# Some Study on the Topological Structure on Semigroups

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*Abstract:* - Some studies related to the topological structure of semigroups are provided. In, [3], considering and investigating the properties of the collection $\mathcal{A}$ of all the proper uniformly strongly prime ideals of a $\Gamma$ -semigroup $S$, such study starts by constructing a topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$ using a closure operator defined in terms of the intersection and inclusion relation among these ideals of $\Gamma$-semigroup $S$, which is a generalization of the semigroup. In this paper, we introduce three other classes of ideals in semigroups called maximal ideals, prime ideals and strongly irreducible ideals, respectively. Investigating properties of the collection $\mathcal{M}$, $\mathcal{B}$ and $\mathcal{S}$ of all proper maximal ideals, prime ideals and strongly irreducible ideals, respectively, of a semigroup $S$, we construct the respective topologies on them. The respective obtained topological spaces are called the structure spaces of the semigroup $S$. We study several principal topological axioms and properties in those structure spaces of semigroup such as separation axioms, compactness and connectedness, etc.

*Key-Words:* - Semigroup, prime ideal, maximal ideal, strongly irreducible ideal, structure space, hull-kernel topology.

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## 1 Introduction and Preliminaries

In [4], the notion of uniformly strongly prime ideals in $\Gamma$-semigroups is introduced. Chattopadhyay and Kar, [3], considering and investigating the properties of the collection $\mathcal{A}$ of all proper uniformly strongly prime ideals of a $\Gamma$-semigroup $S$, a topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$ was constructed by means of a closure operator defined in terms of intersection and inclusion relation among these ideals of $\Gamma$-semigroup $S$. The topological space $(\mathcal{A}, \tau_{\mathcal{A}})$ is often called the structure space of the $\Gamma$-semigroup $H$. Since $\Gamma$ semigroups are generalizations of semigroups, all the results obtained in, [3], hold for semigroups. This kind of topological space has been studied in different algebraic structures, [1], [2], [4], [5], [6], [7], [8], [9], [10], [11]. Several principal topological axioms and properties in this structure space, such as separation axioms, compactness, and connectedness, were studied.

In this paper, we study three other classes of ideals in semigroups called maximal ideal, prime ideal and strongly irreducible ideal, respectively. Properties of the collection $\mathcal{M}$, $\mathcal{B}$ and $\mathcal{S}$ of all proper maximal ideals, prime ideals and strongly irreducible ideals respectively of a semigroup $S$ are investigated. We construct the respective topologies on them using a closure operator defined in terms of the intersection and inclusion relation among these ideals of the semigroup $S$. Some principal topological axioms and properties in those structure spaces of semigroup are investigated.

Recall first the basic terms and definitions.

Let we consider the semigroup $(H,\cdot)$.

A nonempty subset $B$ of a semigroup $H$ is called a *sub-semigroup* of $H$ if $B \cdot B \subseteq B$.

A nonempty subset $I$ of a semigroup $H$ is called a *right* (*left*) *ideal* of $H$ if for all $x \in H$ and $r \in I$, $r \cdot x \in I (x \cdot r \in I)$.

A nonempty subset $I$ of $H$ is called an *ideal* (or *two-sided ideal*) if it is both a left ideal and a right ideal.

An element $e$ in a semigroup $H$ is called *identity* if $x \cdot e = e \cdot x = x, \forall x \in H$.

An element $0$ in a semigroup $H$ is called *zero element* if $x \cdot 0 = 0 \cdot x = 0, \forall x \in H$.

An element $a$ in a semigroup $H$ is called *idempotent element* if $a = a \cdot a$.

The set of all idempotents of the semigroup $S$ is denoted by $E(S)$.

A proper ideal $P$ of a semigroup $S$ is called a *prime ideal* of $S$ if $A \cdot B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for any two ideals $A, B$ of $H$.

An ideal $I$ of a semigroup $H$ is said to be *full* if $E(H) \subseteq I$.

An ideal $I$ of a semigroup $H$ is said to be a prime full ideal if it is both prime and full.

In, [3], the following theorem is proved.

**Theorem 1.1** [3, Theorem 2.8] *Let $H$ be a semigroup and $P$ be an ideal of $H$. Then the following statements are equivalent:*

1. *If $A$ and $B$ are ideals of $H$ such that $A \cdot B \subseteq P$, then either $A \subseteq P$ or $B \subseteq P$.*
2. *If $< a >$ and $< b >$ are principal ideals of $H$ such that $< a > \cdot < b > \subseteq P$, then either $a \in P$ or $b \in P$.*
3. *If $a \cdot H \cdot b \subseteq P$, then either $a \in P$ or $b \in P$ $(a, b \in H)$.*
4. *If $I_1$ and $I_2$ are two right ideals of $H$ such that $I_1 \cdot I_2 \subseteq P$, then either $I_1 \subseteq P$ or $I_2 \subseteq P$.*
5. *If $J_1$ and $J_2$ are two left ideals of $H$ such that $J_1 \cdot J_2 \subseteq P$, then either $J_1 \subseteq P$ or $J_2 \subseteq P$.*

**Definition 1.2** [3], [4] *An ideal $P$ of a semigroup $H$ is called a uniformly strongly prime ideal (usp ideal) if $H$ contains a finite subset $F$ such that for all $x, y \in H$, $x \cdot F \cdot y \subseteq P$ implies that $x \in P$ or $y \in P$.*

**Theorem 1.3** [3], [4] *Let $H$ be a semigroup. Then every uniformly strongly prime ideal is a prime ideal.*

Throughout this paper $H$ will always denote a semigroup with zero.

# 2 On Structure Space of Uniformly Strongly Prime Ideals of Semigroup

In this section, let us recall some of the results and definitions obtained in [3]. The philosophy of them and their proofs will be useful for the results obtained in this paper.

We denote by $\mathcal{A}$ the collection of all uniformly strongly prime ideals of a semigroup $H$. For any subset $A$ of $\mathcal{A}$ (that is, subcollection), we define

$$\overline{A} = \{I \in \mathcal{A}: \bigcap_{J \in A} J \subseteq I\}.$$

It can be easily seen that $\overline{\emptyset} = \emptyset$.

**Theorem 2.1** *Let $A, B$ be any two subsets of $\mathcal{A}$. Then*

1. $A \subseteq \overline{A}$.
2. $\overline{\overline{A}} = \overline{A}$.
3. $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$.
4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

**Definition 2.2** *The closure operator $A \to \overline{A}$ gives a topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$. This topology $\tau_{\mathcal{A}}$ is called the hull-kernel topology and the topological space $(\mathcal{A}, \tau_{\mathcal{A}})$ is called the structure space of the semigroup $H$.*

**Remark 2.3** *Let $\{I_\alpha\}$ be a collection of prime ideals of a semigroup $H$. Then $\bigcap I_\alpha$ is either empty or it is an ideal of $H$ but it need not be a prime ideal of $H$, in general. The following example shows it.*

**Example 2.4,** [13], *Let we consider the semigroup $(M, \cdot)$, where $M = \{m \in Z | m \geq 2\}$. The sets $I(p) = \{p, 2p, 3p, \ldots\}$ ($p$ is prime) are obviously prime ideals of $M$. It is clear that the set $\bigcap\{I(p) | p\ prime\} = \emptyset$.*

In [12] it is proved that the intersection of prime ideals of a semigroup $H$ if it is not empty, is a semiprime ideal of $H$.

We have the following proposition:

**Proposition 2.5** *Let $H$ be a semigroup and $\{I_\alpha\}$ be a collection of prime ideals of $H$ such that $\{I_\alpha\}$ forms a chain. Then $\bigcap I_\alpha$ is a prime ideal of $H$.*

**Definition 2.6** *Let $H$ be a semigroup. The structure space $(\mathcal{A}, \tau_{\mathcal{A}})$ of $H$ is called irreducible if for any decomposition $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are proper closed subsets of $A$ (whether disjoint or non-disjoint), we have either $\mathcal{A} = \mathcal{A}_1$ or $\mathcal{A} = \mathcal{A}_2$.*

**Theorem 2.7** *Let $H$ be a semigroup and $A$ be a closed subset of $\mathcal{A}$. Then $A$ is irreducible if and only if $\emptyset \neq \bigcap_{I_\alpha \in A} I_\alpha$ is a prime ideal of $H$.*

We denote by $\mathcal{C}$ the collection of all uniformly strongly prime full prime ideals of a semigroup $H$. We find that $\mathcal{C}$ is a subset of $\mathcal{A}$ and hence $(\mathcal{C}, \tau_{\mathcal{C}})$ is a structure space, where $\tau_{\mathcal{C}}$ is the subspace topology.

In general, $(\mathcal{A}, \tau_{\mathcal{A}})$ is not compact and connected. But in particular, for the structure space $(\mathcal{C}, \tau_{\mathcal{C}})$, we have the following theorem:

**Theorem 2.8** *Let $H$ be a semigroup. $(\mathcal{C}, \tau_{\mathcal{C}})$ is a compact space.*

**Theorem 2.9** *Let $H$ be a semigroup. $(\mathcal{C}, \tau_{\mathcal{C}})$ is a connected space.*

# 3 On Structure Space of Maximal Ideals of Semigroup

In this section, the structure space of all maximal ideals of a semigroup $H$ with identity 1 is considered and studied.

A proper ideal $I$ of $H$ is maximal in $H$ if for any ideal $J$ of $H$ with $I \subseteq J \subseteq H$, then $J = H$.

**Example 3.1** *Let $p$ be a prime number. Let $H = p\mathbb{Z}$. Then $(H,\cdot)$ is a semigroup. Let $M = p^2\mathbb{Z}$ is a maximal ideal of $H$.*

**Example 3.2** *The set $M = \{m \in \mathbb{N} | m \geq 2\}$ is a maximal ideal of the semigroup* $(\mathbb{N}, +)$.

**Example 3.3** [13], *Let $H = \{a_0, a_1, a_2, \ldots, a_5\}$ be the set of all residue classes mod 6. Then $(H,\cdot)$ is a commutative semigroup with identity, where $a_i \cdot a_k = a_l$ and $l = ik(mod\ 6)$. We consider $M = \{a_0, a_2, a_3, a_4\}$. Then $M$ is a unique maximal ideal of $H$.*

Let $\mathcal{M}$ be the set of all maximal ideals in a semigroup $H$. We define two topologies on $\mathcal{M}$. For every $x \in H$, we denote by $\Delta_x$ the set of all maximal ideals that contain $x$, by $\Gamma_x$ the set $\mathcal{M} - \Delta_x$, that is, the set of all maximal ideals not containing $x$. Let $I$ be an ideal of $H$, we denote by $\Delta_I$ the set of all maximal ideals containing $I$.

We choose the family $\{\Delta_x | x \in H\}$ as a subbase for open sets of $\mathcal{M}$. We shall refer to the resulting topology on $\mathcal{M}$ as $\Delta$-topology (in symbol, $\mathcal{M}_\Delta$). Similarly, we take the family $\{\Gamma_x | x \in H\}$ as a subbase for open sets of $\mathcal{M}$ (in symbol $\mathcal{M}_\Gamma$).

Let $I_1$ and $I_2$ be two ideals of the semigroup $H$. We denote by $I_1 \vee I_2$ the set of finite products of members of $I_1 \cup I_2$.

Let $M_1, M_2$ be two distinct elements of $\mathcal{M}_\Delta$. Then we have $M_1 \vee M_2 = H$. Therefore there are $a, b$ such that $1 = a \cdot b$ and $a \in M_1, b \in M_2$, so we have $\Delta_a \ni M_1, \Delta_b \ni M_2$ and $\Delta_a \cap \Delta_b = \emptyset$. Hence, we have

**Theorem 3.4** *The structure space $\mathcal{M}_\Delta$ is a $T_2$-space.*

Let now $M$ be an element of $\mathcal{M}_\Gamma$, and $M \neq M_1 \in \mathcal{M}_\Gamma$, then there is an element $a$ such that $a \in M_1$ and $a \notin M$. Therefore, $M_1 \notin \Gamma_a$ and $M_1 \notin \bigcap_{x \notin M} \Gamma_x$. This implies $M = \bigcap_{x \notin M} \Gamma_x$. Hence we obtain the following

**Theorem 3.5** *The structure space $\mathcal{M}_\Gamma$ is a $T_1$-space.*

Let $I$ be an ideal of $H$ and $\{a_\lambda\}$ a generator of $I$, then we have

$$\Delta_I = \bigcap_\lambda \Delta_{a_\lambda}.$$

Therefore, the closed sets for the structure space $\mathcal{M}_\Gamma$ have the form $\Delta_{I_1} \cup \Delta_{I_2} \cup \ldots \cup \Delta_{I_n}$, where $I_i$ are ideals of $H$.

Let $I = \bigcap_{i=1}^n I_i$, if $M \in \Delta_{I_i}$ for some $i$, then $M \supset I_i$ and $M \supset I$. This implies $\Delta_I \ni M$ and we have $\bigcup_{i=1}^n \Delta_{I_i} \subset \Delta_I$. Suppose that there is a maximal ideal $M$ such that $M \in \Delta_I \setminus \bigcup_{i=1}^n \Delta_{I_i}$, then $M \in \Delta_I$ and $M \notin \bigcup_{i=1}^n \Delta_{I_i}$. Therefore, $M \supset I$ and $M$ do not contain all $I_i (i = 1,2, \ldots, n)$. Therefore, since $M$ is a maximal ideal, there are elements $a_i \in I_i$ and $m_i \in M$ such that

$$1 = a_i \cdot m_i (i = 1,2, \ldots, n).$$

Thus, we have

$$1 = a_1 \cdot a_2 \cdot \ldots \cdot a_n \cdot m, m \in M$$

and $a_1 \cdot a_2 \cdot \ldots \cdot a_n \in I$. This implies $I \vee M = H$. Hence, by $I \subset M$, we have $M = H$, which is a contradiction. This shows the following relation:

$$\bigcup_{i=1}^n \Delta_{I_i} = \Delta_I$$

and we have the following

**Theorem 3.6** *The closed sets for $\mathcal{M}_\Gamma$ are expressed by sets $\Delta_I$, where $I$ is an ideal of $H$.*

By Theorem 3.5, we prove the following theorem.

**Theorem 3.7** *The space $\mathcal{M}_\Gamma$ is a compact $T_1$-space.*

*Proof.* Let $\{\Delta_{I_\lambda}\}$ be a family of closed sets in $\mathcal{M}_\Gamma$ with the finite intersection property, where $I_\lambda$ are ideals in $H$. Therefore, any finite family of $I_\lambda$ does not contain the semigroup $H$. Hence the ideal $I$ generated by $\{I_\lambda\}$ does not contain the identity 1 of $H$. This shows that $I$ is contained in a maximal ideal $M$. Hence $M \in \bigcap_\lambda \Delta_{I_\lambda}$. Therefore, since $\bigcap_\lambda \Delta_{I_\lambda}$ is non-empty, $\mathcal{M}_\Gamma$ is a compact space. □

# 4 On Structure Space of Prime Ideals of Semigroup

In this section, the structure space $\mathcal{B}$ of all prime ideals of a semigroup $H$ with identity 1 is considered and the relation of $\mathcal{B}$ and the structure space $\mathcal{M}$ of all maximal ideals of $H$ is investigated. Throughout the section, we shall treat a commutative semigroup $H$ with identity 1. An ideal $P$ of $H$ is prime if and only if $a \cdot b \subseteq P$ implies $a \in P$ or $b \in P$. Since $H$ has an identity 1, then any maximal ideal is prime, therefore $\mathcal{B} \supseteq \mathcal{M}$. We notice here that a maximal ideal in a commutative semigroup without identity may not be prime.

**Example 4.1** *The ideal $M$ of Example 3.3 is a maximal ideal of $H$ and it is a prime ideal of $H$.*

The ideals $M$ of Examples 3.1 and 3.2 respectively, are maximal ideals but not a prime ideal of $H$.

To introduce a topology $\tau$ in $\mathcal{B}$, we take $\tau_x = \{P | x \notin P, P \in \mathcal{B}\}$ for every $x \in H$ as an open base of $\mathcal{B}$. We have the following.

**Theorem 4.2** *Let $\mathcal{U}$ be a subset of $\mathcal{B}$, then*

$$\overline{\mathcal{U}} = \{P' \in \mathcal{B} | \bigcap_{P\in\mathcal{U}} P \subset P'\},$$

where $\overline{\mathcal{U}}$ is the closure of $\mathcal{U}$ by the topology $\tau$.

*Proof.* Let $P' \in \{P' \in \mathcal{B} | \bigcap_{P\in\mathcal{U}} P \subset P'\}$ and let $\tau_x$ be a neighbourhood of $P'$, then $x \notin P'$, and we have $x \notin \bigcap_{P\in\mathcal{U}} P$. Therefore, there is a prime ideal $P \in \mathcal{U}$ such that $P$ does not contain $x$ and $\tau_x \ni P$. This shows that $P \in \overline{\mathcal{U}}$. Thus we have proved that the $\overline{\mathcal{U}}$ contains $\{P' \in \mathcal{B} | \bigcap_{P\in\mathcal{U}} P \subset P'\}$.

If a prime ideal $P'$ is not in $\{P' \in \mathcal{B} | \bigcap_{P\in\mathcal{U}} P \subset P'\}$, then $\bigcap_{P\in\mathcal{U}} P - P' \neq \emptyset$. Hence, for $x \in \bigcap_{P\in\mathcal{U}} P - P'$, we have $x \in P, P \in \mathcal{U}$ and $x \notin P'$. This shows $P \notin \tau_x$, $P \in \mathcal{U}$ and $P' \notin \tau_x$. Therefore $\tau_x \cap \mathcal{U} = \emptyset$ and hence $P' \notin \overline{\mathcal{U}}$. The proof is complete. □

A similar argument for $\mathcal{M}$ relative to the Γ-topology implies the following.

**Proposition 4.3** *Let $\mathcal{U}$ be a subset of $\mathcal{M}$, then*

$$\overline{\mathcal{U}} = \{M' \in \mathcal{M} | \bigcap_{M\in\mathcal{U}} M \subset M'\},$$

where $\overline{\mathcal{U}}$ is the closure of $\mathcal{U}$ by the topology Γ.

In a similar way to the proof of the Theorem 2.1, we can prove the following

**Theorem 4.4** *The closure operation $\mathcal{U} \to \overline{\mathcal{U}}$ of $\mathcal{B}$ satisfies the following relations:*

1. $\mathcal{U} \subseteq \overline{\mathcal{U}}$.
2. $\overline{\overline{\mathcal{U}}} = \overline{\mathcal{U}}$.
3. $\overline{\mathcal{U} \cup \mathcal{B}} = \overline{\mathcal{U}} \cup \overline{\mathcal{B}}$.

*Proof.* We shall prove only the last relation (3). By Theorem 4.2, $\mathcal{U} \subset \mathcal{B}$ implies $\overline{\mathcal{U}} \subset \overline{\mathcal{B}}$ and hence $\overline{\mathcal{U}} \cup \overline{\mathcal{B}} \subset \overline{\mathcal{U} \cup \mathcal{B}}$. Let $P \notin \overline{\mathcal{U}} \cup \overline{\mathcal{B}}$, then $P \notin \overline{\mathcal{U}}$ and $P \notin \overline{\mathcal{B}}$. Hence $P \not\supset \bigcap_{P'\in\mathcal{U}} P' = P_{\mathcal{U}}$ and $P \not\supset \bigcap_{P'\in\mathcal{B}} P' = P_{\mathcal{B}}$. The sets $\mathcal{B}_{\mathcal{U}}$ and $\mathcal{B}_{\mathcal{B}}$ are ideals. If $P_{\mathcal{U}} \cdot P_{\mathcal{B}} \subset P$, for any elements $a, b$ such that $a \in P_{\mathcal{U}} - P, b \in P_{\mathcal{B}} - P$, we have $a \cdot b \in P$ and since $P$ is a prime ideal, $a \in P$ or $b \in P$, which is a contradiction. Therefore, $P \not\supset P_{\mathcal{U}} \cdot P_{\mathcal{B}} \supseteq P_{\mathcal{U}} \cap P_{\mathcal{B}} = P_{\mathcal{U}\cup\mathcal{B}}$. Hence $P \notin \overline{\mathcal{U} \cup \mathcal{B}}$. □

**Theorem 4.5** *The structure space $\mathcal{B}$ is a $T_0$-space.*

*Proof.* To prove that the structure space $\mathcal{B}$ is a $T_0$-space, it is sufficient to verify the following conditions:

1. $\mathcal{U} \subseteq \overline{\mathcal{U}}$.
2. $\overline{\overline{\mathcal{U}}} = \overline{\mathcal{U}}$.
3. $\overline{\mathcal{U} \cup \mathcal{B}} = \overline{\mathcal{U}} \cup \overline{\mathcal{B}}$
4. $\overline{P_1} = \overline{P_2}$ implies $P_1 = P_2$.

By the above theorem, it is sufficient to prove that $(\overline{P_1}) = (\overline{P_2})$ implies $P_1 = P_2$. By $P_2 \in (\overline{P_1})$, then $P_2 \supset P_1$. Similarly $P_1 \supset P_2$ and we have $P_1 = P_2$. □

**Theorem 4.6** *The structure space $\mathcal{B}$ is a compact $T_1$-space.*

*Proof.* Let $\mathcal{U}_\lambda$ be a family of closed sets such that $\bigcap_\lambda \mathcal{U}_\lambda = \emptyset$, then we have $\bigvee P_{\mathcal{U}_\lambda} = H$, where $P_{\mathcal{U}_\lambda} = \bigcap_{P\in\mathcal{U}_\lambda} P$. Indeed: Let us suppose that $\bigvee P_{\mathcal{U}_\lambda} \neq H$. Then there is a maximal ideal $M$ containing $\bigvee P_{\mathcal{U}_\lambda}$. Therefore $P_{\mathcal{U}_\lambda} \subset M$ for every $\lambda$. Hence $\mathcal{U}_\lambda \ni M$ for every $\lambda$, and we have $\bigcap_\lambda \mathcal{U}_\lambda \ni M$, which is a contradiction. By $\bigvee P_{\mathcal{U}_\lambda} = H$, we have $1 = a_1 \cdot a_2 \cdot \ldots \cdot a_n, a_i \in P_{\mathcal{U}_{\lambda_i}} (i = 1,2,\ldots,n)$. Hence $\bigvee_{i=1}^{n} P_{\mathcal{U}_{\lambda_i}} = H$. If $\bigcap_{i=1}^{n} \mathcal{U}_{\lambda_i} \neq \emptyset$, then for a prime ideal $P$ of $\bigcap_{i=1}^{n} \mathcal{U}_{\lambda_i}$, we have $P \supset P_{\mathcal{U}_{\lambda_i}} (i = 1,2,\ldots,n)$ and hence $P \supset \bigvee_{i=1}^{n} P_{\mathcal{U}_{\lambda_i}}$. Therefore we have $\bigcap_{i=1}^{n} \mathcal{U}_{\lambda_i} = \emptyset$. □

By the $\mathcal{B}$-*radical* $r(\mathcal{B})$ of the semigroup $H$, we mean the intersection of all prime ideals of $H$, that is, $\bigcap_{P\in\mathcal{B}} P$. By the $\mathcal{M}$-*radical* $r(\mathcal{M})$ of $H$, we mean the intersection of all maximal ideals of $H$, that is, $\bigcap_{M\in\mathcal{M}} M$.

From $\mathcal{M} \subseteq \mathcal{B}$, we have $r(\mathcal{B}) \subseteq r(\mathcal{M})$. In the following proposition we give a condition to be $r(\mathcal{B}) = r(\mathcal{M})$.

**Theorem 4.7** *The subset $\mathcal{M}$ of $\mathcal{B}$ is dense in $\mathcal{B}$, if and only if, $r(\mathcal{B}) = r(\mathcal{M})$.*

*Proof.* Let $\overline{\mathcal{M}} = \mathcal{B}$ for the topology $\tau$. Then we have

$$\{P | \bigcap_{M\in\mathcal{M}} M \subset P\} = \mathcal{B}.$$

Hence

$$r(\mathcal{M}) = \bigcap_{M\in\mathcal{M}} M \subseteq \bigcap_{P\in\mathcal{B}} P = r(\mathcal{B}).$$

Since $r(\mathcal{B}) \subseteq r(\mathcal{M})$, therefore we have $r(\mathcal{B}) = r(\mathcal{M})$.

On the contrary, if $P \in \mathcal{B} - \overline{\mathcal{M}}$, then $P \in \mathcal{B}$ and $P \in \overline{\mathcal{M}}$. Therefore, there is a neighborhood $\tau_x$ of $P$ such that $\tau_x \cap \mathcal{M} = \emptyset$. Hence $r(\mathcal{B}) = \bigcap_{P\in\mathcal{B}} P$ is a proper subset of $\bigcap_{M\in\mathcal{M}} M$. Therefore, $r(\mathcal{B}) \neq r(\mathcal{M})$, which completes the proof. □

**Definition 4.8** *If $r(\mathcal{M})$ is the zero ideal (0), then $A$ is said to be $\mathcal{M} - semisimple$.*

From the Theorem 4.7, we have the following

**Theorem 4.9** *If $H$ is $\mathcal{M}$-semisimple, $\mathcal{M}$ is dense in $\mathcal{B}$.*

## 5 On Structure Space of Strongly Irreducible Ideals of Semigroup

In this section, the structure space $\mathcal{S}$ of all strongly irreducible ideals of a commutative semihypergoup $H$ with identity 1 is investigated.

An ideal $I$ of a semigroup $H$ is called *irreducible*, if and only if $A \cap B = I$ for two ideals $A, B$ implies $A = I$ or $B = I$. An ideal $I$ of a semigroup $H$ is called *strongly irreducible*, if and only if $A \cap B \subset I$ for every two ideals $A, B$ implies $A \subset I$ or $B \subset I$. From $A \cdot B \subset A \cap B$ for any two ideals $A, B$, it follows that any prime ideals are strongly irreducible and any strongly irreducible ideals are irreducible.

Let $\mathcal{S}$ be the set of all strongly irreducible ideals of $H$. From the above, it is clear that $\mathcal{M} \subset \mathcal{B} \subset \mathcal{S}$. To give a topology $\sigma$ on $\mathcal{S}$, we shall take $\sigma_x = \{S \in \mathcal{S} | x \notin S\}$ for every $x \in H$ as an open base of $\mathcal{S}$.

**Theorem 5.1** *Let $\mathcal{U}$ be a subset of $\mathcal{S}$, then we have*

$$\overline{\mathcal{U}} = \{S' \in \mathcal{S} | \cap_{S \in \mathcal{U}} S \subset S'\}$$

where $\overline{\mathcal{U}}$ is the closure of $\mathcal{U}$ by $\sigma$.

*Proof.* Let $\mathcal{F} = \{S' \in \mathcal{S} | \cap_{S \in \mathcal{U}} S \subset S'\}$ and let $S' \in \mathcal{F}$. Let $\sigma_x$ be an open base of $S'$, then, by the definition of the topology $\sigma$, $x \notin S'$. Hence, we have $x \notin \cap_{S \in \mathcal{U}} S$. It follows from this that there is a strongly irreducible ideal $S$ of $\mathcal{U}$ such that $x$ is not contained in $S$. Hence $\sigma_x \ni S$. Therefore $S' \in \overline{\mathcal{U}}$ and $\mathcal{F} \subset \overline{\mathcal{U}}$.

To prove that $\mathcal{F} \supset \overline{\mathcal{U}}$, take a strongly irreducible ideal $S'$ such that $S' \notin \mathcal{F}$. Then $\cap_{S \in \mathcal{U}} S - S' \neq \emptyset$. For an element $x \in \cap_{S \in \mathcal{U}} S - S'$, we have $x \in S (S \in \mathcal{U})$ and $x \in S'$. Hence $S' \in \sigma_x$ and $S \notin \sigma_x$ for all $S$ of $\mathcal{U}$. Therefore, $\mathcal{U} \cap \sigma_x = \emptyset$ and then we have $S' \notin \overline{\mathcal{U}}$. Hence $\mathcal{F} \supset \overline{\mathcal{U}}$. The proof of the theorem is complete. □

We shall prove that the structure space $\mathcal{S}$ for the topology $\sigma$ is a compact $T_0$-space. To prove that $\mathcal{S}$ is a $T_0$-space, it is sufficient to verify the following conditions:

1. $\mathcal{U} \subseteq \overline{\mathcal{U}}$.
2. $\overline{\overline{\mathcal{U}}} = \overline{\mathcal{U}}$.
3. $\overline{\mathcal{U} \cup \mathcal{B}} = \overline{\mathcal{U}} \cup \overline{\mathcal{B}}$
4. $\overline{S_1} = \overline{S_2}$ implies $S_1 = S_2$.

Conditions (1) and (2) are clear and $\mathcal{U} \cup \mathcal{B}$ implies $\overline{\mathcal{U}} \subset \overline{\mathcal{B}}$. From this relation we have $\overline{\mathcal{U}} \cup \overline{\mathcal{B}} \subset \overline{\mathcal{U} \cup \mathcal{B}}$. For some element of $S$ of $\overline{\mathcal{U} \cup \mathcal{B}}$, suppose that $S \notin \overline{\mathcal{U}}$ and $\overline{S} \notin \mathcal{B}$. From Theorem 5.1, we have

$$S \not\supset \cap_{S' \in \mathcal{U}} S' = S_{\mathcal{U}} \text{ and } S \not\supset \cap_{S' \in \mathcal{B}} S' = S_{\mathcal{B}}.$$

$S_{\mathcal{U}}$ and $S_{\mathcal{B}}$ are ideals. If $S_{\mathcal{U}} \cap S_{\mathcal{B}} \subset S$, by the definition of $S$, $S_{\mathcal{U}} \subset S$ or $S_{\mathcal{B}} \subset S$. Hence $S \not\supset S_{\mathcal{U}} \cap S_{\mathcal{B}} = S_{\mathcal{U} \cup \mathcal{B}}$. This shows $S \notin \overline{\mathcal{U} \cup \mathcal{B}}$.

To prove that $\overline{S_1} = \overline{S_2}$ implies $S_1 = S_2$, we shall use condition (1). Then $\overline{S_1} \ni S_2$ and by the definition of closure operation, we have $S_1 \subset S_2$. Similarly, we have $S_1 \supset S_2$ and $S_1 = S_2$. Therefore, we complete the proof that $\mathcal{S}$ is a $T_0$-space.

We shall prove that $\mathcal{S}$ is a compact space. Let $\mathcal{U}_\lambda$ be a family of closed sets with empty intersection. Let $S_{\mathcal{U}_\lambda} = \cap_{S \in \mathcal{U}_\lambda} S$, suppose that $\vee_\lambda S_{\mathcal{U}_\lambda} \neq S$, then there is a maximal ideal $M$ containing the ideal $\vee_\lambda S_{\mathcal{U}_\lambda}$. Therefore, we have $S_{\mathcal{U}_\lambda} \subset M$ for every $\lambda$. By the definition of $S_{\mathcal{U}_\lambda}, \mathcal{U}_\lambda \ni M$ for every $\lambda$. Hence $\cap_\lambda \mathcal{U}_\lambda \ni M$, which contradicts our hypothesis of $\mathcal{U}_\lambda$. Therefore, $\vee_\lambda S_{\mathcal{U}_\lambda} = H$. Therefore, we have $1 = a_1 \cdot a_2 \cdot \ldots \cdot a_n (a_i \in S_{\mathcal{U}_{\lambda_i}} (i = 1,2, \ldots, n))$. Therefore, we have $H = S_{\mathcal{U}_{\lambda_1}} \vee S_{\mathcal{U}_{\lambda_2}} \vee \ldots \vee S_{\mathcal{U}_{\lambda_n}}$. If $\cap_{i=1}^n \mathcal{U}_{\lambda_i} \neq \emptyset$, for all strongly irreducible ideals $S$ of $\cap_{i=1}^n \mathcal{U}_{\lambda_i}$, $S \supset S_{\mathcal{U}_{\lambda_i}} (i = 1,2, \ldots, n)$ and $S \supset \vee_{i=1}^n S_{\mathcal{U}_{\lambda_i}}$. If $\cap_{i=1}^n \mathcal{U}_{\lambda_i} = H$, we can easily prove that $\mathcal{S}$ is a compact space. If $\cap_{i=1}^n \mathcal{U}_{\lambda_i}$ contains a proper strongly irreducible ideal $S$, we have $S \supset \vee_{i=1}^n S_{\mathcal{U}_{\lambda_i}}$, which is a contradiction to $H = \vee_{i=1}^n S_{\mathcal{U}_{\lambda_i}}$. Therefore $\cap_{i=1}^n \mathcal{U}_{\mathcal{U}_{\lambda_i}} = \emptyset$. Hence $\mathcal{S}$ is a compact space. Thus, we have proved the following.

**Theorem 5.2** *The structure space $(\mathcal{S}, \sigma)$ is compact $T_0$-space.*

By $\mathcal{S} - radical\ r(\mathcal{S})$ of a semigroup, we mean the intersection of all strongly irreducible ideals of it, that is, $\cap_{S \in \mathcal{S}} S$. From $\mathcal{M} \subset \mathcal{B} \subset \mathcal{S}$, we have $r(\mathcal{M}) \supset r(\mathcal{B}) \supset r(\mathcal{S})$.

**Theorem 5.3** *The subset $\mathcal{B}$ of $\mathcal{S}$ is dense in $\mathcal{S}$, if and only if $r(\mathcal{B}) = r(\mathcal{S})$.*

*Proof.* Let $\overline{\mathcal{B}} = \mathcal{S}$ for the topology $\sigma$, then we have

$$\{S | \cap_{P \in \mathcal{B}} P \subset S\} = \mathcal{S}.$$

Hence, we have

$$r(\mathcal{B}) = \cap_{P \in \mathcal{B}} P \subset \cap_{S \in \mathcal{S}} S = r(\mathcal{S}).$$

On the other hand, $r(\mathcal{B}) \supset r(\mathcal{S})$. This shows $r(\mathcal{S}) = r(\mathcal{B})$.

Conversely, suppose that $\mathcal{S} - \overline{\mathcal{B}} \neq$, then there is a strongly irreducible ideal $S$ such that $S \notin \overline{\mathcal{B}}$ and $S \in \mathcal{S}$. Therefore, there is a neighborhood $\sigma_x$ of $S$

that does not meet $\mathcal{B}$. Therefore, $r(\mathcal{S}) = \bigcap_{S\in\mathcal{S}} S$ is a proper subset of $\bigcap_{P\in\mathcal{B}} P$, and we have $r(\mathcal{S}) \neq r(\mathcal{B})$. □

**Corollary 5.4** *The subset $\mathcal{M}$ of $\mathcal{S}$ is dense in $\mathcal{S}$, if and only if $r(\mathcal{M}) = r(\mathcal{S})$.*

**Corollary 5.5** *Let H be a semigroup with 0. If H is $\mathcal{M}$-semisimple, then $\mathcal{M}$ and $\mathcal{B}$ are dense in $\mathcal{S}$.*

## 6 Conclusions

In this paper, we investigated three other classes of ideals in semigroups called maximal ideals, prime ideals and strongly irreducible ideals, respectively. Properties of the collection $\mathcal{M}$, $\mathcal{B}$ and $\mathcal{S}$ of all proper maximal ideals, prime ideals and strongly irreducible ideals respectively of a semigroup $S$ were investigated. We constructed the respective topologies on them using a closure operator defined in terms of intersection and inclusion relation among these ideals of the semigroup $S$. Some principal topological axioms and properties in those structure spaces of semigroup were investigated.

In future work, one can develop and extend the study of these structure spaces in Γ-semigroups or further in semihypergroups, Γ-semihypergroups and other kinds of hyperstructures.

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