Dynamic Behaviors of a Stage Structure Commensalism System with Holling type II Commensalistic Benefits

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Abstract: - Noting the fact that commensal species that behave as foragers are subject to the constraints of handling time, a two species commensalism model with Holling type II commensalistic benefits and stage structure is proposed and studied. We first show that among four possible equilibria, host-only equilibrium and positive equilibrium are possible asymptotically stable. Next, we establish a powerful lemma on the global stability property of the single species stage structured model with linear perturbation on mature species. By applying this lemma and the differential inequalities theory, sufficient conditions which ensure the global attractivity of the host-only equilibrium and positive equilibrium are obtained, respectively. Our results generalize some known results.

Key-Words: Stage structure; Commensalism; Comparison theorem; Holling II functional response; Global attractivity

1 Introduction

The aim of this paper is to investigate the dynamic behaviors of the following commensalism system with Holling II functional response and stage structure:

$$
\begin{aligned}
\frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1, \\
\frac{dx_2}{dt} &= \beta x_1 - \delta_2 x_2 - \gamma x_2^2 + d\frac{y}{\eta_1 + \eta_2 y} x_2, \\
\frac{dy}{dt} &= y(b_2 - a_2 y),
\end{aligned}
\tag{1}
$$

where $\alpha, \beta, \delta_1, \delta_2, d, b_2, a_2, \eta$ and γ are all positive constants, $x_1(t)$ and $x_2(t)$ are the densities of the immature and mature commensal species at time t, $y(t)$ is the density of the host species at time t. The following assumptions are made in formulating the model (1):

(A_1) The birth rate of the immature commensal species is proportional to the existing mature population with a proportionality constant α; for the immature commensal species, the death rate and transformation rate of mature are proportional to the existing immature population with proportionality constants β and δ_1;

(A_2) The death rate of the mature commensal species is proportional to the existing mature population with a proportionality constant δ_2. The mature commensal species has density restriction, γ is density dependent coefficient;

(A_3) The host species satisfies the logistic model;

(A_4) The host species only benefits to the mature commensal species with rate $\dfrac{dy}{\eta_1 + \eta_2 y}$, which is of Holling II type.

During the last decade, many scholars investigated the dynamic behaviors of the mutualism model or commensalism model [1]-[39]. By means of commensal interactions, one species benefits from the other without either harming or benefiting the latter. Commensalism is very common in nature, for example, small plants called epiphytes and the large tree branches on which they grow. Epiphytes depend on their hosts for structural support but do not derive nourishment from them or harm them in any way. Another example is the remora (family Echineidae) that rides attached to sharks and other fishes. One could refer to https://www.britannica.com/science/commensalism for more background and examples.

Despite commensalism is a very common relationship in nature, its mathematical modeling lags far behind. In 2013, for the first time, Sun and Sun[42] first time proposed a two species commensalism model. Since then, many scholars worked on this direction. Topics such as the influence of harvesting [2, 3, 5, 9, 10, 11, 25, 26, 32, 37, 38], the existence of periodic solution or almost periodic solution [6, 8, 12, 21, 23], the stability of the system [1, 2, 3, 4, 7, 18, 19, 20, 25, 31, 37, 38, 39, 42], the persistent and extinction property of the system [2, 19], the influence of stage structure [13], the influence of Allee effect [14, 15, 16, 17, 22, 29, 30, 34, 35], the influence of time delays [1, 6, 24], the influence of feedback control [20, 31], the bifurcation phenomenon of the system [26, 28, 29, 30, 34, 35] etc were extensively investigated.

In the real world, almost all animals have the stage structure of immature and mature. In differ-

ent stages, the species have different reactions to the environment. many scholars investigated the dynamic behaviors of the stage structured species, see [1, 13, 32, 40, 44, 45] and the references cited therein. Aiello and Freedman [44] first time proposed the following stage-structured single species model.

$$
\begin{aligned}
\frac{dx_1(t)}{dt} &= \alpha x_2(t) - \gamma x_1(t) \\
&\quad - \alpha e^{-\gamma \tau} x_2(t - \tau), \\
\frac{dx_2(t)}{dt} &= \alpha e^{-\gamma \tau} x_2(t - \tau) - \beta x_2^2(t).
\end{aligned} \tag{2}
$$

They showed that the above system admits a unique positive equilibrium which is globally asymptotically stable. Such a result is similar to the traditional single species Logistic model. However, in above system, the authors did not consider the influence of death rate of mature species. Chen, Xie, Chen [1] proposed and studied the following May type stage-structured cooperation model,

$$
\begin{aligned}
\dot{x}_1(t) &= b_1 e^{-d_{11}\tau_1} x_1(t - \tau_1) - d_{12} x_1(t) \\
&\quad - \frac{a_{11} x_1^2(t)}{c_1 + f_1 x_2(t)} - a_{12} x_1^2(t), \\
\dot{y}_1(t) &= b_1 x_1(t) - d_{11} y_1(t) \\
&\quad - b_1 e^{-d_{11}\tau_1} x_1(t - \tau_1), \\
\dot{x}_2(t) &= b_2 e^{-d_{22}\tau_2} x_2(t - \tau_2) - d_{21} x_2(t) \\
&\quad - \frac{a_{22} x_2^2(t)}{c_2 + f_2 x_1(t)} - a_{21} x_2^2(t), \\
\dot{y}_2(t) &= b_2 x_2(t) - d_{22} y_2(t) \\
&\quad - b_2 e^{-d_{22}\tau_2} x_2(t - \tau_2).
\end{aligned} \tag{3}
$$

In this system, d_{12} and d_{21} represent the death rates of the first and second mature species, respectively. They showed that despite the cooperation between the species, the species may still be driven to the extinction due to the stage structure. Death rate of mature species plays crucial role on the persistence and extinction property of the system.

Instead of consider the influence of delay, some scholars [13], [32], [40], [44] assumed that there are proportional number of immature species becomes mature species. Khajanchi and Banerjee [44] proposed the following stage structure predator-prey

model with ratio dependent functional response

$$
\begin{aligned}
\frac{dx_1}{dt} &= \alpha x_2(t) - \beta x_1(t) - \delta_1 x_1(t), \\
\frac{dx_2}{dt} &= \beta x_1(t) - \delta_2 x_2(t) - \gamma x_2^2(t) \\
&\quad - \frac{\eta(1-\theta)x_2(t)y(t)}{g(1-\theta)x_2(t) + hy(t)}, \\
\frac{dy}{dt} &= \frac{u\eta(1-\theta)x_2(t)y(t)}{g(1-\theta)x_2(t) + hy(t)} - \delta_3 y(t).
\end{aligned} \tag{4}
$$

Here the authors assumed that the prey species is stage structured. The authors only considered the stability property of predator-extinction equilibrium and the positive interior equilibrium. We would like to point out that Xiao and Lei [12] showed that the death rate of the mature species plays important role on the persistence and extinction of the single species stage structured system.

In our opinion, the commensalism models were not well studied in the sense that up today, just one paper [13] considers the influence of the stage structure. A recent study [43] showed that the relationship among Brazil nut and three frog species is commensalism, and it is well known that frog species are stage structured species. Hence, it is necessary to propose some modeling on stage structured commensalism. In [13], Lei proposed the following two species commensalism model

$$
\begin{aligned}
\frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1, \\
\frac{dx_2}{dt} &= \beta x_1 - \delta_2 x_2 - \gamma x_2^2 + dx_2 y, \\
\frac{dy}{dt} &= y(b_2 - a_2 y).
\end{aligned} \tag{5}
$$

Concerned with the global stability property of the equilibria of system (5), by constructing suitable Lyapunov function, the author obtained the following result (Theorem 3.1 and 3.2 in [13]):

Theorem A. (1) if

$$
(\beta + \delta_1)\left(\delta_2 - \frac{db_2}{a_2}\right) - \alpha\beta > 0 \tag{6}
$$

hold, then $A_2(0, 0, \frac{b_2}{a_2})$ is globally asymptotically stable. If

$$
(\beta + \delta_1)\left(\delta_2 - \frac{db_2}{a_2}\right) - \alpha\beta < 0 \tag{7}
$$

hold, then $A_4(\overline{x}_1, \overline{x}_2, \overline{y})$ is globally asymptotically stable, where

$$
\begin{aligned}
\overline{x}_1 &= \frac{\alpha \overline{x}_2}{\beta + \delta_1}, \\
\overline{x}_2 &= \frac{\alpha\beta - \left(\delta_2 - d\overline{y}\right)(\beta + \delta_1)}{(\beta + \delta_1)\gamma}, \qquad (8) \\
\overline{y} &= \frac{b_2}{a_2}.
\end{aligned}
$$

Traditional two species Lotka-Volterra cooperation system, which takes the form

$$
\begin{aligned}
\frac{dx}{dt} &= x(r_1 - a_{11}x + a_{12}y), \\
\frac{dy}{dt} &= y(r_2 - a_{21}y + a_{22}x),
\end{aligned}
$$

is criticized by many biologist. The reason lies in the unrealistic assumption that the benefits of the interaction were unlimited and increased in direct proportion to the density of the mutualistic partner. To overcome this, and inspired by the fact that mutualists that behave as foragers are also subject to the constraints of handling time, for example, (for a species of solitary bee visiting a flower species. The rate of collection of pollen is limited by the handling time per plant), Wright[43] proposed the following two species mutualism model

$$
\begin{aligned}
\frac{dN}{dt} &= N\left(r_1(1 - c_1 N) + \frac{baM}{1 + aT_{11}M}\right), \\
\frac{dM}{dt} &= M\left(r_2(1 - c_2 M) + \frac{cdN}{1 + dT_{22}N}\right),
\end{aligned}
$$

where the author used Holling type II functional response $\frac{baM}{1 + aT_{11}M}$ to describe the feeding rate (items per unit of time, T_{11} is the handling time, b is a coefficient converting M to new units of N. The model is based on the traditional logistic equation with a term added to include the per capita benefits of interacting with the population of the mutualist partner. The model may have none, one or two positive equilibrium, i.e., with the introduction of functional response, the dynamic behaviors of the system becomes more complex.

Probably inspired by the works of Wright [43] or some similar works, several scholars [7],[15],[16],[17],[23],[28],[31],[39] argued that the the relationship of two commensalism model should be described by the suitable functional response. Li, Lin and Chen [23] for the first time adopt the idea of functional response of predator prey system, they proposed the following discrete commensalism model

with Holling II functional response.

$$
\begin{aligned}
x_1(k+1) &= x_1(k)\exp\Big\{a_1(k) - b_1(k)x_1(k) \\
&\quad + \frac{c_1(k)x_2(k)}{e_1(k) + f_1(k)x_2(k)}\Big\}, \\
x_2(k+1) &= x_2(k)\exp\left\{a_2(k) - b_2(k)x_2(k)\right\}.
\end{aligned}
$$

They studied the positive periodic solution of the system.

Wu [7] argued that a relationship of nonlinear type between two species is more feasible, and she established the following two species commensal symbiosis model

$$
\begin{aligned}
\frac{dx}{dt} &= x\Big(a_1 - b_1 x + \frac{c_1 y^p}{1 + y^p}\Big), \\
\frac{dy}{dt} &= y(a_2 - b_2 y),
\end{aligned} \qquad (9)
$$

where $a_i, b_i, i = 1, 2$, p and c_1 are all positive constants, $p \geq 1$. The results of [7] is then generalized by Lei [23] and Wu, Li and Lin [16] to the commensalism model with Allee effect.

Recently, Jawad [26] proposed the following commensalism model with Michaelis-Menten type harvesting and Holling type II functional response:

$$
\begin{aligned}
\frac{du}{dt} &= ru\Big(1 - \frac{u}{k}\Big) + \frac{\beta uv}{\alpha + u} - \frac{qEu}{cE + lu}, \\
\frac{dv}{dt} &= sv\Big(1 - \frac{v}{m}\Big) - dv,
\end{aligned}
$$

where $u(t)$ and $v(t)$ denote the densities of the first and second species at time t, respectively. Topics such as permanence, saddle node bifurcation were discussed in [26].

Chen, Chong and Lin [31] proposed the following commensal symbiosis model with Holling type II functional response and feedback controls:

$$
\begin{aligned}
\frac{dx}{dt} &= x\Big(b_1 - a_{11}x + \frac{a_{12}y}{a_{13} + a_{14}y} - \alpha_1 u_1\Big), \\
\frac{dy}{dt} &= y(b_2 - a_{22}y - \alpha_2 u_2), \\
\frac{du_1}{dt} &= -\eta_1 u_1 + a_1 x, \\
\frac{du_2}{dt} &= -\eta_2 u_2 + a_2 y,
\end{aligned}
$$

$$\qquad (10)$$

where $x(t)$ and $y(t)$ denote the density of the first and second species at time t, and u_1 and u_2 are feedback control variables. By developing some new analytical technique, the authors showed that the unique positive equilibrium is globally attractive.

Xu, Xue, Xie and Lin [39] proposed and studied the commensalism model with Crowley-Martin functional response. i.e.,

$$\frac{dx}{dt} = x\Big(-a_1 - b_1 x + \frac{c_1 y}{d_1 + e_1 x + f_1 y + g_1 xy}\Big),$$

$$\frac{dy}{dt} = y(a_2 - b_2 y).$$

For autonomous case, i.e., when all the coefficients of the system are positive constants, authors investigated the local and global dynamic behaviors of the system. For nonautonomous case, authors investigated the persistent and extinction properties of the system.

Li and Wang [28] argued that a suitable model should include some past state of the system, and they investigated the dynamic behaviors of the following commensalism system

$$\frac{dx}{dt} = x\Big(a_1 - b_1 x + \frac{c_1 y}{1 + e_1 x + f_1 y}\Big),$$

$$\frac{dy}{dt} = y(a_2 - b_2 y(t - \tau)).$$

As was shown above, there are several kinds of functional response used in modelling the commensalism model, however, it seems that the discussion of Wright [43] is one of the most reasonable, and the biological explanation is plausible. Stimulated by the above works, specially stimulated by the work of Wright [43], we propose the system (1). As far as system (1) is concerned, since it seems that the system is similar to system (5), only with the cooperation term $dx_2 y$ in system (5) changed to the term with functional response $d\dfrac{y}{\eta_1 + \eta_2 y} x_2$ in system (1). One may expect the analysis method used in Lei [13] could be directly applied to system (1), i.e., one could investigate the stability property of the positive equilibrium by constructing the Lyapunov function

$$V(x_1, x_2, y)$$

$$= k_1\Big(x_1 - x_1^{**} - x_1^{**} \ln \frac{x_1}{x_1^{**}}\Big)$$

$$+ k_2\Big(x_2 - x_2^{**} - x_2^{**} \ln \frac{x_2}{x_2^{**}}\Big)$$

$$+ k_3\Big(y - y^{**} - y^{**} \ln \frac{y}{y^{**}}\Big),$$

where $x_1^{**}, x_2^{**}, y^{**}$ are defined in (15). However, it is very difficult to deal with the nonlinear term to ensure $D^+V(t) < 0$. By constructing Lyapunov function as above, one could obtain some sufficient conditions to ensure the global stability of the solution to system (1), however, the condition, generally speaking, is not a good one, since in dealing with the term

$d\dfrac{y}{\eta_1 + \eta_2 y} x_2$, some additional conditions are needed, which could not reflect the essential characteristic of system (1). We mention here that in [7], Wu investigated the global stability of the equilibrium of system (9) by using the Dulac criterion, which could only be applied to the two dimensional systems, and could not be applied to three dimensional systems. Recently, in the study of dynamic behaviors of the system (10), Chen, Chong and Lin [31] developed some analytical technique, more precisely, they essentially grasped the characteristics of the commensalism systems, and applied the differential inequalities theory, to obtain some interesting results about system (10). We will try to develop the analytic idea of [31] to investigate the dynamic behaviors of system (1).

The paper is arranged as follows. We investigate the existence and locally stability property of the equilibria of system (1) in Section 2. In Section 3, we establish a stability result about the single species stage structured system. In Section 4, by using the differential inequalities theory and the Lemma established in Section 3, we provide conditions which ensure the global attractivity property of the equilibria. In Section 5, we present some numerical simulations to show the feasibility of the main results. We end this paper by a brief discussion.

2 Local stability

In this section, we will investigate the existence and local stability property of system (1).

The equilibria of system (1) is determined by the following system

$$\alpha x_2 - \beta x_1 - \delta_1 x_1 = 0,$$

$$\beta x_1 - \delta_2 x_2 - \gamma x_2^2 + d\frac{y}{\eta_1 + \eta_2 y} x_2 = 0,$$

$$y(b_2 - a_2 y) = 0.$$

$$(11)$$

The system always admits two boundary equilibria: $A_1(0, 0, 0)$, $A_2(0, 0, \frac{b_2}{a_2})$, if, in addition

$$\alpha\beta > \delta_2(\beta + \delta_1), \qquad (12)$$

then the system admits another boundary equilibrium $A_3\big(x_1^*, x_2^*, 0\big)$, where

$$x_1^* = \frac{\alpha x_2^*}{\beta + \delta_1}, \quad x_2^* = \frac{\alpha\beta - \delta_2(\beta + \delta_1)}{\gamma(\beta + \delta_1)}. \qquad (13)$$

If

$$\alpha\beta - \Big(\delta_2 - \frac{dy^{**}}{\eta_1 + \eta_2 y^{**}}\Big)(\beta + \delta_1) > 0, \qquad (14)$$

then system (1) admits a unique positive equilibrium $A_4\left(x_1^{**}, x_2^{**}, y^{**}\right)$, where

$$
\begin{aligned}
x_1^{**} &= \frac{\alpha x_2^{**}}{\beta + \delta_1}, \\
x_2^{**} &= \frac{\alpha\beta - \left(\delta_2 - \dfrac{dy^{**}}{\eta_1 + \eta_2 y^{**}}\right)(\beta + \delta_1)}{(\beta + \delta_1)\gamma}, \\
y^{**} &= \frac{b_2}{a_2}.
\end{aligned}
\tag{15}
$$

Obviously, x_1^{**}, x_2^{**} and y^{**} satisfies the equations

$$
\alpha x_2^{**} - \beta x_1^{**} - \delta_1 x_1^{**} = 0,
$$

$$
\beta x_1^{**} - \delta_2 x_2^{**} - \gamma(x_2^{**})^2 + d\frac{y^{**}}{\eta_1 + \eta_2 y^{**}}x_2^{**} = 0,
$$

$$
b_2 - a_2 y^{**} = 0.
\tag{16}
$$

We shall now investigate the local stability property of the above equilibria.

The variational matrix of system (1) is

$$
\begin{aligned}
&J(x_1, x_2, y) \\
&= \begin{pmatrix} -\beta - \delta_1 & \alpha & 0 \\ \beta & W_1 & W_2 \\ 0 & 0 & -2a_2y + b_2 \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
W_1 &= -\delta_2 - 2\gamma x_2 + d\frac{y}{\eta_1 + \eta_2 y}, \\
W_2 &= \frac{dx_2}{\eta_1 + \eta_2 y} - \frac{d\eta_2 x_2 y}{(\eta_1 + \eta_2 y)^2}.
\end{aligned}
$$

Theorem 2.1 $A_1(0,0,0)$ *is unstable.*
Proof. The Jacobian matrix of the equilibrium point $A_1(0,0,0)$ is given by

$$
\begin{pmatrix} -\beta - \delta_1 & \alpha & 0 \\ \beta & -\delta_2 & 0 \\ 0 & 0 & b_2 \end{pmatrix}.
$$

The characteristic equation of the above matrix is

$$
(\lambda - b_2)\left(\lambda^2 + (\delta_1 + \delta_2 + \beta)\lambda + \beta\delta_2 + \delta_1\delta_2 - \alpha\beta\right) = 0.
$$

Hence, it has one positive characteristic root $\lambda_1 = b_2$, consequently, $A_1(0,0,0)$ is unstable. This ends the proof of Theorem 2.1.

Theorem 2.2 *Assume that*

$$
(\beta + \delta_1)\left(\delta_2 - \frac{dy^{**}}{\eta_1 + \eta_2 y^{**}}\right) - \alpha\beta > 0,
\tag{17}
$$

then $A_2(0, 0, \frac{b_2}{a_2})$ *is locally asymptotically stable. Assume that*

$$
(\beta + \delta_1)\left(\delta_2 - \frac{dy^{**}}{\eta_1 + \eta_2 y^{**}}\right) - \alpha\beta < 0,
\tag{18}
$$

then $A_2(0, 0, \frac{b_2}{a_2})$ *is unstable.*
Proof. The Jacobian matrix of the system about the equilibrium point $A_2(0, 0, \frac{b_2}{a_2})$ is given by

$$
\begin{pmatrix} -\beta - \delta_1 & \alpha & 0 \\ \beta & \dfrac{dy^{**}}{\eta_1 + \eta_2 y^{**}} - \delta_2 & 0 \\ 0 & 0 & -b_2 \end{pmatrix}.
$$

The characteristic equation of the above matrix is

$$
(\lambda + b_2)\left[\lambda^2 + A_1\lambda + A_2\right] = 0.
\tag{19}
$$

where

$$
\begin{aligned}
A_1 &= \delta_1 + \delta_2 + \beta - \frac{dy^{**}}{\eta_1 + \eta_2 y^{**}}, \\
A_2 &= (\beta + \delta_1)\left(\delta_2 - \frac{dy^{**}}{\eta_1 + \eta_2 y^{**}}\right) - \alpha\beta.
\end{aligned}
$$

Hence, it has one negative characteristic root $\lambda_1 = -b_2 < 0$, the other two characteristic roots are determined by the equation

$$
\lambda^2 + A_1\lambda + A_2 = 0.
\tag{20}
$$

Noting that the two characteristic roots of equation (20) satisfy

$$
\begin{aligned}
\lambda_2 + \lambda_3 &= -A_1, \\
\lambda_2\lambda_3 &= A_2,
\end{aligned}
\tag{21}
$$

under the assumption (18), $\lambda_2\lambda_3 < 0$, hence, λ_2 and λ_3 should be one positive and the other negative, which means that one characteristic root is positive, consequently, $A_2(0, 0, \frac{b_2}{a_2})$ is unstable. Instead, assumption (17) implies that

$$
\delta_2 > \frac{dy^{**}}{\eta_1 + \eta_2 y^{**}},
$$

and so, from (21) and (17), one has $\lambda_2 + \lambda_3 < 0$, $\lambda_2\lambda_3 > 0$. Hence, $\lambda_2 < 0, \lambda_3 < 0$. That is, under the assumption (17), three characteristic roots of the equation (19) are all negative, hence, $A_1(0, 0, \frac{b_2}{a_2})$ is locally asymptotically stable. This ends the proof of Theorem 2.2.

Theorem 2.3 $A_3(x_1^*, x_2^*, 0)$ *is unstable.*

Proof. Obviously, x_1^*, x_2^* (see (13)) satisfy the equations

$$\alpha x_2^* - \beta x_1^* - \delta_1 x_1^* = 0,$$

$$\beta x_1^* - \delta_2 x_2^* - \gamma (x_2^*)^2 = 0.$$

Consequently, the Jacobian matrix at the equilibrium point $A_3(x_1^*, x_2^*, 0)$ is given by

$$\begin{pmatrix} -\beta - \delta_1 & \alpha & 0 \\ \beta & -\delta_2 - 2\gamma x_2^* & \dfrac{dx_2^*}{\eta_1} \\ 0 & 0 & b_2 \end{pmatrix}.$$

The characteristic equation of the above matrix is

$$(\lambda - b_2)\Big(\lambda^2 + B_1\lambda + B_2\Big) = 0, \qquad (22)$$

where

$$\begin{aligned} B_1 &= \delta_1 + 2\gamma x_2^* + \beta + \delta_2, \\ B_2 &= (\beta + \delta_1)(\delta_2 + 2\gamma x_2^*) - \alpha\beta. \end{aligned}$$

Equation (22) has at least one positive root $\lambda_1 = b_2$, consequently, $A_3(x_1^*, x_2^*, 0)$ is unstable. This ends the proof of Theorem 2.3.

Theorem 2.4 *Assume that (14) holds, then $A_4(x_1^{**}, x_2^{**}, y^{**})$, where $x_1^{**}, x_2^{**}, y^{**}$ are expressed in (15), is locally asymptotically stable.*
Proof. The Jacobian matrix of the system about the equilibrium point $A_4(x_1^{**}, x_2^{**}, y^{**})$ is given by

$$\begin{pmatrix} -\beta - \delta_1 & \alpha & 0 \\ \beta & V_1 & V_2 \\ 0 & 0 & -2a_2 y^{**} + b_2 \end{pmatrix}.$$

where

$$\begin{aligned} V_1 &= -\delta_2 - 2\gamma x_2^{**} + d\frac{y^{**}}{\eta_1 + \eta_2 y^{**}}, \\ V_2 &= \frac{dx_2^{**}}{\eta_1 + \eta_2 y^{**}} - \frac{d\eta_2 x_2^{**} y}{(\eta_1 + \eta_2 y^{**})^2}. \end{aligned}$$

Noting that

$$-2a_2 y^{**} + b_2 = -2a_2\frac{b_2}{a_2} + b_2 = -b_2,$$

and, from the second equation of (16), the definitions of x_1^{**} and x_2^{**} (see (15)), we have

$$\begin{aligned} &-\delta_2 - 2\gamma x_2^{**} + d\frac{y^{**}}{\eta_1 + \eta_2 y^{**}} \\ =\ & -\frac{\beta x_1^{**}}{x_2^{**}} - \gamma x_2^{**} = -\frac{\alpha\beta}{\beta + \delta_1} - \gamma x_2^{**}. \end{aligned}$$

So, the characteristic equation of above matrix is

$$\Big(\lambda + b_2\Big)\Big[\lambda^2 + C_1\lambda + C_2\Big] = 0,$$

where

$$C_1 = \beta + \delta_1 + \frac{\beta\alpha}{\beta + \delta_1} + \gamma x_2^{**},$$

$$C_2 = (\beta + \delta_1)\Big(\frac{\beta\alpha}{\beta + \delta_1} + \gamma x_2^{**}\Big) - \alpha\beta.$$

Hence, it has one negative characteristic root $\lambda_1 = -b_2 < 0$, the other two characteristic roots are determined by the equation

$$\lambda^2 + C_1\lambda + C_2 = 0. \qquad (23)$$

Noting that from the expression of x_2^{**} (see (15)) and condition (14), the two characteristic roots of equation (23) satisfy

$$\begin{aligned} &\lambda_2 + \lambda_3 \\ =\ & -C_1 < 0, \\ &\lambda_2\lambda_3 \\ =\ & (\beta + \delta_1)\Big(\frac{\beta\alpha}{\beta + \delta_1} + \gamma x_2^{**}\Big) - \alpha\beta \\ =\ & (\beta + \delta_1)\,\Bigg(\frac{\beta\alpha}{\beta + \delta_1} \\ & +\frac{\alpha\beta - \Big(\delta_2 - d\dfrac{y^{**}}{\eta_1 + \eta_2 y^{**}}\Big)(\beta + \delta_1)}{\beta + \delta_1}\Bigg) \\ & -\alpha\beta \\ =\ & \alpha\beta - \Big(\delta_2 - d\frac{y^{**}}{\eta_1 + \eta_2 y^{**}}\Big)(\beta + \delta_1) > 0, \end{aligned}$$

and so, $\lambda_2 < 0, \lambda_3 < 0$. Therefore, all of the three characteristic roots are negative, consequently, $A_4(x_1^{**}, x_2^{**}, y^{**})$ is locally asymptotically stable. This ends the proof of Theorem 2.4.

Remark 2.1. System (1) admits four equilibria, moreover, the local stability property of this equilibria is similar to the equilibria of system (5). Assume that the inequality (17) holds, then $A_2(0, 0, \frac{b_2}{a_2})$ is locally asymptotically stable, and assume that (14) holds, then the positive equilibrium is locally asymptotically stable. However, $A_1(0, 0, 0)$ and $A_3(x_1^*, x_2^*, 0)$ are all unstable, which means that the second species could not be driven to extinction.

Remark 2.2. Theorem 2.4 shows that if the positive equilibrium exists, it's locally asymptotically stable.

Remark 2.3. Noting that from (13) and (15), one could easily see that $x_2^{**} > x_2^*$, $x_1^{**} > x_1^*$, that is, commensalism increases the final density of the first species.

Remark 2.4. When $\eta_1 = 1, \eta_2 = 0$, system (1) is degenerate to system (5), and Theorems 2.1-2.4 is degenerate to Theorems 2.1-2.4 of Lei [13], thus, we generalize the main results of Lei [13] to the nonlinear case.

3 Dynamic Behaviors of Single Species Stage System with Perturbation

This section will focus on the dynamic behaviors of a single species stage structured system.

Now let's consider the system

$$
\begin{aligned}
\frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1, \\
\frac{dx_2}{dt} &= \beta x_1 - \delta_2 x_2 - \gamma x_2^2,
\end{aligned}
\tag{24}
$$

where $\alpha, \beta, \delta_1, \delta_2$ and γ are all positive constants, $x_1(t)$ and $x_2(t)$ are the densities of the immature and mature members of the species at time t. One could refer to system (1) for more detailed explanation of the biological meaning of those coefficients. From Theorem 4.1 and 4.2 in [32] by Xiao and Lei, we have

Lemma 3.1. *Assume that*

$$
\alpha\beta < \delta_2\big(\beta + \delta_1\big)
\tag{25}
$$

holds, then the boundary equilibrium $O(0,0)$ of system (24) is globally stable. Assume that

$$
\alpha\beta > \delta_2\big(\beta + \delta_1\big)
\tag{26}
$$

holds, then the positive equilibrium $B(x_1^*, x_2^*)$ of system (24) is globally stable, where x_1^* and x_2^* are defined by (13).

Now let's consider the single species system with linear perturbation

$$
\begin{aligned}
\frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1, \\
\frac{dx_2}{dt} &= \beta x_1 - \delta_2 x_2 - \gamma x_2^2 + \varepsilon x_2,
\end{aligned}
\tag{27}
$$

where ε is positive constant. Obviously, if $\varepsilon < \delta_2$, the dynamic behaviors of the system (27) is similar to that

of the system (24). However, if $\varepsilon \geq \delta_2$, then we could not directly apply Lemma 3.1 to system (27). In this case, set $\delta_2' = \varepsilon - \delta_2$, system (27) could be rewritten as follows:

$$
\begin{aligned}
\frac{dx_1}{dt} &= \alpha x_2 - \beta x_1 - \delta_1 x_1, \\
\frac{dx_2}{dt} &= \beta x_1 + \delta_2' x_2 - \gamma x_2^2,
\end{aligned}
\tag{28}
$$

whether system (28) has the similar dynamic behaviors as that of system (24) is still unknown. In (28), δ_2' can not be explained as the death rate of the mature species, indeed it is a sum of the death rate of mature species and the perturbation coefficient, the dynamic behaviors of this system, to the best of our knowledge, is not investigated by the scholars. So, for the sake of completeness, we give here the complete proof of Lemma 3.2 below.

By simple computation, the system admits a positive equilibrium $B_2(x_1^*, x_2^*)$, where

$$
x_1^* = \frac{\alpha(\alpha\beta + \beta\delta_2' + \delta_1\delta_2')}{\gamma(\beta + \delta_1)^2}, \quad x_2^* = \frac{\alpha\beta + \beta\delta_2' + \delta_1\delta_2'}{\gamma(\beta + \delta_1)}.
$$

Concerned with the stability property of this positive equilibrium, we have the following result.

Lemma 3.2. *The positive equilibrium $B_2(x_1^*, x_2^*)$ of system (28) is globally asymptotically stable.*

Proof. We will prove Lemma 3.2 by adapting the idea of Xiao and Lei [32]. More precisely, we will prove Lemma 3.2 by constructing a suitable Lyapunov function as follows. Set

$$
\begin{aligned}
V(x_1, x_2) &= k_1\Big(x_1 - x_1^* - x_1^* \ln \tfrac{x_1}{x_1^*}\Big) \\
&\quad + k_2\Big(x_2 - x_2^* - x_2^* \ln \tfrac{x_2}{x_2^*}\Big),
\end{aligned}
$$

where k_1, k_2 are some positive constants determined later.

One could easily see that the function V is zero at the equilibrium $B_2(x_1^*, x_2^*)$ and is positive for all other positive values of x_1 and x_2. The time derivative of V along the trajectories of (28) is

$$
\begin{aligned}
D^+ &V(t) \\
&= k_1 \frac{x_1 - x_1^*}{x_1}\Big(\alpha x_2 - (\beta + \delta_1)x_1\Big) \\
&\quad + k_2 \frac{x_2 - x_2^*}{x_2}\Big(\beta x_1 + \delta_2' x_2 - \gamma x_2^2\Big).
\end{aligned}
\tag{29}
$$

Since

$$
\alpha x_2^* - \beta x_1^* - \delta_1 x_1^* = 0,
$$

$$
\beta x_1^* + \delta_2' x_2^* - \gamma (x_2^*)^2 = 0,
$$

then

$$\alpha x_2 - (\beta + \delta_1)x_1$$

$$= \frac{\alpha}{x_1^*}\Big(x_2 x_1^* - x_1 x_2^*\Big) + \alpha x_1 \frac{x_2^*}{x_1^*}$$
$$- \Big(\beta + \delta_1\Big)x_1 \qquad (30)$$

$$= \frac{\alpha}{x_1^*}\Big(-x_2(x_1 - x_1^*) + x_1(x_2 - x_2^*)\Big),$$

and

$$\beta x_1 + \delta_2' x_2 - \gamma x_2^2$$

$$= \frac{\beta}{x_2^*}\Big(x_1 x_2^* - x_2 x_1^*\Big) + \beta x_2 \frac{x_1^*}{x_2^*}$$
$$+ \delta_2' x_2 - \gamma x_2^2$$

$$= \frac{\beta}{x_2^*}\Big(x_1 x_2^* - x_1 x_2 + x_1 x_2 - x_2 x_1^*\Big) \qquad (31)$$

$$+ \Big(\frac{\alpha\beta}{\beta + \delta_1} + \delta_2'\Big)x_2 - \gamma x_2^2$$

$$= \frac{\beta}{x_2^*}\Big(x_1(x_2^* - x_2) + x_2(x_1 - x_1^*)\Big)$$
$$+ \gamma x_2^* x_2 - \gamma x_2^2.$$

Applying (30) and (31) to (29), and choosing $k_2 = 1, k_1 = \dfrac{\beta x_1^*}{x_2^* \alpha}$, we finally obtain

$$D^+ V(t)$$

$$= k_1 \frac{x_1 - x_1^*}{x_1}\frac{\alpha}{x_1^*}\Big(-x_2(x_1 - x_1^*) + x_1(x_2 - x_2^*)\Big)$$

$$+ k_2 \frac{x_2 - x_2^*}{x_2}\frac{\beta}{x_2^*}\Big(x_1(x_2^* - x_2) + x_2(x_1 - x_1^*)\Big)$$

$$+ k_2 \frac{x_2 - x_2^*}{x_2}\Big(\gamma x_2^* x_2 - \gamma x_2^2\Big)$$

$$= -\frac{\beta}{x_2^*}\Big[\sqrt{\frac{x_2}{x_1}}(x_1 - x_1^*) - \sqrt{\frac{x_1}{x_2}}(x_2 - x_2^*)\Big]^2$$

$$- \gamma\Big(x_2 - x_2^*\Big)^2.$$

that is, $D^+ V(t) < 0$ strictly for all $x_1, x_2 > 0$ with the exception of the positive equilibrium $B_2(x_1^*, x_2^*)$, where $D^+ V(t) = 0$. Thus, $V(x_1, x_2)$ satisfies Lyapunov's asymptotic stability theorem, and the positive equilibrium $B_2(x_1^*, x_2^*)$ of system (28) is globally asymptotically stable.

This completes the proof of Lemma 3.2.

4 Global attractivity

We had shown in Section 2 that $A_2(0, 0, \frac{b_2}{a_2})$ and $A_4(x_1^{**}, x_2^{**}, y^{**})$ could be locally asymptotically stable under some suitable assumptions. One interesting issue is to investigate the global stability property of the equilibria. In this section we will try to obtain some sufficient conditions to ensure the global attractivity of the equilibria A_2 and A_4 of system (1).

Theorem 4.1 *Assume that*

$$(\beta + \delta_1)\Big(\delta_2 - \frac{dy^{**}}{\eta_1 + \eta_2 y^{**}}\Big) - \alpha\beta > 0 \qquad (32)$$

holds, then $A_2(0, 0, \frac{b_2}{a_2})$ is globally attractive.

Proof. For $\varepsilon > 0$ enough small, condition (32) implies that

$$(\beta + \delta_1)\Big(\delta_2 - \frac{dy^{**} + \varepsilon)}{\eta_1 + \eta_2(y^{**} + \varepsilon)}\Big) - \alpha\beta > 0. \quad (33)$$

Since the third equation of system (1) is a Logistic equation, then

$$\lim_{t \to +\infty} y(t) = \frac{b_2}{a_2}. \qquad (34)$$

In corresponding to the above chosen $\varepsilon > 0$, there exists a $T > 0$ such that

$$y(t) < \frac{b_2}{a_2} + \varepsilon \stackrel{\text{def}}{=} y^{**} + \varepsilon \quad \text{for all} \quad t > T.$$

From above inequality and the first and second equations of system (1), also, noting that the function $\dfrac{y}{\eta_1 + \eta_2 y}$ is monotonically increasing, for $t > T$, we have

$$\frac{dx_1}{dt} = \alpha x_2 - \beta x_1 - \delta_1 x_1,$$

$$\frac{dx_2}{dt} = \beta x_1 - \delta_2 x_2 - \gamma x_2^2 + d\frac{y}{\eta_1 + \eta_2 y}x_2$$

$$\leq \beta x_1 - \delta_2 x_2 - \gamma x_2^2$$

$$+ d\frac{y^{**} + \varepsilon}{\eta_1 + \eta_2(y^{**} + \varepsilon)}x_2.$$

$$(35)$$

Now let's consider the system

$$\frac{dv_1}{dt} = \alpha v_2 - \beta v_1 - \delta_1 v_1,$$

$$\frac{dv_2}{dt} = \beta v_1 - \delta_2 v_2 - \gamma v_2^2 \qquad (36)$$

$$+ d\frac{y^{**} + \varepsilon}{\eta_1 + \eta_2(y^{**} + \varepsilon)}v_2.$$

It follows from (33) and Lemma 3.1 that the boundary equilibrium $O(0, 0)$ of system (36) is globally stable.

That is, for any positive solution $(v_1(t), v_2(t))$ of the system (36), one has

$$\lim_{t \to +\infty} v_1(t) = 0, \quad \lim_{t \to +\infty} v_2(t) = 0.$$

Let $(x_1(t), x_2(t), x_3(t))$ be any positive solution of system (1) with initial condition $(x_1(T), x_2(T), y(T)) = (x_{10}, x_{20}, y_0)$, and let $(v_1(t), v_2(t))$ be the positive solution of system (36) with the initial condition $(v_1(T), v_2(T)) = (x_{10}, x_{20})$, then it follows from (35), (36) and the differential inequality theory that

$$x_i(t) \le v_i(t) \quad \text{for all} \quad t \ge T, \quad i = 1, 2,$$

and therefore that

$$\limsup_{t \to +\infty} x_i(t) \le \lim_{t \to +\infty} v_i(t) = 0, \quad i = 1, 2.$$

Thus, from the positivity of the solution of system (1), it immediately follows that

$$0 \le \liminf_{t \to +\infty} x_i(t) \le \limsup_{t \to +\infty} x_i(t) \le 0, \quad i = 1, 2.$$

Therefore

$$\lim_{t \to +\infty} x_i(t) = 0, \quad i = 1, 2. \tag{37}$$

(34) together with (37) shows that $A_2(0, 0, \frac{b_2}{a_2})$ is globally attractive.

This completes the proof of Theorem 4.1.

Theorem 4.2 *Assume that*

$$\alpha\beta - \left(\delta_2 - \frac{dy^{**}}{\eta_1 + \eta_2 y^{**}}\right)(\beta + \delta_1) > 0 \tag{38}$$

holds, then $A_4(x_1^{**}, x_2^{**}, y^{**})$, where $x_1^{**}, x_2^{**}, y^{**}$ are defined in (15), is globally attractive.

Proof. For $\varepsilon > 0$ enough small, without loss of generality, we may assume that $\varepsilon < \frac{b_2}{2a_2}$, so condition (38) implies that

$$\alpha\beta - \left(\delta_2 - \frac{d(y^{**} + \varepsilon)}{\eta_1 + \eta_2(y^{**} + \varepsilon)}\right)(\beta + \delta_1) > 0 \tag{39}$$

and

$$\alpha\beta - \left(\delta_2 - \frac{d(y^{**} - \varepsilon)}{\eta_1 + \eta_2(y^{**} - \varepsilon)}\right)(\beta + \delta_1) > 0 \tag{40}$$

hold. The third equation of system (1) is a Logistic equation, thus

$$\lim_{t \to +\infty} y(t) = \frac{b_2}{a_2}. \tag{41}$$

In correspondence to the above chosen $\varepsilon > 0$, there exists a $T > 0$ such that

$$\frac{b_2}{a_2} - \varepsilon < y(t) < \frac{b_2}{a_2} + \varepsilon \quad \text{for all} \quad t > T, \quad i = 1, 2. \tag{42}$$

From (42) and the first and second equation of system (1), also, noting that the function $\frac{y}{\eta_1 + \eta_2 y}$ is strictly increasing, for $t > T$, we have

$$\frac{dx_1}{dt} = \alpha x_2 - \beta x_1 - \delta_1 x_1,$$

$$\frac{dx_2}{dt} \le \beta x_1 - \delta_2 x_2 - \gamma x_2^2 \tag{43}$$

$$+ d\frac{y^{**} + \varepsilon}{\eta_1 + y^{**} + \varepsilon} x_2.$$

Now let's consider the system

$$\frac{dv_1}{dt} = \alpha v_2 - \beta v_1 - \delta_1 v_1,$$

$$\frac{dv_2}{dt} = \beta v_1 - \left(\delta_2 - d\frac{y^{**} + \varepsilon}{\eta_1 + \eta_2(y^{**} + \varepsilon)}\right)v_2$$

$$- \gamma v_2^2. \tag{44}$$

There are two subcases.
(i)

$$\delta_2 > d\frac{y^{**} + \varepsilon}{\eta_1 + \eta_2(y^{**} + \varepsilon)}.$$

In this case, it follows from (39) and Lemma 3.1 that the system (44) admits a unique positive equilibrium which is globally asymptotically stable.
(ii)

$$\delta_2 \le d\frac{\frac{b_2}{a_2} + \varepsilon}{\eta + \frac{b_2}{a_2} + \varepsilon}.$$

In this case, it follows from Lemma 3.2 that the system (44) admits a unique positive equilibrium which is globally stable.

Hence, in any case, system (44) admits a unique positive equilibrium $(v_1(\varepsilon), v_2(\varepsilon))$, which is globally asymptotically stable, where

$$v_1(\varepsilon) = \frac{\alpha v_2(\varepsilon)}{\beta + \delta_1},$$

$$v_2(\varepsilon) = \frac{\alpha\beta - \left(\delta_2 - d\frac{y^{**} + \varepsilon}{\eta_1 + \eta_2(y^{**} + \varepsilon)}\right)(\beta + \delta_1)}{(\beta + \delta_1)\gamma}. \tag{45}$$

Therefore, let $(v_1(t), v_2(t))$ be any positive solution of the system (44), one has

$$\lim_{t \to +\infty} v_1(t) = v_1(\varepsilon), \quad \lim_{t \to +\infty} v_2(t) = v_2(\varepsilon). \tag{46}$$

Let $(x_1(t), x_2(t), x_3(t))$ be any positive solution of system (1) with initial condition $(x_1(T), x_2(T), y(T)) = (x_{10}, x_{20}, y_0)$, and let $(v_1(t), v_2(t))$ be the positive solution of system (44) with the initial condition $(v_1(T), v_2(T)) = (x_{10}, x_{20})$, it then follows from (43), (44) and the differential inequality theory that

$$x_i(t) \le v_i(t) \ \text{ for all } \ t \ge T, \ \ i = 1, 2. \quad (47)$$

The positivity of the solution of system (1) and (47) lead to

$$\limsup_{t \to +\infty} x_i(t) \le \lim_{t \to +\infty} v_i(t) = v_i(\varepsilon), \ \ i = 1, 2. \quad (48)$$

From (42) and the first and second equation of system (1), since $\dfrac{y}{\eta_1 + \eta_2 y}$ is strictly increasing, for $t > T$, we have

$$\frac{dx_1}{dt} = \alpha x_2 - \beta x_1 - \delta_1 x_1,$$

$$\frac{dx_2}{dt} \ge \beta x_1 - \delta_2 x_2 - \gamma x_2^2 \quad (49)$$

$$+ d \frac{y^{**} - \varepsilon}{\eta_1 + \eta_2(y^{**} - \varepsilon)} x_2.$$

Now let's consider the system

$$\frac{dw_1}{dt} = \alpha w_2 - \beta w_1 - \delta_1 w_1,$$

$$\frac{dw_2}{dt} = \beta w_1 - \left(\delta_2 - d\frac{y^{**} - \varepsilon}{\eta_1 + \eta_2(y^{**} - \varepsilon)}\right) w_2$$

$$- \gamma w_2^2. \quad (50)$$

There are two subcases.
(i)

$$\delta_2 > d\frac{y^{**} - \varepsilon}{\eta_1 + \eta_2(y^{**} - \varepsilon)}. \quad (51)$$

In this case, it follows from (40) and Lemma 3.1 that the system (50) admits a unique positive equilibrium which is globally asymptotically stable.
(ii)

$$\delta_2 \le d\frac{y^{**} - \varepsilon}{\eta_1 + \eta_2(y^{**} - \varepsilon)}.$$

In this case, it follows from Lemma 3.2 that the system (50) admits a unique positive equilibrium which is globally stable.

Hence, in any case, system (50) admits a unique positive equilibrium $(w_1(\varepsilon), w_2(\varepsilon))$, which is glob-

ally asymptotically stable, where

$$w_1(\varepsilon) = \frac{\alpha w_2(\varepsilon)}{\beta + \delta_1},$$

$$w_2(\varepsilon) = \frac{\alpha\beta - \left(\delta_2 - d\dfrac{y^{**} - \varepsilon}{\eta_1 + \eta_2(y^{**} - \varepsilon)}\right)(\beta + \delta_1)}{(\beta + \delta_1)\gamma}. \quad (52)$$

Therefore, let $(w_1(t), w_2(t))$ be any positive solution of the system (50), one has

$$\lim_{t \to +\infty} w_1(t) = w_1(\varepsilon), \quad \lim_{t \to +\infty} w_2(t) = w_2(\varepsilon). \quad (53)$$

Let $(x_1(t), x_2(t), y(t))$ be any positive solution of system (1) with initial condition $(x_1(T), x_2(T), y(T)) = (x_{10}, x_{20}, y_0)$, and let $(w_1(t), w_2(t))$ be the positive solution of system (50) with the initial condition $(w_1(T), w_2(T)) = (x_{10}, x_{20})$, it then follows from (49), (50) and the differential inequality theory that

$$x_i(t) \ge w_i(t) \ \text{ for all } \ t \ge T, \ \ i = 1, 2. \quad (54)$$

The positivity of the solution of system (1) and (53), (54) lead to

$$\liminf_{t \to +\infty} x_i(t) \ge \lim_{t \to +\infty} w_i(t) = w_i(\varepsilon), \ \ i = 1, 2. \quad (55)$$

Inequality (48) together with (55) leads to

$$\begin{aligned} w_i(\varepsilon) &= \lim_{t \to +\infty} w_i(t) \\ &\le \liminf_{t \to +\infty} x_i(t) \\ &\le \limsup_{t \to +\infty} x_i(t) \\ &\le \lim_{t \to +\infty} v_i(t) = v_i(\varepsilon), \ \ i = 1, 2. \end{aligned} \quad (56)$$

From (45) and (52) one can easily see that

$$w_i(\varepsilon) \to x_i^{**}, \ v_i(\varepsilon) \to x_i^{**}, \ \text{as} \ \varepsilon \to 0, \ i = 1, 2. \quad (57)$$

Noting that ε is any enough small positive constant, letting $\varepsilon \to 0$ in (56), (57) leads to

$$\lim_{t \to +\infty} x_i(t) = x_i^{**}, \ \ i = 1, 2. \quad (58)$$

So, (41) together with (58) shows that $A_4(x_1^{**}, x_2^{**}, y^{**})$ is globally attractive.
This completes the proof of Theorem 4.2.

Remark 4.1. Theorem 4.1 and 4.2 depicts a very intuitive biological phenomenon. From Zhang, Chen and Neumann [40], for single species stage structured system (24), we can regard $\dfrac{\alpha}{\delta_2}$ as a relative birth rate of

the fist mature species, $\dfrac{\beta}{\beta + \delta_1}$ as a relative transformation rate of the first immature species. Then condition (32) and (38) are equivalent to

$$\frac{\alpha}{\delta_2 - d\dfrac{y^{**}}{\eta_1 + \eta_2 y^{**}}} \frac{\beta}{\beta + \delta_1} < 1, \qquad (59)$$

and

$$\frac{\alpha}{\delta_2 - d\dfrac{y^{**}}{\eta_1 + \eta_2 y^{**}}} \frac{\beta}{\beta + \delta_1} > 1, \qquad (60)$$

respectively. Hence, with the help of the host species, the relative birth rate of the mature commensal species is increasing from $\dfrac{\alpha}{\delta_2}$ to $\dfrac{\alpha}{\delta_2 - d\dfrac{y^{**}}{\eta_1 + \eta_2 y^{**}}}$, thus finally increasing the chance of the survival of the first species.

Remark 4.2. When $\eta_1 = 1, \eta_2 = 0$, system (1) is degenerate to system (5), and Theorem 4.1-4.2 are degenerate to Theorem 3.1-3.2 of Lei[13], respectively. Thus, we generalize the main results of Lei [13] to the nonlinear case.

5 Examples
Example 5.1. Consider the following stage structured commensalism system

$$\begin{aligned}
\frac{dx_1}{dt} &= 3x_2 - x_1 - x_1, \\
\frac{dx_2}{dt} &= x_1 - x_2 - x_2^2 + \frac{dy}{\eta_1 + \eta_2 y} x_2, \qquad (61) \\
\frac{dy}{dt} &= y(1 - y).
\end{aligned}$$

Here, corresponding to system (1), we take $\alpha = 3, \beta = \delta_1 = \delta_2 = \gamma = b_2 = a_2 = 1$. Note that

$$\alpha\beta = 3 > 2 = \delta_2(\beta + \delta_1). \qquad (62)$$

In this case, without the commensalism of the second species, the first species is globally stable. For any positive constant d, η_1 and η_2, inequality (38) holds, so it follows from Theorem 4.2 that $A_4(x_1^{**}, x_2^{**}, y^{**})$ is globally attractive. Take $d = \eta_1 = \eta_2 = 1$, then the positive equilibrium $A_4(\frac{3}{2}, 1, 1)$ is globally attractive, noting that $\lim\limits_{t\to+\infty} y(t) = 1$, hence, we only give the numeric simulations of immature species x_1 and mature species x_2. Also, here we fixed $x_1(0) + x_2(0) = 2.5, y(0) = 2$. Fig. 1 shows that $\lim\limits_{t\to+\infty} x_1(t) = \frac{3}{2}$, though the solutions are finally approaching to $\frac{3}{2}$, moreover for the initial conditions

$x_1(0) < \frac{3}{2}$, the solution is strictly increasing at first, then, after it reaches the extremum, which is larger than $\frac{3}{2}$, after that, the solutions becomes strictly decreasing. On the contrary, for the initial conditions $x_1(0) > \frac{3}{2}$, the solution is strictly decreasing at first, then, after it reach the extremum, which is smaller than $\frac{3}{2}$, then it becomes strictly increasing. Fig.2 shows that $\lim\limits_{t\to+\infty} x_2(t) = 1$. However, the dynamic behaviors is relative simple, for solutions with initial conditions larger than 1.5, the solutions is strictly decreasing, and for solutions with initial conditions smaller than 1, the solutions is strictly increasing.
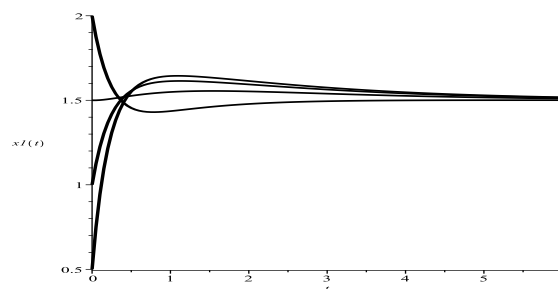

Figure 1: Dynamic behaviors of the first components x_1 in system (61) with the initial condition $(x_1(0), x_2(0), y(0)) = (1, 1.5, 2), (0.5, 2, 2), (1.5, 1, 2)$ and $(2, 0.5, 2)$, respectively.
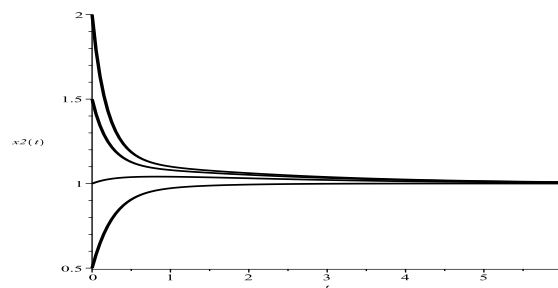

Figure 2: Dynamic behaviors of the second components x_2 in system (61) with the initial condition $(x_1(0), x_2(0), y(0)) = (1, 1.5, 2), (0.5, 2, 2), (1.5, 1, 2)$ and $(2, 0.5, 2)$, respectively.

Example 5.2. Consider the following stage structured

commensalism system

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_2 - x_1 - x_1, \\
\frac{dx_2}{dt} &= x_1 - x_2 - x_2^2 + \frac{3y}{\eta_1 + \eta_2 y}x_2, \\
\frac{dy}{dt} &= y(1 - y).
\end{aligned}
\tag{63}
$$

Here, corresponding to system (1), we take $\alpha = \beta = \delta_1 = \delta_2 = \gamma = b_2 = a_2 = 1, d = 3$. For $\eta_1 = 1, \eta_2 = 0$, the system degenerates to the case 2 in system 4.1 of Lei [13]. From [13], without the commensal of the second species, the first species, which satisfies the equations

$$
\begin{aligned}
\frac{dx_1}{dt} &= x_2 - x_1 - x_1, \\
\frac{dx_2}{dt} &= x_1 - x_2 - x_2^2,
\end{aligned}
\tag{64}
$$

will be driven to extinction. Also, with the commensal of the second species, the system admits a unique positive equilibrium $A_4(\frac{3}{4}, \frac{3}{2}, 1)$, which is globally asymptotically stable.

Noting that from the third equation of (63), we have $y^{**} = 1$. By simple computation, if

$$
\begin{aligned}
&(\beta + \delta_1)\left(\delta_2 - \frac{3}{\eta_1 + \eta_2}\right) - \alpha\beta \\
&= 2\left(1 - \frac{3}{\eta_1 + \eta_2}\right) - 1 > 0,
\end{aligned}
\tag{65}
$$

which is equivalent to $\eta_1 + \eta_2 > 6$ holds true, then it follows from Theorem 4.1 that $A_2(0, 0, 1)$ is globally attractive. Let's take $\eta_1 = 9, \eta_2 = 1$. Numeric simulations reported in Fig. 3 and 4 supports this assertion. By a similarly discussion, if $\eta_1 + \eta_2 < 6$ holds, then it follows from Theorem 4.2 that $A_4(x_1^{**}, x_2^{**}, 1)$ is globally attractive. Let's take $\eta_1 = 1, \eta_2 = 1$, then $A_4(0.5, 1, 1)$ is globally attractive. Numeric simulations reported in Fig. 5 and 6 supports this assertion. In Fig. 3-6, we choose the initial conditions $x_1(0) + x_2(0) = 2, y(0) = 1$.

6 Conclusion

Stimulated by the work of Wright [41], we argued that in the commensal relationship between two species, the commensal species is also subjected to the constraints of handling time. For example, consider a species of solitary bee visiting a flower species. The rate of collection of pollen is limited by the handling time per plant. This finally motivated us to propose a two species commensalism model with Holling type II functional response and stage structure. If $\eta_1 =$
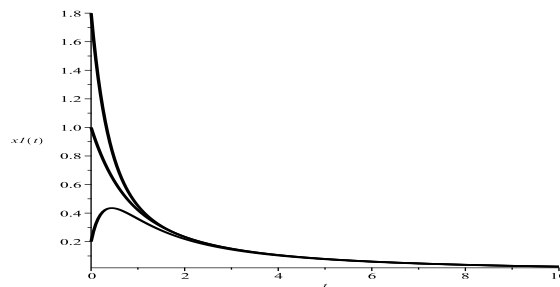

Figure 3: Dynamic behaviors of the first component x_1 in system (64) with $\eta_1 = 9, \eta_2 = 1$, and the initial condition $(x_1(0), x_2(0), y(0)) = (1, 1, 1), (1.8, 0.2, 1)$ and $(0.2, 1.8, 1)$, respectively.
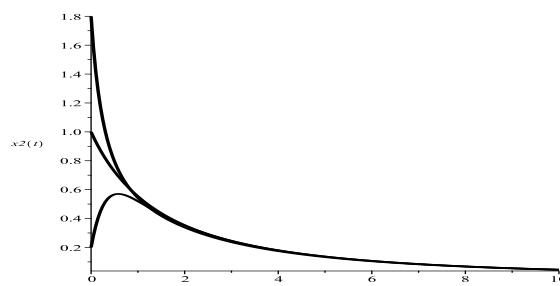

Figure 4: Dynamic behaviors of the second component x_2 in system (64) with $\eta_1 = 9, \eta_2 = 1$, and the initial condition $(x_1(0), x_2(0), y(0)) = (1, 1, 1), (1.8, 0.2, 1)$ and $(0.2, 1.8, 1)$, respectively.

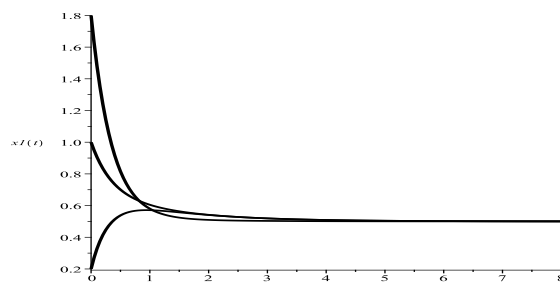

Figure 5: Dynamic behaviors of the first component x_1 in system (64) with $\eta_1 = 1, \eta_2 = 1$, and the initial condition $(x_1(0), x_2(0), y(0)) = (1, 1, 1), (1.8, 0.2, 1)$ and $(0.2, 1.8, 1)$, respectively.
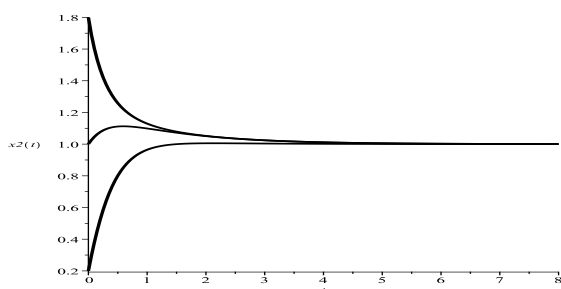
Figure 6: Dynamic behaviors of the second component x_2 in system (64) with $\eta_1 = 1, \eta_2 = 1$, and the initial condition $(x_1(0), x_2(0), y(0)) = (1, 1, 1)$, $(1.8, 0.2, 1)$ and $(0.2, 1.8, 1)$, respectively.

$1, \eta_2 = 0$, then our model is degenerate to the model considered by Lei [13].

We first showed that the system can admit four equilibria, however, only two of them can be locally asymptotically stable.

From Lemma 3.1 we know that for the stage structured single species system, depending on the relationship of the coefficients, the species may be driven to extinction or survival in the long run. Theorem 4.2 shows that if the species without commensalism of the second species could survive in the long run, then for the commensalism system, two species could coexist in a stable state. If the first species will be driven to extinction without the commensalism, then, Theorem 4.1 shows that limited commensalism still could not avoid the extinction of the first species. However, Theorem 4.2 shows that if the commensalism effect is large enough, then two species can coexist in a stable state.

One can easily see that if $\eta_1 = 1, \eta_2 = 0$, then Theorems 2.1-2.4 degenerate to Theorems 2.1-2.4, Theorems 4.1-4.2 degenerate to Theorems 3.1-3.2 in Lei [13], hence, we generalize the main result of Lei [13].

What we really concern is the influence of Holling type II response, which, from the point of Wright [43], can be explained as the handling time of commensal time. Example 5.2 shows that handling time has negative effect on the persistent property of the commensal species. If $\eta_1 + \eta_2 y^{**}$ is large enough, the influence of the host will be reduced, and with the influence of stage structure, despite the commensalism of host species, the commensal species will still be driven to extinction.

We mention here that the following two aspects need to be studied. The first one is about the influence of delay. A more plausible model should include some past state of the system [28], this leads to the delayed modelling. A recent study [28] shows that delay could change the dynamic behaviors of the system greatly. Up today still no scholars propose and study the delayed stage structured commensal model, we will try to do some work in this direction. The second one is to investigate the influence of Allee effect. Already, several scholars [14], [15], [16], [17], [34], [35] studied the commensalism system with Allee effect, however, all of those works did not consider the stage structure of the species. We will try to do some works on this direction in the future.

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