$|v|^{p(.)-2}v\ln|v| \in L^\infty\left((0,T), L^2(\Omega)\right)$$

$$\subset L^2(\Omega\times(0,T))$$

Thus for every $v \in L^\infty\left((0,T), H_0^1(\Omega)\backslash\{0\}\right)$, there is a unique u such that

$$\begin{gathered}u \in L^\infty\left((0,T), H_0^1(\Omega)\right), \\ u_t \in L^\infty\left((0,T), L^2(\Omega)\right) \cap L^{m(.)}(\Omega\times(0,T)),\end{gathered} \tag{3.21}$$

solve the nonlinear problem

$$\left\{\begin{gathered}u_{tt} - \Delta u + |u_t|^{m(.)-2}u_t = |v|^{p(.)-2}v\ln|v|, \\ \text{in } \Omega\times(0,T) \\ u(x,t) = 0, \\ \text{on } \partial\Omega\times(0,T) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \\ \text{in } \Omega.\end{gathered}\right. \tag{3.22}$$

Let R_0 be a positive real number such that

$$R_0 = \sqrt{2\left(|u_1|^2 + |\nabla u_0|^2\right)},$$

for a sufficiently small time $T > 0$ we define the space $B_T(R_0)$ by

$$B_T(R_0) = \left\{\begin{gathered}v(t) \in L^\infty\left((0,T), H_0^1(\Omega)\right), \\ v_t(t) \in L^\infty\left((0,T), L^2(\Omega)\right)), \\ |v'(t)|^2 + |\nabla v(t)|^2 \leq R_0^2 \text{ on } [0,T], \\ v(0) = v_0,\ v'(0) = u_1.\end{gathered}\right\}$$

We introduce the metric d on the space $B_T(R_0)$

$$\mathrm{d}(u,v) = \sup_{0\leq t\leq T}\left(|u_t(t) - v_t(t)|^2 + |\nabla u(t) - \nabla v(t)|^2\right)$$

$$\text{for } u,\ v \in B_T(R_0).$$

Obviously the space $B_T(R_0)$ is the complete metric space. Let $v \in B_T(R_0)$. Then $|\nabla v(t)| \leq R_0$, $|v'(t)| \leq R_0$ for all $t \in [0,T]$. Define the mapping Φ

$$\Phi(v) = u,$$

where u satisfies (3.21) and (3.22). Then we have

$$\Phi(v) = u \in B_T(R_0) \text{ for } v \in B_T(R_0), \quad (3.23)$$

$$\Phi : B_T(R_0) \to B_T(R_0) \text{ is a contractive mapping.} \quad (3.24)$$

For showing (3.23), multiply (3.22) by u_t

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u_t^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x\right) + \int_\Omega |u_t|^{m(x)} \mathrm{d}x$$

$$= \int_\Omega |v|^{p(x)-2} v (\ln|v|) u_t \mathrm{d}x \qquad (3.25)$$

From Young's inequality, (3.19) and (3.20) for all $\varepsilon > 0$ the following estimates hold:

$$\begin{aligned}
&\left|\int_\Omega v^{p(x)-2} v (\ln|v|) u_t \mathrm{d}x\right| \\
&\leq \int_\Omega u_t^2 \mathrm{d}x + \tfrac{1}{4}\int_\Omega |v|^{2p(x)-2} (\ln|v|)^2 \mathrm{d}x \\
&\leq \int_\Omega u_t^2 \mathrm{d}x + \tfrac{1}{4}\Big[\int_\Omega |v|^{2p_2-2} (\ln|v|)^2 \mathrm{d}x \\
&\quad + \int_\Omega |v|^{2p_1-2} (\ln|v|)^2 \mathrm{d}x\Big] \\
&\leq \int_\Omega u_t^2 \mathrm{d}x \\
&\quad + \tfrac{1}{4}\Big[2\tfrac{|\Omega|}{e^2} + \tfrac{1}{e^2} C_s^\sigma \left(\tfrac{2}{\sigma+2-2p_1}\right)^2 \|\nabla v\|_2^\sigma \\
&\quad + \tfrac{1}{e^2} C_s^\sigma \left(\tfrac{2}{\sigma+2-2p_2}\right)^2 \|\nabla v\|_2^\sigma\Big].
\end{aligned}$$

So (3.25) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t\|_2^2 + \|\nabla u\|_2^2\right)$$

$$\leq \frac{1}{e^2}|\Omega| + \frac{2}{e^2} C_s^\sigma \left(\frac{2}{\sigma+2-2p_2}\right)^2 R_0^\sigma + \|u_t\|_2^2.$$

Thus, we have

$$\begin{aligned}
&\psi_v(u)(t) \leq \psi_v(u)(0) \\
&+ \int_0^t \left(\tfrac{1}{e^2}|\Omega| + \tfrac{2}{e^2} C_s^\sigma \left(\tfrac{2}{\sigma+2-2p_2}\right)^2 R_0^\sigma + \psi_v(u)(t)\right) \mathrm{d}s \\
&\leq \tfrac{1}{2} R_0^2 + \beta_0 \int_0^t (1 + \psi_v(u)(t)) \mathrm{d}s,
\end{aligned}$$

where $\beta_0 = \max\left(\frac{1}{e^2}|\Omega| + \frac{2}{e^2} C_s^\sigma \left(\frac{2}{\sigma+2-2p_2}\right)^2 R_0^\sigma, 1\right)$ and

$$\psi_v(u)(t) = \|u_t\|_2^2 + \|\nabla u\|_2^2.$$

By the Gronwall inequality and simple calculations we have

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 \leq \left(\frac{1}{2}R_0^2 + \beta_0 T_0\right) e^{\beta_0 T_0} < R_0^2, \quad 0 \leq t \leq T_0,$$

for sufficiently small $0 < T_0 \leq T$. Thus (3.23) is fulfilled.

Next we show (3.24). Let $w = u_1 - u_2$, where $u_1 = \Phi(v_1)$, $u_2 = \Phi(v_2)$ with v_1, $v_2 \in B_T(R_0)$. Then we have

$$\begin{aligned}
&(w_{tt}, v) - (\Delta w, v) \\
&+ \left(|u_{1t}(t)|^{m(x)-1} u_{1t}(t) - |u_{2t}(t)|^{m(x)-1} u_{2t}(t), v\right) \\
&= \left(|v_1|^{p(x)-2} v_1 \ln|v_1| - |v_2|^{p(x)-2} v_2 \ln|v_2|, v\right), \\
&\quad \text{in } L^2\left(0, T_1; H^{-1}(\Omega)\right).
\end{aligned} \qquad (3.26)$$

Now, set

$$\beta_v(w)(t) = |w_t(t)|^2 + |\nabla w(t)|^2.$$

Multiplying (3.26) by w_t and using (3.5) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|w_t(t)|^2 + |\nabla w(t)|^2\right)$$

$$\leq \left(|v_1|^{p(x)-2} v_1 \ln|v_1| - |v_2|^{p(x)-2} v_2 \ln|v_2|, w_t\right).$$

Now we estimate

$$I = \int_\Omega |F(v_1(s)) - F(v_2(s))| |w_t| \mathrm{d}x$$

$$= \int_\Omega |F'(\xi)||v||w_t| \mathrm{d}x,$$

where

$$v = v_1 - v_2 \text{ and } \xi = a v_1 + (1-a) v_2, \ 0 \leq a \leq 1.$$

By Holders, Youngs inequalities and Lemma (3.3) we have

$$\begin{aligned}
&I^2 \leq \int_\Omega w_t^2 \mathrm{d}x \int_\Omega |F'(\xi)|^2 |v|^2 \mathrm{d}x \\
&\leq 4\int_\Omega w_t^2 \mathrm{d}x \Big[\left(\tfrac{2(p_2-1)}{e((p_1-2)-k_1)}\right)^2 \\
&\quad \int_\Omega \left(|\alpha v_1 + (1-\alpha) v_2|^{2k_1}\right) |v|^2 \mathrm{d}x \\
&+ \left(\tfrac{2(p_2-1)}{e(k_2-(p_2-2))}\right)^2 \int_\Omega \left(|\alpha v_1 + (1-\alpha) v_2|^{2k_2}\right) |v|^2 \mathrm{d}x \\
&+ 4\int_\Omega \left(|\alpha v_1 + (1-\alpha) v_2|^{2(p_1-2)}\right) |v|^2 \mathrm{d}x \\
&+ 4\int_\Omega \left(|\alpha v_1 + (1-\alpha) v_2|^{2(p_2-2)}\right) |v|^2 \mathrm{d}x\Big] \\
&\leq c_* \left(\int_\Omega w_t^2 \mathrm{d}x\right) \left(\int_\Omega |v|^{\frac{2n}{n-2}} \mathrm{d}x\right)^{\frac{n-2}{n}} \\
&\quad \Big[\left(\int_\Omega |\alpha v_1 + (1-\alpha) v_2|^{k_1 n}\right)^{\frac{2}{n}} \mathrm{d}x \\
&+ \left(\int_\Omega |\alpha v_1 + (1-\alpha) v_2|^{n k_2}\right)^{\frac{2}{n}} \mathrm{d}x \\
&+ \int_\Omega \left(|\alpha v_1 + (1-\alpha) v_2|^{2(p_1-2)}\right) \mathrm{d}x \\
&+ \int_\Omega \left(|\alpha v_1 + (1-\alpha) v_2|^{2(p_2-2)}\right) \mathrm{d}x\Big],
\end{aligned}$$

If we recall (3.1) and (3.12) we come to

$$\begin{aligned} I^2 & \leq c_* c_s \left(\int_\Omega w_t^2 \mathrm{d}x\right) \|\nabla v\|_2^2 \Big[\|\nabla v_1\|_2^{2k_1} + \|\nabla v_1\|_2^{2k_2} \\ & \quad + \|\nabla v_2\|_2^{2k_1} + \|\nabla v_2\|_2^{2k_2} \\ & \quad + \|\nabla v_1\|_2^{2(p_1-2)} + \|\nabla v_1\|_2^{2(p_2-2)} \\ & \quad + \|\nabla v_2\|_2^{2(p_1-2)} + \|\nabla v_2\|_2^{2(p_2-2)}\Big] \\ & \leq 8 c_* c_s R_0^{2(k_2+p_2-2)} \mathrm{d}(v_1, v_2)\, \beta_{v_1}(w)(t), \end{aligned}$$

where $c_* = c(e, p_1, p_2, k_1, k_2)$ and c_s the Sobolev embedding $H_0^1(\Omega) \to L^{\frac{2n}{n-2}}(\Omega)$.

If we combine, it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta_v(w)(t) \leq \xi \mathrm{d}(v_1, v_2)^{\frac{1}{2}} \beta_v(w)(t)^{\frac{1}{2}}.$$

Since $\beta_v(w)(0) = 0$, by the Gronwall lemma

$$\mathrm{d}(u_1, u_2) \leq \frac{\xi^2 T}{4} \mathrm{d}(v_1, v_2)\, e^T.$$

Choose a $0 < T_1 \leq T$ small enough to satisfy

$$\frac{\xi^2}{4} T_1 e^{T_1} < 1.$$

Thus, according to Banach's contraction mapping theorem, there exists a fixed point $u = \Phi(u) \in B_{T_1}(R_0)$, which is a locally weak solution in time to (1.1).

2. Uniqueness. Suppose we have two solutions u and v and set

$$w(s) = \begin{cases} u_1(s) - u_2(s), & s \in [0,t] \\ 0, & s \in [t, T], \end{cases}$$

then

$$w \in L^2\left(0, T; W_0^{1,p(.)}(\Omega)\right),$$

$$w_t \in L^2\left(0, T; H_0^1(\Omega)\right)$$

and w fulfilled

$$\frac{1}{2}\int_\Omega w_t^2 \mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla w|^2 \mathrm{d}x$$

$$\leq \int_0^t \int_\Omega (F(u) - F(v))\, w_t \mathrm{d}x$$

Consequently, the uniqueness results from the local Lipschitz continuity of $F : \mathbb{R}^* \to \mathbb{R}$ and the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$. This completes the proof of the theorem.

■

4 Blow-up of weak solutions

Finally, we give the sufficient conditions for $m(.)$ for inflating weak solutions of the problem (1.1) in finite time if

$$2 < m_1 \leq m(x) \leq m_2$$

$$< p_1 \leq p(x) \leq p_2 < 2\frac{n-1}{n-2},\ n \geq 3, \tag{4.1}$$

holds, and $E(0) < 0$, where

$$\begin{aligned} E(t) = \ & \tfrac{1}{2}\int_\Omega \left(|u_t(x,t)|^2 + |\nabla u(x,t)|^2\right) \mathrm{d}x \\ & - \int_\Omega \tfrac{1}{p(x)} |u(x,t)|^{p(x)} \ln(|u(x,t)|) \mathrm{d}x \\ & + \int_\Omega \tfrac{1}{p^2(x)} |u(x,t)|^{p(x)} \mathrm{d}x. \end{aligned} \tag{4.2}$$

For our purpose we need to the following lemma showing the decrease in energy E.

Lemma 4.1 The energy associated with the problem (1.1) given by (4.2) satisfies the

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\int_\Omega |u_t|^{m(x)} \mathrm{d}x \leq 0, \tag{4.3}$$

and the inequality $E(t) \leq E(0)$ holds, where

$$\begin{aligned} E(0) = \ & \tfrac{1}{2}\int_\Omega \left(|u_1|^2 + |\nabla u_0|^2\right) \mathrm{d}x \\ & - \int_\Omega \tfrac{1}{p(x)} |u_0|^{p(x)} \ln(|u_0|) \mathrm{d}x \\ & + \int_\Omega \tfrac{1}{p^2(x)} |u_0|^{p(x)} \mathrm{d}x. \end{aligned} \tag{4.4}$$

Let

$$H(t) = -E(t) \text{ for } t \geq 0, \tag{4.5}$$

since $E(t)$ is absolutely continuous, hence $H'(t) \geq 0$ and

$$0 < H(0) \leq H(t) \leq \int_\Omega \frac{1}{p(x)} |u(x,t)|^{p(x)} \ln(|u|) \mathrm{d}x.$$

Lemma 4.2 Let the assumptions (2.1) be fulfilled and let u be the solution of (1.1). Then,

$$\int_\Omega |u|^{p(x)} \mathrm{d}x \geq \int_{\Omega_2} |u|^{p_1} \mathrm{d}x := \|u\|_{p_1,\Omega_2}^{p_1}, \tag{4.6}$$

where

$$\Omega_2 = \{x \in \Omega / |u(x,t)| \geq 1\}.$$

proof. Let

$$\Omega_1 = \{x \in \Omega / |u(x,t)| < 1\},$$

so, we have

$$\begin{aligned} \int_\Omega |u|^{p(x)} \mathrm{d}x & = \int_{\Omega_2} |u|^{p(x)} \mathrm{d}x + \int_{\Omega_1} |u|^{p(x)} \mathrm{d}x \\ & \geq \int_{\Omega_2} |u|^{p_1} \mathrm{d}x + \int_{\Omega_1} |u|^{p_2} \mathrm{d}x \\ & \geq \int_{\Omega_2} |u|^{p_1} \mathrm{d}x := \|u\|_{p_1,\Omega_2}^{p_1}. \end{aligned}$$

Thus (4.6). ■

Lemma 4.3 Under the assumptions of Theorem (3.1), the function $H(t)$ presented above yields the following estimates:

$$0 < H(0) \leq H(t) \leq \frac{|\Omega|}{p_1 e} + \frac{B_s}{(s-p_2)\, ep_1} \|\nabla u\|_2^s,\ t \geq 0, \quad (4.7)$$

where s is chosen sufficiently small such that

$$\begin{aligned} p_1 &\leq p_2 < s \leq \frac{2n}{n-2}, \text{ for } n \geq 3, \quad (4.8)\\ p_1 &\leq p_2 < s < \infty \text{ for } n = 1, 2, \end{aligned}$$

and B_s is a positive constant of embedding $H_0^1(\Omega)$ in $L^s(\Omega)$ such that

$$\|u\|_s \leq B_s \|\nabla u\|_2,\ \forall u \in H_0^1(\Omega). \quad (4.9)$$

proof. By Lemma (4.1), $H(t)$ is nondecreasing in t. Thus

$$H(t) \geq H(0) = -E(0) > 0,\ t \geq 0. \quad (4.10)$$

Combining (4.2), (4.3), (4.5) and using the fact that $\ln \zeta \leq \frac{1}{e\sigma}\zeta^\sigma$ for any $\sigma > 0$ we have

$$\begin{aligned} 0 < H(t) &< \tfrac{1}{p_1} \textstyle\int_\Omega |u(x,t)|^{p(x)} \ln(|u(x,t)|)\mathrm{d}x \\ &= \tfrac{1}{p_1} \textstyle\int_{\{x\in\Omega:|u(x)|<1\}} |u(x,t)|^{p(x)-1} (|u(x,t)| (\ln(|u(x,t)|)))\, \mathrm{d}x \\ &+ \tfrac{1}{p_1} \textstyle\int_{\{x\in\Omega:|u(x)|\geq 1\}} |u(x,t)|^{p(x)} \ln(|u(x,t)|)\mathrm{d}x \\ &\leq \tfrac{|\Omega|}{p_1 e} + \tfrac{1}{\sigma e p_1} \int_{\{x\in\Omega:|u(x)|\geq 1\}} |u|^{p_2+\sigma}\, \mathrm{d}x \\ &\leq \tfrac{|\Omega|}{p_1 e} + \tfrac{1}{\sigma e p_1} \|u\|_{p_2+\sigma}^{p_2+\sigma} \\ &\leq \tfrac{|\Omega|}{p_1 e} + \tfrac{B_s}{(s-p_2)ep_1} \|\nabla u\|_2^s, \end{aligned} \quad (4.11)$$

and (4.7) follows. ■

Theorem 4.4 Suppose the conditions of Theorem (3.1) are satisfied. Moreover, let (4.1) hold as well as $E(0) < 0$. Then the solution of problem (1.1) given by Theorem (3.1) blows up in finite time.

proof. for each t in $[0, T)$ let define

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_\Omega u(x,t)u_t(x,t)\mathrm{d}x, \quad (4.12)$$

with $\varepsilon > 0$ is small enough to be chosen later and α such that

$$0 < \alpha \leq \min\left\{ \frac{p_1 - 2}{2p_1}, \frac{p_1 - m_2}{p_1(m_2 - 1)}, \frac{2(p_1 - m_1)}{s(m_1 - 1)p_1}, \frac{2(p_1 - m_1)}{s(m_2 - 1)p_1} \right\}. \quad (4.13)$$

A straightforward derivation of (4.12) using Eq. (1.1), we obtain

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega \left[u_t^2 - |\nabla u|^2\right] \\ &+ \varepsilon \int_\Omega |u|^{p(x)} (\ln|u|) - \varepsilon \int_\Omega |u_t|^{m(x)-2} u u_t \end{aligned} \quad (4.14)$$

On the right-hand side of (4.14) by adding and subtracting $\varepsilon(1-\eta)p_1 H(t)$ with $0 < \eta < \frac{p_1-2}{p_1}$, we obtain

$$\begin{aligned} L'(t) = (1-\alpha)&H^{-\alpha}(t)H'(t) + \varepsilon(1-\eta)p_1 H(t) \\ &+ \eta \textstyle\int_\Omega |u|^{p(x)} (\ln|u|)\, \mathrm{d}x \\ +\varepsilon &\left(\tfrac{(1-\eta)p_1}{2} + 1\right) \|u_t\|_2^2 + \varepsilon \left(\tfrac{(1-\eta)p_1}{2} - 1\right) \|\nabla u\|_2^2 \\ &- \varepsilon \textstyle\int_\Omega u u_t |u_t|^{m(x)-2}\, \mathrm{d}x, \end{aligned} \quad (4.15)$$

Due to the fact that (4.6), taking into account

$$\frac{1}{p_2^2} \int_\Omega |u(x,t)|^{p(x)}\mathrm{d}x < \frac{1}{p_1} \int_\Omega |u|^{p(x)} (\ln|u|)\, \mathrm{d}x,$$

(4.15) result in

$$\begin{aligned} L'(t) &\geq (1-\alpha)H^{-\alpha}(t)H'(t) - \varepsilon \textstyle\int_\Omega |u_t|^{m(x)-2} u u_t \mathrm{d}x \\ &+ \varepsilon\beta \left[H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + \textstyle\int_\Omega |u(x,t)|^{p(x)}\mathrm{d}x\right] \\ &\geq (1-\alpha)H^{-\alpha}(t)H'(t) - \varepsilon \textstyle\int_\Omega |u_t|^{m(x)-2} u u_t \mathrm{d}x \\ &+ \varepsilon\beta \left[H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + \|u\|_{p_1,\Omega_2}^{p_1}\right], \end{aligned} \quad (4.16)$$

where

$$\beta = \min\left\{ (1-\eta)p_1, \frac{p_1}{p_2^2}\eta, \frac{(1-\eta)p_1}{2} + 1, \frac{(1-\eta)p_1}{2} - 1 \right\} > 0.$$

Now, using Young's inequality, we estimate the last term in (4.14) in the manner shown below

$$\int_\Omega |u_t|^{m(x)-1} |u|\mathrm{d}x \leq \frac{1}{m_1} \int_\Omega \zeta^{m(x)} |u|^{m(x)}\mathrm{d}x + \frac{m_2 - 1}{m_2} \int_\Omega \zeta^{-\frac{m(x)}{m(x)-1}} |u_t|^{m(x)}\, \mathrm{d}x,\ \forall \zeta > 0. \quad (4.17)$$

Consequently, by taking δ such that

$$\zeta^{-\frac{m(x)}{m(x)-1}} = kH^{-\alpha}(t),\ k > 0,$$

By putting it in (4.17) with k large enough to be determined later, we obtain

$$\int_\Omega |u_t|^{m(x)-1}|u|\mathrm{d}x \le \frac{1}{m_1}\int_\Omega k^{1-m(x)}|u|^{m(x)}H^{\alpha(m(x)-1)}(t)\mathrm{d}x + \frac{(m_2-1)k}{m_2}H^{-\alpha}(t)H'(t). \tag{4.18}$$

The result of joining (4.16) with (4.18)

$$\begin{aligned} L'(t) \ge & \left[(1-\alpha)-\varepsilon\left(\frac{m_2-1}{m_2}\right)k\right]H^{-\alpha}(t)H'(t) \\ &+\varepsilon\beta\left[H(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+\|u(t)\|_{p_1}^{p_1}\right] \\ &-\varepsilon\frac{k^{1-m_1}}{m_1}H^{\alpha(m_2-1)}(t)\int_\Omega |u|^{m(x)}\mathrm{d}x. \end{aligned} \tag{4.19}$$

Applying lemma (4.3) we have

$$\begin{aligned} & H^{\alpha(m_2-1)}(t)\int_\Omega |u(t)|^{m(x)}\mathrm{d}x \\ &\le C\Big[2^{\alpha(m_2-1)-1}\left(\frac{|\Omega|}{p_1e}\right)^{\alpha(m_2-1)} \\ &+2^{\alpha(m_2-1)-1}\frac{1}{(s-p_2)ep_1}\|\nabla u\|_2^{s\alpha(m_2-1)} \\ &\left(\|u\|_{p_1,\Omega_2}^{m_1}+\|u\|_{p_1,\Omega_2}^{m_2}\right)\Big] \\ &\le 2^{\alpha(m_2-1)-1}C\left(\frac{|\Omega|}{p_1e}\right)^{\alpha(m_2-1)} \\ &\times\left(\left(\|u\|_{p_1,\Omega_2}^{p_1}\right)^{\frac{m_1}{p_1}}+\left(\|u\|_{p_1,\Omega_2}^{p_1}\right)^{\frac{m_2}{p_1}}\right) \\ &+2^{\alpha(m_2-1)-1}C\frac{1}{(s-p_2)ep_1}\|\nabla u\|_2^{s\alpha(m_2-1)} \\ &\times\left(\|u\|_{p_1,\Omega_2}^{m_1}+\|u\|_{p_1,\Omega_2}^{m_2}\right). \end{aligned} \tag{4.20}$$

We are to analyze the terms on the right-hand side of (4.20). By using Young's inequality, we have

$$\begin{aligned} \|\nabla u\|_2^{s\alpha(m_2-1)}\|u\|_{p_1,\Omega_2}^{m_1} &\le \frac{m_1}{p_1}\|u(t)\|_{p_1,\Omega_2}^{p_1} + C\frac{p_1-m_1}{p_1}\|\nabla u\|_2^{\frac{s\alpha(m_2-1)p_1}{p_1-m_1}} \\ &= \frac{m_1}{p_1}\|u(t)\|_{p_1,\Omega_2}^{p_1} + C\frac{p_1-m_1}{p_1}\left(\|\nabla u\|_2^2\right)^{\frac{s\alpha(m_2-1)p_1}{2(p_1-m_1)}}, \end{aligned}$$

similarly

$$\|\nabla u\|_2^{s\alpha(m_2-1)}\|u(t)\|_{p_1,\Omega_2}^{m_2} \le \frac{m_2}{p_1}\|u(t)\|_{p_1,\Omega_2}^{p_1} + C\frac{p_1-m_2}{p_1}\left(\|\nabla u\|_2^2\right)^{\frac{s\alpha(m_2-1)p_1}{2(p_1-m_2)}}.$$

Using the following well-known algebraic inequality:

$$z^\tau \le z+1 \le \left(1+\frac{1}{d}\right)(z+d),\ \forall z\ge 0,\ 0<\tau\le 1,\ d\ge 0, \tag{4.21}$$

with $z=\|u(t)\|_{p_1,\Omega_2}^{p_1}$, $a=1+\frac{1}{H(0)}$, $d=H(0)$ and $\tau=\frac{m_1}{p_1}$ $\left(\tau=\frac{m_2}{p_1}\right)$, respectively, then the condition (4.1) implies that $0<\tau\le 1$ and therefore

$$\left(\|u(t)\|_{p_1,\Omega_2}^{p_1}\right)^{\frac{m_1}{p_1}}+\left(\|u(t)\|_{p_1,\Omega_2}^{p_1}\right)^{\frac{m_2}{p_1}} \le 2a\left(\|u(t)\|_{p_1,\Omega_2}^{p_1}+H(0)\right) \le 2a\left(\|u(t)\|_{p_1,\Omega_2}^{p_1}+H(t)\right),$$

similarly, with $z=\|\nabla u\|_2^2$, $b=1+\frac{1}{H(0)}$, $d=H(0)$ and $\tau=\frac{s\alpha(m_2-1)p_1}{2(p_1-m_1)}$, then the condition (4.13) implies that $0<\tau\le 1$ and therefore

$$\left(\|\nabla u\|_2^2\right)^{\frac{s\alpha(m_2-1)p_1}{2(p_1-m_1)}} \le b\left(\left(\|\nabla u\|_2^2+H(0)\right)\right) \le b\left(\left(\|\nabla u\|_2^2+H(t)\right)\right),$$

also, with $z=\|\nabla u\|_2^2$, $h=1+\frac{1}{H(0)}$, $d=H(0)$ and $\tau=\frac{s\alpha(m_2-1)p_1}{2(p_1-m_2)}$,

$$\left(\|\nabla u\|_2^2\right)^{\frac{s\alpha(m_2-1)p_1}{2(p_1-m_2)}} \le h\left(\left(\|\nabla u\|_2^2+H(t)\right)\right),$$

therefore, (4.20) leads to

$$H^{\alpha(m_2-1)}(t)\int_\Omega |u(t)|^{m(x)}\mathrm{d}x \le C\left(\|u(t)\|_{p_1,\Omega_2}^{p_1}+H(t)+\|\nabla u\|_2^2\right),\ \forall t\in[0,T]. \tag{4.22}$$

where C to indicate a generic positive constant depending on $(\Omega, e, h, p_{1,2}, m_{1,2})$ only. Combining (4.19) and (4.22) yields

$$\begin{aligned} L'(t) &\ge \left((1-\alpha)-\varepsilon\left(\frac{m_2-1}{m_2}\right)k\right)H^{-\alpha}(t)H'(t) \\ &+\varepsilon\left(\beta-\frac{k^{1-m_1}}{m_1}C\right) \\ &\times\left[H(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+\|u(t)\|_{p_1,\Omega_2}^{p_1}\right]. \end{aligned} \tag{4.23}$$

At this point we pick $\gamma=\beta-\frac{k^{1-m_1}}{m_1}C>0$, (it is the case when $k>\left(\frac{\beta m_1}{C}\right)^{\frac{1}{1-m_1}}$).

Once k is fixed we pick $\varepsilon>0$ sufficient small so that

$$(1-\alpha)-\varepsilon\left(\frac{m_2-1}{m_2}\right)k\ge 0$$

and $L(0)=H^{1-\alpha}(0)+\varepsilon\int_\Omega u_0(x)u_1(x)\mathrm{d}x>0$.

Hence (4.23) takes the form

$$L'(t) \geq \gamma \left(H(t) + \|u_t\|_2^2 + \|\nabla u\|_2^2 + \|u(t)\|_{p_1,\Omega_2}^{p_1} \right). \tag{4.24}$$

Therefore, we have

$$L(t) \geq L(0) > 0, \text{ for all } t \geq 0$$

On the other hand from (4.12),

$$L^{\frac{1}{1-\alpha}}(t) \leq 2^{1/(1-\alpha)} \left(H(t) + \left| \int_\Omega u u_t(x,t) \mathrm{d}x \right|^{\frac{1}{1-\alpha}} \right), \tag{4.25}$$

By applying Holder's inequality we see that

$$\left| \int_\Omega u u_t(x,t) \mathrm{d}x \right| \leq C\|u\|_{p_1} \|u_t\|_2 \leq 2C\|u\|_{p_1,\Omega_2} \|u_t\|_2 .$$

Again, algebraic inequality (4.21), with $z = \|u\|_{p_1,\Omega_2}^{p_1}$, $h = 1 + \frac{1}{H(0)}$, $d = H(0)$ and $0 < \tau = \frac{2}{(1-2\alpha)p_1} \leq 1$ (see (4.13)), gives

$$\left(\|u\|_{p_1,\Omega_2}^{p_1} \right)^{\frac{2}{(1-2\alpha)p_1}} \leq C \left(\|u\|_{p_1,\Omega_2}^{p_1} + H(t) \right),$$

Thus, Young's inequality gives

$$\begin{aligned} &\left| \textstyle\int_\Omega u u_t(x,t) \mathrm{d}x \right|^{1/(1-\alpha)} \\ &\leq C \left[\|u\|_{p_1,\Omega_2}^{\frac{2(1-\alpha)}{1-2\alpha}} + \|u_t\|_2^{2(1-\alpha)} \right]^{1/(1-\alpha)} \\ &\leq C \left[\left(\|u\|_{p_1,\Omega_2}^{p_1} \right)^{\frac{2}{(1-2\alpha)p_1}} + \|u_t\|_2^2 \right] \\ &\leq C \left[\|u\|_{p_1,\Omega_2}^{p_1} + H(t) + \|u_t\|_2^2 \right], \text{ for all } t \geq 0, \end{aligned}$$

joining it with (4.24) and (4.25) yields

$$L'(t) \geq \delta L^{\frac{1}{1-\alpha}}(t), \text{ for all } t \geq 0, \tag{4.26}$$

where δ is a positive constant depending on (ε, γ, C). With a simple integration of (4.26) over $(0,t)$ we infer that

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{L^{\frac{\alpha}{1-\alpha}}(0) - \frac{\alpha}{1-\alpha}\delta t}. \tag{4.27}$$

Consequently, $L(t)$ blows up in a finite time $\widehat{T}$

$$\widehat{T} \leq \frac{1-\alpha}{\delta\alpha L^{\frac{\alpha}{1-\alpha}}(0)}.$$

■

Data Availability

No data is used in the manuscript.

Disclosure statement

No potential conflict of interest was reported by the author

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