A Reliable Algorithm for Solving System of Multi-Pantograph Equations

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Abstract. In this article, a new series solution of a system of pantograph equations is established using the residual power series method (RPSM). The proposed method produces the solution in terms of a convergent infinite series, requiring no linearization, perturbation or discretization, in some cases it reproduces the exact solutions. We apply the RPSM to solve the multi-pantograph equations, and we show that the outcomes are very accurate. Some examples are given to demonstrate the simplicity and efficiency of the proposed method. Comparisons to the Laplace decomposition approach are made to verify the efficiency and applicability of the presented method in solving similar problems.

Key-words: Residual power series method; Pantograph equations; System of initial value problems.

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1 Introduction

The pantograph equation, which is one of the most important kinds of delay differential equations, [1], [2], [3], [4], [5] and [6], has been studied extensively owing to the numerous applications in which these equations arise. The name pantograph originated from the work of the researchers, [1], on the collection of current by the pantograph head of an electric locomotive, this equation appeared in modeling various problems in engineering and sciences such as biology, economy, control, population studies and electrodynamics, [7], [8], [9], [10].

In the last years, extensive work dealt with the pantograph equation. Several methods have been used to solve different types of the pantograph equation, such as Adomian's decomposition method, [5], [6], the homotopy perturbation method [7], Variational iteration method, [8], [9], Runge–Kutta methods, [10], the reproducing kernel space method, [11], Taylor polynomials approach, [12], [13], one-leg θ-methods [14], Spectral methods, [15], differential transformation method, [16], Discontinuous Galerkin methods, [17], Bessel matrix and collocation methods, [18], [19], Chebyshev polynomials method, [20], Laplace decomposition algorithm (LDA) [21], [22], and so on [23], [24], [25], [26], [27], [28], [29], [30].

The purpose of this paper is to extend the application of the residual power series (PSR) method [31], [32] to provide a symbolic approximate solution for a system of multi-pantograph equations:

$$\begin{aligned}
z_1'(t) &= \beta_1 z_1(t) + g_1\big(t, z_1(\alpha_{11}t), z_2(\alpha_{12}t), \dots, z_n(\alpha_{1n}t)\big),\\
z_2'(t) &= \beta_2 z_2(t) + g_2\big(t, z_1(\alpha_{21}t), z_2(\alpha_{22}t), \dots, z_n(\alpha_{2n}t)\big),\\
&\ \vdots\\
z_n'(t) &= \beta_n z_n(t) + g_n\big(t, z_1(\alpha_{n1}t), z_2(\alpha_{n2}t), \dots, z_n(\alpha_{nn}t)\big),
\end{aligned}$$
(1.1)

Subject to the initial conditions

$$z_i(0) = z_{i,0},\ i = 1,2,3,\dots,n, \quad (1.2)$$

Where $\beta_i, u_{i,0}$ are constants, f_i are analytical functions, and $0 < \alpha_{ij} \le 1$.

The RPS was developed in [31] as an efficient method for determining the coefficients of the power series solution of the first and second-order fuzzy differential equation. It has been successfully applied in the numerical solution of the generalized Lane-Emden equation, which is a highly nonlinear problem, [32]. The RPS method is effective and easy to construct a power-series solution for strongly linear and nonlinear equations without linearization, perturbation, or discretization. In contrast to the classical power series (CPS) methods, the RPS method does not need to compare the coefficients of the corresponding terms, and a recursion relation is not required. This method computes the coefficients of the power series by a chain of linear equations of one variable. The RPS method is an alternative procedure for obtaining an analytic Taylor series solution of the system of multi-pantograph equations. By using residual error concept, we get a series solution, in practice a truncated series solution. For linear problems, the exact solution can be obtained by a few terms of the RPS method solution. As we shall see later, the exact solution is available when the solution is polynomial. Moreover, the solutions and all their derivatives are applicable for each arbitrary point in the given interval. It does not require any converting while switching from the first order to the higher order; as a result, the method can be applied directly to the given problem by choosing an appropriate value for the initial guess approximation.

This paper is organized as follows: in the next section, we state some definitions and theorems that help us to construct the proposed method. In section 3, we present the basic idea of the power series method. In section 4 we extend the PSR method to provide a symbolic approximate series solution for a system of multi-pantograph equations. In section 5, numerical examples are given to illustrate the capability of the proposed method. Section 6 is the brief conclusion of this paper. Finally, some references are listed at the end.

2 Preliminaries

In this section, we introduce some definitions and theorems related to Taylor's series and analytic functions.

Definition 2.1. A function g is called analytic at $t_0 \in I$, where I is an open interval, if it can be represented in a form of a power series as

$$g(t) = \sum_{n=0}^{\infty} c_n (t-t_0)^n. \qquad (2.1)$$

Taking $t_0 = 0$,we get the Maclaurin series

$$g(t) = \sum_{n=0}^{\infty} c_n t^n, \forall\ t \in I.$$

Theorem 2.1 [22] There are only three possibilities for the convergence conditions of the power series (2.1):

(i) The series converges only when $t = t_0$, and the radius of convergence is zero..

(ii) The series converges for all $t > t_0$, and the radius of convergence is ∞.

(iii) There is a positive number $R > 0$ such that the series converges if $|t-t_0| < R$ and diverges if $|t-t_0| > R$.

Here R is called the radius of convergence.

Theorem 2.2. **[22]** If g has a power series representation as follows:

$$g(t) = \sum_{n=0}^{\infty} c_n (t-t_0)^n, |t-t_0| < R,$$

Then its coefficients c_n are given by the formula:

$$c_n = \frac{g^{(n)}(t_0)}{n!}, n = 0,1,2, ...$$

Theorem 2.3 (Convergence Analysis) [22]

If we have $0 < K < 1$, and $\|g_{n+1}(t)\| \leq K\|g_n(t)\|$, for all $n \in N$ and $0 < t < R < 1$, then the series of the numerical solutions converges to the exact solution.

3 Adapting RPSM to Solve Multi-Pantograph Equations

In this section, we introduce the procedure of using RPSM in solving multi pantograph systems (1.1) and (1.2).

We present a simple algorithm that explains the method and illustrates the steps of the RPSM in handling the proposed problem.

To apply the RPSM, we firstly assume that the solutions of system (1.1) and (1.2) have the forms:

$$z_i(t) = \sum_{m=0}^{\infty} c_{i,m} t^m, i = 1,2, \dots, n, \quad (3.1)$$

Where $c_{i,0} = z_{i,0}$, $i = 1,2,3, \dots, n$.

Since $u_i(t)$ satisfies the initial conditions (1.2), $u_{i_{\text{init}}}(t) = u_{i,0}$ are the zeroth RPS solutions of the IVP (1.1) and (1.2).Thus, the solutions have the form:

$$z_i(t) = z_{i,0} + \sum_{m=1}^{\infty} c_{i,m} t^m, i = 1,2, \dots, n, (3.2)$$

And the kth-approximate solutions will be:

$$z_{i,k}(t) = z_{i,0} + \sum_{m=1}^{k} c_{i,m} t^m, i = 1,2, \dots, n. (3.3)$$

Following that, we define the kth-residual functions of system (1.1) as:

$$\mathrm{Res}_i^k(t) = z'_{i,k}(t) - \beta_i z_{i,k}(t) - g_i\begin{pmatrix} t, z_{1,k}(\alpha_{i2}t), z_{2,k}(\alpha_{i2}t), \\ \dots, z_{n,k}(\alpha_{in}t) \end{pmatrix} \quad (3.4)$$

$, i = 1,2,\dots,n,$

And the following residual functions:

$$\mathrm{Res}_i(t) = \lim_{n\to\infty} \mathrm{Res}_i^k(t) = z'_i(t) - \beta_i z_i(t) - g_i\big(t, z_1(\alpha_{i2}t), z_2(\alpha_{i2}t), \dots, z_n(\alpha_{in}t)\big), i = 1,2,\dots,n. \quad (3.5)$$

It is obvious that: $\mathrm{Res}_i(t) = 0$ for each $t \in (-R_i, R_i)$ where R_i is the radius of convergence of the power series (3.1). This shows that these residual functions are infinitely many times differentiable at $t = 0$. On the other hand,

$$\frac{d^m}{dt^m}\mathrm{Res}_i(0) = \frac{d^m}{dt^m}\mathrm{Res}_i^k(0) = 0, m = 0,1,2,\dots,k. \quad (3.6)$$

In fact, these relations are fundamental rules in RPSM, for the proof and more details, see [31], [32]. Moreover, a special case of (3.6) is:

$$\frac{d^{k-1}}{dt^{k-1}}\mathrm{Res}_i^k(0) = 0, i = 1,2,\dots,n, k = 1,2,\dots. \quad (3.7)$$

In order to obtain the kth-approximate solutions of system (1.1) and (1.2), we substitute the kth-truncated series (3.3) into Eq. (3.4) to get:

$$\mathrm{Res}_i^k(t) = \sum_{m=1}^{k} m c_{i.m} t^{m-1} - \beta_i \sum_{m=0}^{k} c_{i.m} t^m - g_i\left(\begin{matrix} t, \sum_{m=0}^{k} c_{1.m}{\alpha_{i1}}^m t^m, \sum_{m=0}^{k} c_{2.m}{\alpha_{i2}}^m t^m, \dots, \\ \sum_{m=0}^{k} c_{p.m}{\alpha_{in}}^m t^m \end{matrix} \right)$$

$i = 1,2,\dots,n.$ (3.8)

To obtain the first approximate solution, we substitute $t = 0$ and $k = 1$ into Eq. (3.8), and using (3.7):

$\mathrm{Res}_i^1(0) = 0$, $i = 1,2,\dots,n$, we get:

$$c_{i,1} = \beta_i c_{i,0} + g_i(0, c_{1.0}, c_{2.0}, \dots, c_{n.0}) = \beta_i z_{i,0} + g_i(0, z_{1.0}, z_{2.0}, \dots, z_{n.0}), \quad i = 1,2,\dots,n.$$

Thus, the first approximation for the system of multi-pantograph equations (1.1) and (1.2) can be expressed as:

$$z_{i,1}(t) = z_{i,0} + \big(\beta_i z_{i,0} + f_i(0, z_{1.0}, z_{2.0}, \dots, z_{n.0})\big)t, i = 1,2,\dots,n. \quad (3.4)$$

Similarly, to find the second approximation, we differentiate both sides of (3.8) with respect to t and substitute $t = 0$ and $k = 2$, to get:

$$\left(\frac{d}{dt}\mathrm{Res}_i^2\right)(0) = 2c_{i,2} - \beta_i c_{i,1} - \frac{d}{dt}\left(g_i\left(\begin{matrix} t, \sum_{m=0}^{2} c_{1.m}{\alpha_{i1}}^m t^m, \sum_{m=0}^{2} c_{2.m}{\alpha_{i2}}^m t^m, \dots, \\ \sum_{m=0}^{2} c_{p.m}{\alpha_{in}}^m t^m \end{matrix} \right)\right),$$

$i = 1,2,\dots,n.$

According to (3.7), we have the values of $c_{i,2}$ as follows:

$$c_{i,2} = \frac{1}{2}\left[+\beta_i c_{i,1} + \frac{d}{dt}\left(g_i\left(\begin{matrix} t, \sum_{m=0}^{2} c_{1.m}{\alpha_{i1}}^m t^m, \\ \sum_{m=0}^{2} c_{2.m}{\alpha_{i2}}^m t^m, \dots, \sum_{m=0}^{2} c_{p.m}{\alpha_{in}}^m t^m \end{matrix} \right)\right)\right],$$

$i = 1,2,\dots,n.$

Thus, the second approximation for the system of multi-pantograph equations (1.1) and (1.2) will be:

$$z_{i,2}(t) = z_{i,0} + z_{i,0} + \big(\beta_i z_{i,0} + f_i(0, z_{p,0})\big)t + \frac{1}{2}\left[+\beta_i c_{i,1} + \frac{d}{dt}\left(g_i\left(\begin{matrix} t, \sum_{m=0}^{2} c_{1.m}{\alpha_{i1}}^m t^m, \sum_{m=0}^{2} c_{2.m}{\alpha_{i2}}^m t^m, \dots, \\ \sum_{m=0}^{2} c_{p.m}{\alpha_{in}}^m t^m \end{matrix} \right)\right)\right] t^2$$

$i = 1,2,\dots,n.$

Completing in the same manner, we can obtain the rest of the coefficients recursively. Thus the series solution of

of the multi-pantograph equations (1.2) and (1.2) are obtained. Moreover, higher accuracy can be achieved by evaluating more components of the solution.

4 Numerical Example and Discussion

In this section, we consider four interesting examples of the multi pantograph equations, we apply the RPSM to solve them and analyze the results. The results demonstrate the efficiency and accuracy of the presented technique. We mention that all numerical computations are performed using Mathematica 11.0 software package.

Example 4.1. Consider the two-dimensional pantograph equations:

$$z_1'(t) = z_1(t) - z_2(t) + z_1\left(\frac{t}{2}\right) - e^{\frac{t}{2}} + e^{-t},$$
$$z_2'(t) = -z_1(t) - z_2(t) - z_2\left(\frac{t}{2}\right) + e^{-\frac{t}{2}} + e^{t}, \quad (4.1)$$

Subject to the initial conditions:

$$z_1(0) = 1, z_2(0) = 1. \quad (4.2)$$

The exact solution of system (4.1) and (4.2) is:

$$z_1(t) = e^t, z_2(t) = e^{-t}.$$

According to the residual functions (3.5), we obtain:

$$\mathrm{Res}_1(t) = z_1'(t) - z_1(t) + z_2(t) - z_1\left(\frac{t}{2}\right) + e^{\frac{t}{2}} - e^{-t},$$
$$\mathrm{Res}_2(t) = z_2'(t) + z_1(t) + z_2(t) + z_2\left(\frac{t}{2}\right) - e^{-\frac{t}{2}} - e^{t}.$$
(4.3)

According to the initial conditions (4.2), we can determine the first coefficients of the power series as:

$$c_{1,0} = z_{1,0} = z_1(0) = 1$$
and
$$c_{2,0} = z_{2,0} = z_2(0) = 1.$$

Hence, the power series solution of system (4.1) can be expressed as:

$$z_1(t) = 1 + c_{1,1}t + c_{1,2}t^2 + c_{1,3}t^3 + \cdots,$$
$$z_2(t) = 1 + c_{2,1}t + c_{2,2}t^2 + c_{2,3}t^3 + \cdots.$$

It is clear that the first approximations of the series solution for system (4.1) and (4.2) is of the form:

$$z_1(t) = 1 + c_{1,1}t,$$
$$z_2(t) = 1 + c_{2,1}t. \quad (4.4)$$

To find the values of the coefficients $c_{1,1}$ and $c_{2,1}$, we substitute the equations in system (4.4) into (4.3) to get the following 1st-residual functions of Eqs. (4.1):

$$\mathrm{Res}_1^1(t) = c_{1,1}\left(1 - \frac{3}{2}t\right) + c_{2,1}t + e^{\frac{t}{2}} - e^{-t} - 1,$$
$$\mathrm{Res}_2^1(t) = c_{2,1}\left(1 + \frac{3}{2}t\right) + c_{1,1}t - e^{-\frac{t}{2}} - e^{t} + 3.$$
(4.5)

Setting $t = 0$ in (4.5) and use the fact (3.6), then we obtain $c_{1,1} = 1$, and $c_{2,1} = -1$.

Thus, the first approximations of the series solution of (4.1) and (4.2) are:

$$z_1(t) = 1 + t$$
$$z_2(t) = 1 - t$$

The second approximations of the series solution of (4.1) and (4.2) have the forms:

$$z_1(t) = 1 + t + c_{1,2}t^2 \quad (4.6)$$
$$z_2(t) = 1 - t + c_{2,2}t^2$$

In order to find the values of the coefficients $c_{1,2}$, and $c_{2,2}$, we substitute (4.6) into (4.3) to get the form of the 2nd-residual functions of (4.1) which is:

$$\mathrm{Res}_1^2(t) = \left(2c_{1,2} - \frac{5}{2}\right)t + \left(c_{2,2} - \frac{5}{4}c_{1,2}\right)t^2 + e^{\frac{t}{2}} - e^{-t},$$
$$\mathrm{Res}_2^2(t) = \left(2c_{2,2} - \frac{1}{2}\right)t + \left(c_{1,2} + \frac{5}{4}c_{2,2}\right)t^2 + 2 - e^{-\frac{t}{2}} - e^{t}. \quad (4.7)$$

Differentiate the both sides of Eqs. (4.7) with respect to t as follows:

$$\mathrm{Res}_1'(t) = \left(2c_{1,2} - \frac{5}{2}\right) + \left(2c_{2,2} - c_{1,2}\frac{5}{2}\right)t + \frac{1}{2}e^{\frac{t}{2}} + e^{-t},$$
$$\mathrm{Res}_2'(t) = \left(2c_{2,2} - \frac{1}{2}\right) + \left(2c_{1,2} + c_{2,2}\frac{5}{2}\right)t + \frac{1}{2}e^{-\frac{t}{2}} - e^{t}. \quad (4.8)$$

Substituting $t = 0$ in (4.8) and using the fact in (3.6) leads to $c_{1,2} = \frac{1}{2}$, and $c_{2,2} = \frac{1}{2}$.

Thus, the second approximations of the series solution of (4.1) and (4.2) can be written as:

$$z_1(t) = 1 + t + \frac{1}{2}t^2,$$
$$z_2(t) = 1 - t + \frac{1}{2}t^2. \quad (4.9)$$

Continuing with similar fashion, the series solutions of $u_1(t)$ and $u_2(t)$ will be:

$$z_1(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \cdots,$$
$$z_2(t) = 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \cdots (4.10)$$

The closed form of above solutions, when $k \to \infty$ are $u_1(t) = e^t$, $u_2(t) = e^{-t}$ which are the exact solutions.

Example 4.2. Consider the system of multi-pantograph equations [21]:

$$z_1'(t) = -z_1(t) - e^{-t}\cos\left(\frac{t}{2}\right) z_2\left(\frac{t}{2}\right) - 2e^{-\frac{3t}{4}}\cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{4}\right) z_1\left(\frac{t}{4}\right),$$
$$z_2'(t) = e^t z_1^2\left(\tfrac{t}{2}\right) - z_2^2\left(\tfrac{t}{2}\right). \quad (4.11)$$

Subject to the initial conditions:
$u_1(0) = 1, u_2(0) = 0$. (4.12)
According to residual functions in (3.5), we obtain:

$$\mathrm{Res}_1(t) = z_1'(t) + z_1(t) + e^{-t}\cos\left(\frac{t}{2}\right) z_2\left(\frac{t}{2}\right) + 2e^{-\frac{3t}{4}}\cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{4}\right) z_1\left(\frac{t}{4}\right),$$
$$\mathrm{Res}_2(t) = z_2'(t) - e^t z_1^2\left(\frac{t}{2}\right) + z_2^2\left(\frac{t}{2}\right). \quad (4.13)$$

The first approximations of the series solution of (4.11) and (4.12) have the form:

$$z_1(t) = 1 + c_{1,1}t,$$
$$z_2(t) = c_{2,1}t. \quad (4.14)$$

To find the values of the coefficients $c_{1,1}$ and $c_{2,1}$, substitute Eqs. (5.14) into Eqs. (5.13) to obtain the 1st-residual function which of the form:

$$\mathrm{Res}_1^1(t) = c_{1,1} + 1 + c_{1,1}t + c_{2,1}\frac{t}{2}e^{-t}\cos\left(\frac{t}{2}\right) + 2e^{-\frac{3t}{4}}\cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{4}\right)\left(1 + c_{1,1}\frac{t}{4}\right),$$
$$\mathrm{Res}_2^1(t) = c_{2,1} - e^t\left(1 + c_{1,1}\tfrac{t}{2}\right)^2 + \left(c_{2,1}\tfrac{t}{2}\right)^2. \quad (4.15)$$

If we set $t = 0$ in Eq. (5.15) and use the fact $\mathrm{Res}_i^1(0) = 0, i = 1,2$, then we obtain $c_{1,1} = -1$, and $c_{2,1} = 1$. Thus, the first approximations of the series solution for Eqs. (5.11) and (5.12) are:

$$z_1(t) = 1 - t,$$
$$z_2(t) = t. \quad (4.16)$$

By continuing with the similar arguments of Example (4.1), we get the series solutions of $z_1(t)$ and $z_2(t)$ as follows:

$$z_1(t) = 1 - t + \frac{t^3}{3} - \frac{t^4}{6} + \frac{t^5}{30} - \frac{t^7}{630} + \frac{t^8}{2520} - \frac{t^9}{22680} + \cdots$$
$$z_2(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880} - \cdots \quad (4.17)$$

Which are the expansions of the exact solutions:
$u_1(t) = e^{-t}\cos t$, $u_2(t) = \sin t$.

Example 4.3. Consider the following system of multi-pantograph equations:

$$z_1'(t) = z_1(t) - tz_1(2t) + 3z_2(3t) - 2 - 38t + 22t^2 + 4t^3,$$
$$z_2'(t) = -3z_2(t) - tz_1\left(\tfrac{t}{2}\right) - z_2(3t) + 1 + 31t - 13t^2 + \tfrac{t^3}{4}. \quad (4.18)$$

Subject to the initial conditions:
$z_1(0) = 3, z_2(0) = -1$. (4.19)
Which have the exact solution:
$z_1(t) = t^2 - 2t + 3, z_2(t) = -t^2 + 5t - 1.$

As in the previous examples, the initial guesses approximation as:
$z_{1_{\text{init}}}(t) = 3$
And
$z_{2_{\text{init}}}(t) = -1,$

Then the power series expansions of the solution take the form:

$$z_1(t) = 3 + c_{1,1}t + c_{1,2}t^2 + c_{1,3}t^3 + \cdots,$$
$$z_2(t) = -1 + c_{2,1}t + c_{2,2}t^2 + c_{2,3}t^3 + \cdots. \quad (4.20)$$

Consequently, the first approximations of the series solution of (4.18) and (4.19) are:

$$z_1(t) = 3 + c_{1,1}t,$$
$$z_2(t) = -1 + c_{2,1}t, \quad (4.21)$$

and the1st-residual functions of Eqs. (4.19) are:

$$\mathrm{Res}_1^1(t) = 2 + 41t - 22t^2 - 4t^3 + c_{1,1}(1 - t + 2t^2) - 9tc_{2,1},$$
$$\mathrm{Res}_2^1(t) = -5 - 28t + 13t^2 - \tfrac{t^3}{4} + c_{1,1}\tfrac{t^2}{2} + c_{2,1}(1 + 6t). \quad (4.22)$$

Setting $t = 0$ in (4.21) and using the fact in (3.7), one can get $c_{1,1} = -2$, and $c_{2,1} = 5$.

Thus, the second approximations of the series solutions of (4.18) and (4.19) are:

$$z_1(t) = 3 - 2t + c_{1,2}t^2,$$
$$z_2(t) = -1 + 5t + c_{2,2}t^2, \quad (4.23)$$

and the 2nd –residual functions of (4.18) are:

$$\mathrm{Res}_1^2(t) = -2t - 26t^2 - 4t^3 + c_{1,2}t(2 - t + 4t^2) - 27t^2c_{2,2},$$
$$\mathrm{Res}_2^2(t) = 2t + 12t^2 - \tfrac{t^3}{4} + c_{1,2}\tfrac{t^3}{4} + c_{2,2}t(2 + 12t). \quad (4.24)$$

Using the fact in (3.7) for $k = 2$ reduces a system of two linear equations with two variables $c_{1,2}$ and $c_{2,2}$. The solution of this system gives $c_{1,2} = 1$, and $c_{2,2} = -1$.

It is easy to discover that each of the coefficients $c_{1.m}$ and $c_{2.m}$ for $m > 2$ in the expansions (4.20) vanished. In other words, we have:

$\sum_{m=0}^{\infty} c_{i.m} t^m = \sum_{m=0}^{3} c_{i.m} t^m, i = 1,2.$ (4.25)

Thus, the analytic approximate solution of system (4.18) and (4.19) coincide with the exact solution, which is a powerful merit in RPSM, that is it gives the exact solution if it is a polynomial.

Example 4.4. Consider the three-dimensional pantograph equations :

$$z_1'(t) = 2z_2\left(\frac{t}{2}\right) + z_3(t) - t\cos\left(\frac{t}{2}\right),$$
$$z_2'(t) = 1 - t\sin(t) - 2z_3^2\left(\frac{t}{2}\right),$$
$$z_3'(t) = z_2(t) - z_1(t) - t\cos(t). \quad (4.26)$$

Subject to the initial conditions:
$z_1(0) = -1, z_2(0) = 0, z_3(0) = 0$. (4.27)
Which has the exact solution $z_1(t) = -\cos t$, $z_2(t) = t\cos t$ and $z_3(t) = \sin t$.
Repeating the same steps in the previous examples, we can find the numerical solution of system (4.26) and (4.27) as:

$$z_1(t) = -1 + \frac{t^2}{2} - \frac{t^4}{24} + \frac{t^6}{720} - \frac{t^8}{40320} + \frac{t^{10}}{3628800} - \cdots,$$
$$z_2(t) = t - \frac{t^3}{2} + \frac{t^5}{24} - \frac{t^7}{720} + \frac{t^9}{40320} - \cdots,$$
$$z_3(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880} - \cdots.$$

(4.28)

For the third example which are the exact solutions $z_1(t) = -\cos t$, $z_2(t) = t\cos t$ and , $z_3(t) = \sin t$.
To show the accuracy of the presented method, we report two types of errors. The first one is the residual error,Re_i and defined as:

$$\mathrm{Re}_i(t) = \left| \frac{d}{dt} z_{i,\mathrm{RPS}}^k(t) - \beta_i z_{i,\mathrm{RPS}}^k(t) - g_i\left(\begin{pmatrix} t, z_{1,\mathrm{RPS}}^k(\alpha_{i2}t), z_{2,\mathrm{RPS}}^k(\alpha_{i2}t), \\ \dots, z_{n,\mathrm{RPS}}^k(\alpha_{in}t) \end{pmatrix}\right) \right| \quad (4.29)$$

While the exact error,Ex_k is defined, by:
$\mathrm{Ex}_i(t) := \left|z_{i,\mathrm{Exact}}(t) - z_{i,\mathrm{RPS}}^k(t)\right|$. (4.30)

Where, $u_{i,\mathrm{RPS}}^k$ is the kth-order approximation of $z_i(t)$ obtained by the RPS method, and $z_{i,\mathrm{Exact}}(t)$ is the exact value of $z_i(t)$, $i = 1,2,\dots,n$. We introduce Table 1, Table 2 and Table 3, below to show the related errors of $z_1(t), z_2(t)$ *and* $z_3(t)$.

Without loss of generality, we will test the accuracy of the presented method for the fourth example.
In Table 1,2 and 3, the residual errors, exact errors and the exact errors obtained by the Laplace decomposition algorithm (LDA), [21], have been calculated for various values of t in [0,1] to compare the 10th-order approximate RPS method solution with LDA. From the tables, it can be seen that the RPS method provides us with the accurate approximate solution of system (4.26) and (4.27). Moreover, we can control the error also by evaluating more components of the solution.

Table 1. Exact and residual error of $z_1(t)$ of Example (4.4)

t	Exact Error(LDA)	Exact Error(RPS)	Residual Error(RPS)
0.2	8.904×10^{-5}	0	0
0.4	1.511×10^{-3}	1.1102×10^{-6}	0
0.6	8.051×10^{-3}	0	0
0.8	2.665×10^{-2}	0	1.1102×10^{-16}
1.0	6.766×10^{-2}	1.1102×10^{-16}	1.1102×10^{-16}

Table 2. Exact and residual error of $z_2(t)$ of Example (4.4)

t	Exact Error(LDA)	Exact Error(RPS)	Residual Error(RPS)
0.2	5.496×10^{-6}	0	0
0.4	1.808×10^{-4}	5.5511×10^{-17}	0
0.6	1.408×10^{-3}	0	1.1102×10^{-16}
0.8	6.069×10^{-3}	1.1102×10^{-16}	0
1.0	1.890×10^{-2}	1.1102×10^{-16}	2.2204×10^{-16}

Table 3. Exact and residual error of $z_3(t)$ of Example (4.4)

t	Exact Error(LDA)	Exact Error(RPS)	Residual Error(RPS)
0.2	6.4558×10^{-5}	2.7755×10^{-17}	2.7755×10^{-17}
0.4	9.9595×10^{-4}	0	5.5511×10^{-17}
0.6	4.8397×10^{-3}	0	5.5511×10^{-17}
0.8	1.4613×10^{-2}	0	0
1.0	3.3917×10^{-2}	0	1.1102×10^{-16}

5 Conclusion

The aim of this work is to propose an efficient algorithm of the solution of the system of pantograph equations. We extended the RPS method to solve this class of systems of IVPs. We conclude that the RPS method is a powerful and efficient technique in constructing approximate series solutions of linear and nonlinear IVPs of different types. The proposed algorithm produced a rapidly convergent series without requiring perturbations, discretization, or other restrictive assumptions which may change the structure of the problem being solved. We believe that the efficiency of the RPS method gives it a much wider applicability. In the future, we will expand the applications of the presented method to solve more physical and engineering problems.

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