On Adaptive Grid Approximations in the Weight Norm

YURI K.DEM'YANOVICH
Parallel Algorithm Department,
Saint Petersburg State University,
Universitetskaya nab.,7/9, Saint Petersburg,
RUSSIA

Abstract: - The purpose of this paper is to develop an algorithm construction of adaptive variation-grid methods for classes of one-dimensional boundary value problems of the second order. Classes of non-degenerate problems are considered, as well as classes of problems with weak and strong degeneracy. The results obtained are suitable for learning computer systems designed to solve problems of the aforementioned classes. To achieve the set goals, the corresponding approximation theorems are established with degeneration. Ways of adaptive choice approximation space at a variation-grid method in a one-dimensional boundary value problem are considered. The locality of the approximation is substantially used. The considerations are reduced to an iterative process, while building an adaptive grid. Numerical examples illustrating the effectiveness of the proposed approach are given.

Key-Words: - adaptive approximation, artificial intelligence, mathematical physics, numerical calculations

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1 Introduction

Research in the field of artificial intelligence is widespread and covers many areas of the development of science and practice. These studies make it possible, to a certain extent, to optimize efforts aimed at the effective solution of urgent and difficult problems that require the resources of powerful computer systems. Let us give an example of some studies in the mentioned area.

The authors in [1] predict the future of excitation energy transfer with artificial intelligence-based quantum dynamics.

In [2], the authors propose a distributed approximate Newton-type algorithm with a fast convergence rate for communication-efficient federated edge learning.

The authors of [3] show that pool-based active classification can be improved.

The technological troubles with the design of Processing-in-Memory are discussed in [4].

In [5], the authors propose a new algorithm of machine learning techniques. The mentioned algorithm has less complexity than known machine learning algorithms.

Based on the recent advancements in music structure analysis, the authors of [6] automate the evaluation process by introducing a collection of metrics that can objectively describe structural properties of the music signal.

The development of a disagreement-based online learning algorithm is given in [7].

In [8], the authors first construct a graph-based network model as well as a Poisson process-based traffic model in the context of 5G mobile networks.

In [9], the authors develop a deep learning-based approach to model and predict the designers' sequential decisions in the systems design context.

Everyone knows the computational difficulties of solving problems of mathematical physics. These difficulties are overcome with the sophisticated use of additional information about the tasks under consideration.

Let us mention some works in which it is possible to effectively use learning systems and the means of artificial intelligence.

The article [10] is devoted to the solution of the nonstationary integro-differential equation with a degenerate elliptic differential operator. The Galerkin method with cubic spline wavelets is employed for spatial discretization combined with the Crank-Nicolson scheme and Richardson extrapolation for time discretization.

It seems that adaptive methods developed with the help of artificial intelligence greatly simplify the solution of problems of mathematical physics. Such problems are considered in a large number of papers.

The authors in [11] investigate the approximate solution to a nonlinear Volterra integro-differential equation in the complex plane by applying the iterative method in each iteration.

In [12], the authors proposed a highly efficient and accurate collocation method based on the Haar wavelet for the parameter identification in multidimensional hyperbolic partial differential equations. It can also be assumed that the mentioned means would be useful in solving the problems considered in the works, [13] and [14].

Computational stability issues are investigated in [15]. A comprehensive study of the approximation in variational grid methods in the case of uniform grids, is given in the monograph, [16]. A generation to irregular grids is available, [17].

As can be seen from the previous discussion, the problem of building an adaptive algorithm in classes of boundary value problems is very relevant. The relevance of this approach is determined by two circumstances. On the one hand, the properties of the solutions to the boundary value problems are investigated in many problems. On the other hand, trainable computer systems (artificial intelligence) have been developed.

These achievements give hope for a significant increase in the efficiency of the numerical solution of boundary value problems.

The purpose of this paper is to develop an algorithm construction of adaptive - variation grid methods for classes of one-dimensional boundary value problems of the second order. Classes of non-degenerate problems are considered, as well as classes of problems with weak and strong degeneracy. The results obtained are suitable for learning computer systems designed to solve the problems of the mentioned classes.

Thus this work is aimed at developing adaptive approximations suitable for teaching intelligent computer systems. In this paper, the adaptive approximations are *applied* to variation-grid methods in a one-dimensional boundary problem. The basis of this study is the works of S.G. Mikhlin which are related to the variation-grid method for an one-dimensional boundary problem with degeneration (see [16]). In this study, we consider an adaptive choice of the approximation space when certain information of the behavior of the solution of the mentioned problem is available.

The results of this work are adaptive numerical methods suitable for the teaching of intellectual computer systems using representatives of the class of the mentioned problems. This work makes it possible to train a computer system using a set of problems with a known exact or approximate solution. Numerical examples illustrating the effectiveness of the proposed approaches are given at the end of this paper.

2 Background

This paragraph provides the necessary information about a boundary value problem with degeneracy for an ordinary differential equation. Here are the results obtained by Professor S.G. Mikhlin, [16].

Consider the differential equation

$$-\big(p(x)u'(x)\big)' + q(x)u(x) = f(x), x \in (a,b), \quad (1)$$

where $p \in C[a,b] \cap C^1$, $f \in L_2$, and the function $q(x) \geq 0$ is measurable and limited. Suppose $p(x) > 0$ for $x \in (a,b]$.

When setting the boundary value problem for equation (1), we will distinguish the following cases.

1. The function $p(x)$ at point a is nonzero,,

$$p(a) > 0. \quad (2)$$

2. The function $p(x)$ at ~~the~~ point a is equal to zero,

$$p(a) = 0. \quad (3)$$

In the second case, one has to consider (see [16]) three subcases.

2a. Integral over interval (a,b) of the function $1/p(x)$ converges at point a,

$$\int_a^b \frac{dx}{p(x)} < +\infty, \quad (4)$$

2b. The integral of function $1/p(x)$ diverges, but the integral of function $x/p(x)$ converges. In other words, the relations

$$\int_a^b \frac{dx}{p(x)} = +\infty, \qquad \int_a^b \frac{xdx}{p(x)} < +\infty, \quad (5)$$

2c. Integral over interval (a,b) of the function $x/p(x)$ diverges at point a,

$$\int_a^b \frac{xdx}{p(x)} = +\infty. \quad (6)$$

In cases 1. and 2a. (see formulas (2) -- (4)) we add the boundary conditions

$$u(a) = u(b) = 0 \quad (7)$$

to equation (1).

In cases 2b. and 2c. (see formulas (5) -- (6)) we will assume that $q(x) \geq q_0 = \text{const} > 0$. In these cases, we add the boundary condition

$$u(b) = 0 \quad (8)$$

to equation (1).

Under these conditions, we can assume that the problems under consideration have the form $Au = f$, where $f \in L_2$, operator A acts in the space $L_2(a,b)$ and it is extended to a positive-definite self-adjoint operator. Hence it follows that the solution u_* of the problem lies in the space $L_2(a,b)$. The approximate solution of the problem is the solution $\tilde{u}_*$ of the problem to a minimum of energy functional $F(u)$ on a suitable subspace S of the energy space H_A:

$\min_{\tilde{u}\in S}F(\tilde{u}),\quad F(\tilde{u}) = \mathrm{I}u_* - \tilde{u}\mathrm{I}^2 - \mathrm{I}u_*\mathrm{I}^2,$ (9)

where

$$\mathrm{I}u\mathrm{I}^2 = \int_a^b (p(x)|u'(x)|^2 + q(x)|u(x)|^2)dx. \quad (10)$$

The inequality

$$\mathrm{I}u_* - \tilde{u}_*\mathrm{I} \le \mathrm{I}u_* - \tilde{u}\mathrm{I} \quad \forall \tilde{u} \in S \quad (11)$$

follows from (9) - (10). Relation (11) is usually used to obtain estimates of the convergence rate for the method (see, for example, [17]).

In this paper, inequality (11) is used for the adaptive approximation. The last one reduces to a suitable choice of subspaces S. Here the simplest variants of S subspaces, namely, spaces of piecewise linear continuous functions are considered.

For what follows, we need the estimates' approximations in the spaces with a weighted norm.

3 Approximation Estimates

Let u be an element of the space $C[c,d]$. Let's put $U_{c,d}(x) = U_{c,d}[u](x) = u(x) - u(c) - \frac{u(d)-u(c)}{d-c}(x-c)$.

Lemma 1. *The following statements are true.*

1. For the function $u \in C^1[c,d]$, $c < d$, a representation

$$u(x) - U_{c,d}(x) = (d-c)^{-1}\int_c^x d\xi \int_c^d \big(u'(\xi) - u'(\zeta)\big)\, d\zeta \quad (12)$$

is true. For the function $u \in C^2[c,d]$, $c < d$, the formula

$$u(x) - U_{c,d}(x) = (d-c)^{-1}\int_c^x d\xi \int_c^d d\zeta \int_\zeta^\xi u''(\eta)\, d\eta \quad (13)$$

is valid.

Proof. 1. Writing the left side of formula (12), using the Leibniz formula, we have

$u(x) - u(c) - \frac{u(d)-u(c)}{d-c}(x-c) = \int_c^x u'(\xi)d\xi - -\frac{x-c}{d-c}\int_c^d u'(\eta)d\,\eta.$

To prove formula (12), it remains to take out the multiplier $(d-c)^{-1}$ from the brackets and combine the difference of the derivatives using the double integral indicated on the right side formulas (12).

2. Formula (13) is obtained from formula (12) by applying the Leibniz formula for the difference $u(\xi) - u(\zeta)$,.

This concludes the proof.

Lemma 2. *For $u \in L_2(c,d)$, $x \in [c,d]$, $c < d$, the inequality*

$|\int_c^x d\xi \int_c^d \big(u'(\xi) - u'(\zeta)\big)\, d\zeta| \le$
$2(d-c)^{\frac{3}{2}} \quad (\int_c^d |u'(\xi)\,|^2 d\xi)^{1/2}$ (14)

is correct.

Proof. The following chain of relations is obvious

$$\int_c^x d\xi \int_c^d \big(u'(\xi) - u'(\zeta)\big)\, d\zeta| \le$$

$\int_c^x d\xi \int_c^d |u'(\xi) - u'(\zeta)|\, d\zeta \le$
$\int_c^x d\xi \int_c^d |u'(\xi)|\, d\zeta + \int_c^x d\xi \int_c^d |u'(\zeta)|\, d\zeta =$
$2(d-c)\int_c^d |u'(\xi)|\, d\xi$
$\le 2(d-c)^{\frac{3}{2}}(\int_c^d |u'(\xi)|^2\, d\xi)^{\frac{1}{2}}.$

This concludes the proof.

Lemma 3. *If $u \in L_2(c,d), c < d$, and $x \in (c,d)$, then we have the estimate*

$$|u(x) - U_{c,d}(x)| \le 2(\mathrm{d}-\mathrm{c})^{\frac{1}{2}}||u'||_{L_2(c,d).} \quad (15)$$

Proof. Under the assumptions of the lemma, the function $u(x)$ is continuous. We use relation (12) and inequality (14). As a result, we arrive at estimate (15). This completes the proof.

Corollary 1. *Under the conditions of Lemma 3 L_2-estimate*

$||u \text{-} U_{c,d}\,||_{L_2(c,d)} \le 2(d-c)||u'||_{L_2(c,d)}$

is valid.

The proof is obtained by squaring the relation (15) and integrating the result over the interval (c,d).

4 Approximation Estimates

Theorem 1. *If $u \in W_2^2(c,d)$ then the inequality*

$||q(u\text{-}U_{c,d})||^2_{L_2(c,d)} \le$
$1/3(d-c)^3 \int_c^d q(x)dx\, ||u''||^2_{L_2(c,d)}$ *(16)*

is correct.

Proof. We introduce the notation

$$\mathrm{I} = ||q(u\text{-}U_{c,d})||^2_{L_2(c,d)}. \quad (17)$$

In accordance with formula (13), for (17) we have

$$\mathrm{I} = \int_c^d q(x)|(d-c)^{-1} \times$$
$$\int_c^x d\xi \int_c^d \mathrm{d}\zeta \int_\zeta^\xi u''(\eta)d\eta\,|^2 dx. \quad (18)$$

Using the Cauchy-Bunyakovsky inequality, we obtain

$$| \int_c^x d\xi \int_c^d d\zeta \int_\zeta^\xi u''(\eta) d\eta |^2$$
$$\leq \int_c^x d\xi \int_c^d d\zeta \, | \int_\zeta^\xi d\eta|$$
$$\times \int_c^x d\xi \int_c^d d\zeta \, | \int_\zeta^\xi |u''(\eta)|^2 d\eta|. \qquad (19)$$

Simple calculations give

$$\int_c^x d\xi \int_c^d d\zeta \, | \int_\zeta^\xi d\eta| \leq \frac{(d-c)^3}{3}. \qquad (20)$$

For $x \in (c, d)$ from (19) - (20) we have

$$| \int_c^x d\xi \int_c^d d\zeta \int_\zeta^\xi u''(\eta) d\eta |^2$$
$$\leq \frac{(d-c)^5}{3} \int_c^d |u''(\eta)|^2 d\eta. \qquad (21)$$

Now from (18) with the help of (21) we obtain

$$I \leq \frac{(d-c)^3}{3} \int_c^d |q(x)|^2 dx \times \int_c^d |u''(\eta)|^2 d\eta.$$

This concludes the proof.

Theorem 2. *If* $u \in W_2^2(c,d)$ *then the inequality*

$$||p(u' - U'_{c,d})||^2_{L_{2(c,d)}}$$
$$\leq (d-c) \int_c^d p(x)\, dx \times ||u''||^2_{L_{2(c,d)}} \qquad (22)$$

is fulfilled.

Proof. It is easy to see that the relation $u'(x) - U'_{c,d}(x) = (d-c)^{-1} \int_c^d d\xi \int_\xi^x u''(\eta) d\eta$ holds.

Let's put

$$I^1 = ||p(u' - U'_{c,d})||_{L_2(c,d)}$$
$$= \int_c^d p(x) |(d-c)^{-1} \times$$
$$\int_c^d d\,\xi \int_\xi^x u''(\eta) d\eta\,|^2 dx \qquad (23)$$

Because the

$$\int_c^d d\xi \int_\xi^x u''(\eta) d\eta = \int_c^x (\eta - c) u''(\eta) d\eta, \qquad (24)$$

then for $x \in (c, d)$ from (24) we have

$$| \int_c^d d\,\xi \int_\xi^x u''(\eta)\, d\eta| \leq (x-c) \int_c^x |u''(\eta)|\, d\eta$$

So

$$| \int_c^d d\,\xi \int_\xi^x u''(\eta)\, d\eta|^2$$
$$\leq (d-c)^3 \int_c^d |u''(\eta)|^2\, d\eta \qquad (25)$$

From formulas (23) - (25) we obtain

$$I^1 \leq (d-c) \int_c^d p(x)\, dx \int_c^d |u''(\eta)|^2\, d\eta \qquad (26)$$

Relation (26) is equivalent to inequality (22).

This concludes the proof.

5 Approximation Estimates without Degeneracy

Consider problem (1), (7), assuming that condition (2) is fulfilled.

Here we confine ourselves to the case $q(x)$=0.

In the future, the solution of the problems under consideration will be denoted with $u(x)$.

Let's introduce a grid

$$X: a = x_0 < x_1 < \cdots < x_{M-1} < x_M = b. \qquad (27)$$

Let $S = S(X)$ be the space of the splines of the first degree, whose coordinate splines are the continuous functions ω_j,

$$\omega_j(x) = \frac{x - x_{j-1}}{x_j - x_{j-1}} \ \textit{for}\ x \in (x_{j-1}, x_j),$$
$$\omega_j(x) = \frac{x_{j+1} - x}{x_{j+1} - x_j} \ \textit{for}\ x \in (x_j, x_{j+1}),$$
$$\omega_j(x) = 0 \ \textit{for}\ x \in [a, b] \setminus [x_j, x_{j+1}]. \qquad (28)$$

where $j = 1, 2, \dots, M-1$. In this way

$$S(X) = \{s | s = \textstyle\sum_{j=1}^{M-1} v_j \omega_j \quad \forall v_j \in R^1. \qquad (29)$$

Theorem 3. *Let u be an element of the space* $W_2^2(a,b) \cap W_2^{0,1}(a,b)$ *, and* s_0 *is an interpolant for function u given by the formula*

$$s_0 = s_0(x) = \textstyle\sum_{j=1}^{M-1} u(x_j) \omega_j(x). \qquad (30)$$

Then the inequality

$$\mathbf{I}u - s_0\mathbf{I}^2 \leq \textstyle\sum_{k=0}^{M-1} (x_{k+1} - x_k) \int_{x_k}^{x_{k+1}} p(x) dx \times \int_{x_k}^{x_{k+1}} u''(\eta) d\eta \qquad (31)$$

is right.

Proof. Consider $\mathbf{I}u - s_0\mathbf{I}$. Taking into account the ratio

$$s_0(x) = U_{x_k, x_{k+1}}(x) \ \ \forall x \in (x_k, x_{k+1}),$$

we have

$$\mathbf{I}u - s_0\mathbf{I}^2 =$$
$$\int_a^b p(x)(u'(x) - s_0(x))^2 dx =$$
$$\textstyle\sum_{k=0}^{M-1} \int_{x_k}^{x_{k+1}} p(x)(u'(x) - s_0(x))^2 dx. \qquad (32)$$

Using relations (22) in formula (32), we obtain inequality (31). This concludes the proof.

6 Approximation Estimate in a Problem with Weak Degeneracy

Consider the solution of problem (1), (7) under conditions (3) -- (4). Here we use the approximation defined by formulas (27) - (29).

The difference from the previous point is that in this case the solution u belongs to spaces $W_2^2(a + \varepsilon, b) \cap W_2^{0,1}(a,b) \ \ \forall \varepsilon \in (0, b-a)$. In view of this, the estimate changes on the interval (x_0, x_1).

Lemma 4. *An inequality*

$\int_a^{x_1} p(x)\,|u'(x) - U'_{a,x_1}(x)|^2 dx \le$
$4max_{x\in[a,x_1]} p(x) \int_a^{x_1} |u'(x)|^2 dx$ (33)
is correct.

Proof. Introduce the notation

$J_0 = \int_a^{x_1} p(x)\,|u'(x) - U'_{a,x_1}(x)|^2 dx,$ (34)

$J_{00} = \int_a^{x_1} p(x)\,|u'(x)|^2 dx,$

$$J_{01} = \int_a^{x_1} p(x)\,dx |U'_{a,x_1}|^2 . \quad (35)$$

The inequalities are obvious

$J_{00} \le \max_{x\in[a,x_1]} p(x) \int_a^{x_1} |u'(x)|^2 dx$ (36)

$J_{01} \le \max_{x\in[a,x_1]} p(x)(x_1 - a)^{-1} |u(x_1)|^2$ (37)

Using the ratio

$$|u(x_1)|^2 \le (x_1 - a) \int_a^{x_1} |u'(x)|^2 dx,$$

from inequality (37) we have

$J_{01} \le \max_{x\in[a,x_1]} p(x) \int_a^{x_1} |u'(x)|^2 dx$ (38)

Taking into account formulas (34) - (38) and the obvious inequality $J_0 = 2(J_{00} + J_{01})$ we obtain relation (33). This completes the proof.

Theorem 4. *Let u be the solution to problem (1), (7) under conditions (3) -- (4). Let s_0 be an interpolant for the function u, given by formula (30). Then the inequality*

$\mathbf{I}u\text{-}s_0\mathbf{I}^2 \le 4max_{x\in[a,x_1]} p(x) \int_a^{x_1} |u'(x)|^2 dx +$
$\sum_{k=1}^{M-1} (x_{k+1} - x_k) \times$
$\int_{x_k}^{x_{k+1}} p(x)dx \int_{x_k}^{x_{k+1}} |u''(x)|^2 dx$ (39)
is correct.

The proof of this theorem differs from the proof of Theorem 3 only by the summand corresponding to the index $k = 0$. For the above term, estimate (33) should be applied, and found in Lemma 4. As a result, we obtain inequality (39).

7 Approximation Estimate in a Problem with Strong Degeneracy

In the case of strong degeneracy, consider problem (1), (8) under conditions (3), (5), assuming that $q(x) \ge q_0 = const > 0$. Because of this assumption the operator of the problem under consideration is a positive definite operator. The difference from the previous point is that in this case, the solution u of the problem under consideration belongs to spaces $W_2^2(a+\varepsilon,b) \cap W_{*2}^{01}(a,b)$ $\forall\varepsilon \in (0, b-a)$, where $W_{*2}^{01}(a,b)$ is the subspace of space $W_2^1(a,b)$ consisting of functions which equal to zero for $x = b$. In view of this, as an approximation, S.G. Mikhlin proposed (see [16]) to take a piecewise linear approximation, which is equal to a constant on the interval (a, x_0).

Theorem 5. *Let u be the solution problem (1), (7) under conditions (3), (5). If*
$\tilde{s}_0 = u(x_1)$ *for* $x \in (a, x_1]$
$\tilde{s}_0 = U_{x_k,x_{k+1}}(x)$ *for* $x \in (x_k, x_{k+1}]$
$\forall k \in \{1,2,\dots, M-1\}$,
then the inequality

$\mathbf{I}u\text{-}\tilde{s}_0\mathbf{I}^2 \le \max_{x\in[0,x_1]} \big(p(x) + q(x)(x -$
$a)\big) \int_a^{x_1} |u'(\xi)|^2 d\,\xi + \sum_{k=1}^{M-1} \int_{x_k}^{x_{k+1}} \{(x_{k+1} -$
$x_k)p(x) + \frac{1}{3}(x_{k+1} - x_k)^3 q(x)\} dx$
$\times \int_{x_k}^{x_{k+1}} |u''(\eta)|^2\, d\eta$ (40)
is fulfilled.

Proof. Consider $\mathbf{I}\ u\text{-}\tilde{s}_0\mathbf{I}$. We have

$\mathbf{I}u\text{-}\tilde{s}_0\mathbf{I}^2 = \int_a^{x_1} (p(x)|u'(x)|^2 + q(x)|u(x) -$
$u(x_1)|^2) dx + \sum_{k=1}^{M-1} \int_{x_k}^{x_{k+1}} p(x)(u'(x) -$
$U'_{x_k,x_{k+1}})^2 + \sum_{k=1}^{M-1} \int_{x_k}^{x_{k+1}} q(x)(u(x) -$
$-U_{x_k,x_{k+1}})^2.$ (41)

For the first term in (41) we use the obvious inequality

$|u(x) - u(x_1)|^2 \le (x_1 - x) \int_x^{x_1} |u'(\xi)|^2 d\xi.$

So

$$\int_a^{x_1} (p(x)|u'(x)|^2 + q(x)|u(x) - u(x_1)|^2) dx \le$$

$$\max_{x\in[0,x_1]} (p(x) + q(x)(x - a)) \int_a^{x_1} |u'(\xi)|^2 d\xi\,. \quad (42)$$

The second term in (41) is estimated in the same way as in Theorems 3 and 4,

$$\sum_{k=1}^{M-1} \int_{x_k}^{x_{k+1}} p(x)(u'(x) - U'_{x_k,x_{k+1}})^2\, dx$$
$$\le \sum_{k=1}^{M-1} (x_{k+1} - x_k) \int_{x_k}^{x_{k+1}} p(x)dx \int_{x_k}^{x_{k+1}} |u''(x)|^2 dx. \quad (43)$$

Finally, when estimating the third term on the right side of formula (41), we use inequality (16). As a result we get

$$\sum_{k=1}^{M-1} \int_{x_k}^{x_{k+1}} q(x)(u(x) - U_{x_k,x_{k+1}})^2\, dx \le$$
$1/3 \sum_{k=1}^{M-1} (x_{k+1} -$
$x_k)^3 \int_{x_k}^{x_{k+1}} q(x)dx \int_{x_k}^{x_{k+1}} |u''(x)|^2 dx$ (44)

Adding relations (43) and (44), we find
$\sum_{k=1}^{M-1} \int_{x_k}^{x_{k+1}} p(x)(u'(x) -$
$U'_{x_k,x_{k+1}})^2 dx + \int_{x_k}^{x_{k+1}} q(x)(u(x) -$
$U_{x_k,x_{k+1}})^2\, dx \le \sum_{k=1}^{M-1} \int_{x_k}^{x_{k+1}} \{(x_{k+1} - x_k)p(x) +$
$1/3(x_{k+1} - x_k)^3 q(x)\} dx \int_{x_k}^{x_{k+1}} |u''(x)|^2 dx.$ (45)

Adding inequality (42) to inequality (45), we arrive at relation (40).

This concludes the proof.

8 Choosing an Adaptive Grid

For the fixed $a\in \boldsymbol{R}^1$ and $M\in \mathbf{N}$, $M>1$, consider the set $\boldsymbol{K}=\boldsymbol{K}(a,M)$ of grids $Y=Y(a,M)$ of the form

$$Y:\ y_0 = a < y_1 < y_2 < \cdots < y_{M-1} < y_M. \quad (46)$$

On set $\boldsymbol{K}$, consider the function $\Phi(y)$ of the form

$$\Phi(y) = \Phi(y_1,\dots,y_M) = \sum_{k=0}^{M-1} \Phi_k(y_k, y_{k+1}) \quad (47)$$

where the functions $\Phi_k(\xi,\eta)$ are defined and differentiable on the set

$$\mathbf{D} = \{(\xi,\eta) | a < \xi < \eta < +\infty\}.$$

Let us discuss the conditions

(B)

$$\frac{\partial\Phi_k}{\partial\xi}(\xi,\eta) > 0,\ \ \frac{\partial\Phi_k}{\partial\eta}(\xi,\eta) < 0\ \forall(\xi,\eta)\in\mathbf{D} \quad (48)$$

and for a fixed $\xi > a$ the functions $\frac{\partial\Phi_k}{\partial\eta}(\xi,\eta)$, $\xi<\eta$, are continuous in the set of variables ξ,η, and for fixed $\xi\in(a,+\infty)$ are strictly monotone decreasing if $\eta\to+\infty$, $k=1,2,\dots,M-1$.

In addition, it is assumed that

$$\lim_{\eta\to\xi+0}\frac{\partial\Phi_k}{\partial\eta}(\xi,\eta)=0,\ \lim_{\eta\to\xi+\infty}\frac{\partial\Phi_k}{\partial\eta}(\xi,\eta)=-\infty \quad \forall\xi>a. \quad (49)$$

Consider the conditions under which the mentioned derivatives are equal to zero. So we are interested in the situation when the relations

$$\frac{\partial\Phi}{\partial y_k}=0 <=>$$

$$\frac{\partial\Phi_{s-1}}{\partial y_s}(y_{s-1},y_s) = -\frac{\partial\Phi_s}{\partial y_s}(y_s,y_{s+1}) \quad (50)$$

are right.

Theorem 6. *If conditions (B) are fulfilled (see formulas (46) -- (49)), then the following properties hold.*

1. For any $y_1>a$ there are numbers y_j^*, $j=2,3,\dots,M$ such that for $y_j=y_j^*$ the relations (50) turn into true equalities. Numbers y_j^* are determined uniquely by the given $y_1>a$ and can be treated as functions of the argument $y_1>a$, $y_j^*=y_j^*(y_1)$, j=2,3,... ,M. Wherein

$$a < y_1 < y_2^*(y_1) < \cdots < y_M^*(y_1) \quad (51)$$

2. The functions $y_j^*=y_j^*(y_1)$ depend continuously on $y_1\in(a,+\infty)$.

3. As $y_1\to a+0$ the functions $y_j^*=y_j^*(y_1)$ tend to $a+0$, keeping relations (51).

4. For $y_1\to+\infty$ the functions $y_j^*=y_j^*(y_1)$ tend to $+\infty$, keeping relation (51).

Proof. 1. Let us fix $y_1>y_0$. In view of the (B) conditions, we have

$$\frac{\partial\Phi_0}{\partial y_1}(y_0,y_1)>0.$$

We suppose y_2 increase from y_1 to $+\infty$. Due to the same conditions, the value

$$-\frac{\partial\Phi_1}{\partial y_1}(y_0,y_1)$$

increases monotonically from 0 to $+\infty$. Therefore, there is a value $y_2=y_2^*$ that satisfies the relation

$$\frac{\partial\Phi_0}{\partial y_1}(y_0,y_1) = -\frac{\partial\Phi_1}{\partial y_1}(y_1,y_2^*).$$

Similarly, we conclude that there is y_3^* such that for $y_2=y_2^*$, $y_3=y_3^*$ formula (50), written for $s=2$, turns into the correct equality.

Continuing this process, successive enumeration of relations (50) uniquely define the values $y_j=y_j^*$, $j=4,5,\dots,M$. Thus, y_j^* can be considered as single-valued argument functions of $y_1\in R_+^1$, $y_j=y_j^*$, $j=2,3,\dots,M$, so relation (50) holds.

Item 1 of the lemma to be proved is established.

It remains to note that the continuity of the functions $y_j=y_j^*(y_1)$ follows from the implicit function theorem, and points 3, 4 are obvious consequences of conditions (B).

Lemma 5. *Whatever the number $c>0$ a unique point y^c of the form $y^c=(y_1^c,y_2^c,\dots,y_{M-1}^c,c)$ exists in set $\boldsymbol{K}$ with properties*

$$\frac{\partial\Phi}{\partial y_s}(y^c)=0,\ \ s=1,2,\dots,M-1. \quad (52)$$

Proof. Of the above properties 1 - 4 it follows that the monotonic increase of y_1 from $+0$ to $+\infty$ leads to a monotonic increase in y_M within the same limits.

Thus, there is a unique value $y_1=y_1^*$, where $y_M=c$. In this case, we set $y_1^c=y_1^*$, $y_s^c=y_s^*(y_1^c)$, $s=2,3,\dots,M-1$, so relation (52) is fulfilled.

This concludes the proof.

Consider the set Z_c of vectors $z=(z_1,z_2,\dots,z_{M-1})$ satisfying ratios

$$0<z_1<z_2<\cdots<z_{M-1}<c. \quad (53)$$

Obviously, Z_c is an open set in the Euclidean space $\boldsymbol{R}^{M-1}$.

The right-hand sides of estimates (31), (39), and (40) are functions of the form $\Phi(x_1,x_2,\dots,x_M)$ satisfying the (B) condition.

Taking into account their dependence on $p(x)$ and $q(x)$, we introduce a common designation

$$\psi_c(P,Q;\mathbf{x}) = \psi_c(P,Q;x_1,x_2,\dots,x_{M-1}) = \Phi(x_1,x_2,\dots,x_{M-1},c),\ \ c=x_M. \quad (54)$$

Note the following obvious property of the function $\psi_c(P,Q;z)$

$$P(x)\le\tilde{P}(x),\ \ Q(x)\le\tilde{Q}(x)\ \ \forall x\epsilon\boldsymbol{R}^1\ ==> \psi_c(P,Q;z)\le\psi_c(\tilde{P},\tilde{Q};z). \quad (55)$$

Theorem 7. *The function $\psi_c(P,Q;z)$ is positive on set Z_c, it has a unique critical point in this set, and at this point the function $\psi_c(P,Q;z)$ reaches its minimum.*

The proof follows from Lemma 5 for

$\Phi(x_1,x_2,\dots,x_{M-1},c)=\psi_c(P,Q;x_1,x_2,\dots,x_{M-1})$.

Theorem 8. *For the right-hand side of each of the estimates (31), (39) and (40) for a fixed x_M ($x_M>x_0$, M is fixed) there is a unique set of nodes $x_1<x_2<\dots>x_{M-1}$, for which the right side takes the minimum value. This set of nodes is the only critical point on the right side.*

The proof is a direct consequence of Theorem 7.

9 On the Choice of Grid in the Case of Strong Degeneracy

We use estimate (40). Let's put

$\Phi_0(x_0,x_1)=\int_a^{x_1}p(x)|u'(x)|^2dx+$
$(x_1-a)\int_a^{x_1}p(x)\,dx\int_a^{x_1}|u'(x)|^2dx,$
(57)

$\Phi_k(x_k,x_{k+1})=\int_{x_k}^{x_{k+1}}\{(x_{k+1}-x_k)p(x)+$
$\frac{1}{3}(x_{k+1}-x_k)^3q(x)\}\,dx\int_{x_k}^{x_{k+1}}|u''(x)|^2dx,$
(58)

$k=1,2,\dots,M-1.$

Without loss of generality, we will assume that the integrands functions in relations (57) - (58) are extended to $(b,+\infty)$ positive constants, so that the conditions (B) are satisfied.

An example of problem (1), (8) is the problem (see [16])

$$-(x^\alpha u')'+qu=\frac{(x^{3-\alpha}-1)q-2(3-\alpha)x}{3-\alpha},x\epsilon(0,1), \quad (59)$$

$$u(1)=0, \quad (60)$$

where $1\le\alpha\le2$, $q=const>0$.

The solution to problem (59) -- (60) is function

$$u(x)=\frac{x^{3-\alpha}-1}{3-\alpha}. \quad (61)$$

Taking into account (61) we have (58) in the form

$$\Phi_k(x_k,x_{k+1})=$$
$$\{(x_{k+1}-x_k)\frac{x_{k+1}^{\alpha+1}-x_k^{\alpha+1}}{1+\alpha}+\frac{q}{3}(x_{k+1}-x_k)^4\}$$
$$\times\frac{x_{k+1}^{3-2\alpha}-x_k^{3-2\alpha}}{3-2\alpha}. \quad (62)$$

Consider the case α =1. Formula (62) takes the form

$$\Phi_k(x_k,x_{k+1})=(x_{k+1}-x_k)^3[\frac{(x_{k+1}+x_k)}{2}+$$
$$q(x_{k+1}-x_k)^2/3]. \quad (63)$$

We introduce the notation $x=x_{k+1}$, $y=x_k$, $z=x_{k+1}$. By (63) equations (50) can be represented in the form

$$(y-x)^2\left[x+2y+\frac{5q(y-x)^2}{3}\right]=$$
$$(z-y)^2\left[2y+z+\frac{5q(z-y)^2}{3}\right], \quad (64)$$

where x and y are known values, x <y, and z is the desired unknown, $z>y$. The obvious root of the equation is $z=x$, but this solution should be excluded, since the condition $z>y$ is not satisfied.

Using elementary transformations, we eliminate the factor $z-x$ and from equation (64) we pass to the cubic equation

$$\frac{5q}{3}[z^3+z^2(x+2y)+z(6y^2-4xy+x^2)+x^2-$$
$$4xy+6xy^2-4y^3]+z^2+zx+x^2-3y^2=$$
$$0 \quad (65)$$

Setting the numbers x= a and y= x_1 and solving equation (65), we find the solution $x_2=z$. Then we solve equation (65) for x= x_1 and y= x_2. As a result, we find $x_3=z$. The process can be stopped when the end of the interval is reached (a, b]. The resulting number of nodes on the mentioned interval may not satisfy the user. In this case calculations should be repeated with the changed value of the number x_1, $x_1\epsilon(a,b)$.

10 Numerical Experiment Results

For a numerical experiment, the problem was considered with strong degeneracy, namely, problem (59) - (60) for $\alpha=1$, $q=1$:

$-(xu')'+u=(x^2-4x-1)/2 \qquad u(1)=0.$

The problem was solved with the variation-grid method described above.

To determine the grid, we set $q=1$ in equation (65) and use it in the recurrent process mentioned above.

The initial data for this process are $x=0$, $y=0.001$, $M=10^4$. The results are shown in Table 1.

The problem represented by an illustrative example of the previous point, was solved on an HP 27 p251ur computer using the system Maple 2017.0 with $M=20,50,100,1000,$ 10000.

For values 100, 1000, 10000 the accuracy achieved was of the order of 10^{-4}, 10^{-6}, 10^{-7} respectively. On a uniform grid with the same number of nodes, known calculations (see [16]) in some cases gave less accurate results (see Table 1).

Although the comparison seems difficult because of the differences in the types of computers used to (calculations presented in [16] carried out at BESM-6). Control calculations using the same program Maple 2021.1 systems gave the same result.

The computation time using the proposed algorithm was 92 seconds.

Table 1. Calculation Error

No.	Value of argument	Error at BESM-6	Error Maple-2021
1.	0.0	$0.20\cdot 10^{-3}$	$2.94\cdot 10^{-7}$
2.	0.1	$0.18\cdot 10^{-3}$	$2.01\cdot 10^{-9}$
3.	0.2	$0.13\cdot 10^{-3}$	$1.41\cdot 10^{-9}$
4.	0.3	$0.87\cdot 10^{-4}$	$1.13\cdot 10^{-9}$
5.	0.4	$0.65\cdot 10^{-4}$	$1.03\cdot 10^{-9}$
6.	0.5	$0.48\cdot 10^{-4}$	$9.33\cdot 10^{-10}$
7.	0.6	$0.34\cdot 10^{-4}$	$7.67\cdot 10^{-10}$
8.	0.7	$0.23\cdot 10^{-4}$	$7.67\cdot 10^{-10}$
9.	0.8	$0.14\cdot 10^{-4}$	$7.05\cdot 10^{-10}$
10.	0.9	$0.67\cdot 10^{-4}$	$7.30\cdot 10^{-10}$

11 Conclusion

It is widely known that the numerical solutions of problems of mathematical physics have many difficulties. These difficulties are associated with the constant expansion of the set of such tasks. In addition, the requirements to the accuracy of the numerical solutions increase constantly. Using priori information on such problems, we divide their set into classes of tasks. Its circumstance allows the use of trained computer systems for the effective solution of problems. As you know, the training of a computer system is carried out on several problems of this class with a known solution. The result of learning is determination of the parameters of the adaptive method. Since the solution of problems of mathematical physics is often carried out at various grid methods (FEM, finite difference method), the natural parameters of these methods are grid knots. The adaptive choice of knots seems to be very important in the case of problems with particular features of the solution. Mortgaged into the system, the method uses a priori information about solution properties for adaptive positioning of grid knots.

Note that in most complex computational problems the desired function has regions of fast and slow changes (in tasks related to earthquakes, tsunamis, weapons tests, etc.) The approximation of the desired function using a grid method on an adaptive non-uniform grid wins significantly when compared to its approximation on a uniform grid (see, for example, Table 1).

In areas of rapid change function, the grid should be dense, and in areas of slow change, a sparse mesh can be used. Training a computer system based on available information about the decision makes it possible to indicate the mentioned areas. For this, goals require approximation estimates with known constants (see, for example, the theorems of this paper). In practice, we have to solve many similar problems. In this case, the training of the computer system is carried out on several problems of this type with a known decision. The result is an averaged adaptive mesh, which can be used in the future for other tasks of the mentioned type.

In this work, this approach is implemented in the case of the variation-grid method for a boundary value problem in the one-dimensional differential equation of the second order. Numerical results of computer implementation of such an approach are done.

In the future, it is planned to use this approach for more complex boundary problems. The perceived difficulties are of a technical nature. One of these difficulties is that the process of learning a computer system requires significant time resources. However, this should not stop research of this kind, because the power of the emerging computing systems are growing exponentially.

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