The Stationary Problem of Two-Phase Filtration by the Monte-Carlo Method: Solutions and Applications

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Abstract: - This article is devoted to solving the problems of applying Monte-Carlo algorithms to filtration problems. The "random walk by spheres" and "random walk by boundary" algorithms of Monte-Carlo methods are used to solve the stationary problem of filtration of two immiscible inhomogeneous incompressible liquids in a porous medium. Estimates constructed using the "random walk by spheres" and "random walk by boundary" algorithms of Monte-Carlo methods will bemostly ε −biased. Unbiased estimates are in most cases unrealizable on a computer, since with a probability of1 they do not go to the boundary of the region, and therefore are of little use. In practice, they are usually limited to only the first two points of evaluation.

Key-Words: Monte-Carlo method, continuity equation, Dirichlet problem, Markov chains, algorithms ofrandom walk by spheres, random walk by boundary, estimation of the solution and derivatives of the solution.

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1 Introduction

Numerical implementation of mathematical models challenges the construction of effective algorithms for the solution of engineering problems related to filtration of liquids. To date, such challenges are approached by state-of-the-art computational methods based on the Monte Carlo algorithm. The algorithms of the Monte-Carlo methods well implement, firstly, multidimensional problems, and secondly, with the help of the algorithms of the Monte-Carlo methods, it is possible to find a solution at a single point in a complex area, which is a very relevant problem in underground hydraulic mechanics (for example, determining the point of greatest pressure in the area). In this paper, solutions and derivatives of solutions to the above two-phase filtration problems are evaluated by Monte- Carlo methods. Estimates constructed using the "random walk by spheres" and "random walk by boundary" algorithms of Monte-Carlo methods will be mostly ε −biased. Unbiased estimates in most cases are unrealizable on a computer, since with a probability of 1 they do not go to the boundary of the region, and therefore are of little use. In practice, they are usually limited toonly the first two points of evaluation [1-5].

We consider the mathematical model of filtration process of two immiscible liquids (water and oil) through the porous medium.

2 Formulation of the Problem in Saturations and Pressure (s, p)

From the equation of continuity follows that

$$div\vec{V} = 0,\ \vec{V} = \vec{V}_1 + \vec{V}_2 \qquad (1)$$

Here $\vec{V}-$ velocity vector of mixture filtration and $|\vec{V}|$- total flow intensity of mixture.Let introduce the new function of “reduced” pressure

$$p = p_1 - \int \frac{\partial p_c}{\partial s}\frac{k_{02}}{k} d\xi + \rho_1 \cdot g \cdot h, \qquad (2)$$

where $k = k_{01} + k_{02}$, $g\nabla h = \vec{g}$, we present vector $\vec{V}$ through ∇p and S, and it is independent of s:

$$\vec{V} = -(K\nabla p + \vec{f}) \equiv \vec{V}(s, p),\quad K = k \cdot K_0, \qquad (3)$$

where

$$\vec{f} = K\int_s^1 \nabla_x \frac{\partial p_c}{\partial s}\frac{k_{02}}{k} d\xi + K_2\nabla p_c + K_2(\rho_2 - \rho_1)\cdot\vec{g}.$$

Here $K_i = K_0 \cdot k_{0i}(s)-$ symmetric tensor of phase permeability, $K_0(x)-$ filtration tensor for homogeneous liquid. In the same way, using (2), we have

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$$-\vec{V}_1 = K_1(\nabla p_1 + p_1 \cdot \vec{g}) = K(\nabla p - \frac{\partial p_c}{\partial s}\frac{k_{02}}{k}\cdot\nabla s + \int_s^1 \nabla_x \frac{\partial p_c}{\partial s}\frac{k_{02}}{k}d\xi).$$

By $a = -\frac{\partial p_c}{\partial s}\cdot\frac{k_{01}k_{02}}{k_{01}+k_{02}}$ and $\vec{f}_0 = K\int_s^1 \nabla_x \frac{\partial p_c}{\partial s}\cdot\frac{k_{02}}{k}d\xi$ we receive

$$-\vec{V}_1 = K_0 a\nabla s + K_1\nabla p + \vec{f}_0 \equiv -\vec{V}_1(s,p). \quad (4)$$

Using (3) we find $K_1\nabla p = -K_1K^{-1}(\vec{V}+\vec{f})$ and considering $K_1 = k_{01}K_0$ and $K = k\cdot K_0$ we receive $K_1K^{-1} = k_{01}K^{-1} \equiv b(s)$. Then (4) will be as follows

$$-\vec{V}_1 = K_0 a\nabla s - b\vec{V} + \vec{F}, \vec{F} = \vec{f}_0 - b\vec{f}. \quad (5)$$

In the equation of continuity (1), substituting for the first phase of expression (4), we come to the system relative to *{s, p}:*

$$\begin{cases} m\frac{\partial s}{\partial t} = div(K_0 a\nabla s + K_1\nabla p + \vec{f}_0) \equiv -div\vec{V}_1(s,p), & (6) \\ div(K\nabla p + \vec{f}) \equiv -div\vec{V}(s,p) = 0. & (7) \end{cases}$$

For the initial boundary value problem (6)-(7), we consider the filtration flow in the fixed finite area Ω with piecewise-smooth border $\partial\Omega$ [6-7]. Let $Q = \Omega\times[0,T]$, $S_i = \partial\Omega_i\times[0,T]$, $\vec{n}$ – external normal to $\partial\Omega$. We determine boundary data in relation to functions *s, p*. The impermeability conditions on $\partial\Omega_0$ for both phases are as follows

$$\vec{V}\cdot\vec{n} = \vec{V}_1\cdot\vec{n} = 0, \ (x,t)\in S_0 = \partial\Omega_0\times[0,T]. \quad (8)$$

The boundary conditions are respectively rearranged to the form

$$p = p_0(x,t), s = s_0(x,t), \ (x,t)\in S_2 = \partial\Omega_2\times[0,T], \quad (9)$$

$$-(K_0\nabla p + \vec{f})\cdot\vec{n} \equiv \vec{V}\cdot\vec{n} = R(x,t), \ (x,t)\in S_1 = \partial\Omega_1\times[0,T], \quad (10)$$

$$-(K_0 a\nabla s + K_1\nabla p + \vec{f}_0)\cdot\vec{n} \equiv \vec{V}_1\cdot\vec{n} = b\cdot R(x,t), \ (x,t)\in S_1 \quad (11)$$

At *R(x,t)=0* equality (10) and (11) are equivalent (8), and $\partial\Omega_0$ add in $\partial\Omega_1$ and assume that $\partial\Omega_1$ consists of several components, on parts as *R=0*.

Therefore, $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$. The initial condition is set only for saturation *s(x,t)*:

$$s(x,0) = \tilde{s}^0(x), x\in\Omega, \quad (12)$$

Components $\partial\Omega_1$or $\partial\Omega_2$ on $\partial\Omega_0$ can absent, i.e. $\partial\Omega \equiv \partial\Omega_1$ or $\partial\Omega \equiv \partial\Omega_2$. In case of $\partial\Omega \equiv \partial\Omega_1$ the law of conservation of mass on the area Ω need for following necessary condition:

$$\int_\Omega p(x,t)dx = \int_{\partial\Omega} R(x,t)dx = 0, \ t\in[0,T]. \quad (13)$$

Coefficients of the equation and boundary conditions (8) – (12) have the form:

$$a = a(x,s) = -\frac{\partial p_c(x,s)}{\partial s}\cdot\frac{k_{01}(s)\cdot k_{02}(s)}{k(s)},$$

$$k(s) = k_{01}(s) + k_{02}(s), b = b(s) = -\frac{k_{01}(s)}{k(s)} \equiv K_1K^{-1},$$

$$K_i = K_0(x)\cdot k_{0i}(s),$$

$$K = kK_0 = K_0(x)(k_{01}(s) + k_{02}(s)),$$

$$\vec{f} = \vec{f}(x,s) = K(x,s)\cdot K_1^{-1}(x,s)\cdot\vec{f}_0(x,s) +$$

$$+ K_2(x,s)\nabla_x p_c(x,s) + K_2(x,s)(\rho_2 - \rho_1)\cdot\vec{g},$$

$$K = K(x,s) = K_1(x,s) + K_2(x,s),$$

$$\vec{F} = \vec{F}(x,s) = \vec{f}_0(x,s) - b(s)\cdot\vec{f}(x,s) = -k_{01}k_{02}k^{-1},$$

$$K_0[\nabla_x p_c + (\rho_2 - \rho_1)\cdot\vec{g}],$$

$$K_0(x,s) = k_{0i}(s)\cdot K_0(x), \qquad i = 1,2,$$

$$\vec{f}_0 = \vec{f}_0(s) = K_0\int_s^1 \nabla_s \frac{\partial p(x,s)}{\partial s}\Big|_{s=\xi}\cdot\frac{k_{02}(\xi)}{k(\xi)}d\xi. \quad (14)$$

Further assume that total velocity of filtration $\vec{V} = \vec{V}_1 + \vec{V}_2$ or $\vec{V} = -(K\nabla p + \vec{f}) \equiv \vec{V}(s,p)$ is independent of s, in other words. if coefficients $K = K_0(s)k(s)$ and $\vec{f}(x,s)$ are independent of s, and system of equation (6), (7) is separated and can be consistently determined $\vec{V}$ and $s_i(x,t)$. From

$$\vec{f}(x,s) = k \cdot K_0 \int_s^1 \nabla_x \frac{\partial p_c(x,s)}{\partial s}\bigg|_{s=\xi} \cdot \frac{k_{02}(\xi)}{k(\xi)} d\xi + K_2 \cdot \nabla p_c + K_2(\rho_2 - \rho_1) \cdot \vec{g}.$$

follows that this assumption is right:

1) as $k=const$ for miscible liquid. For immiscible liquids the significant deviation k from a constant is observed only near limit values $s=0,1$ given saturation;

2) as $p_c = p_c(s),$ in other words

$$\frac{\partial p_c}{\partial x_i} = 0 \text{ or } \nabla_x p_c = 0 \left(\nabla_x \frac{\partial p_c}{\partial s} = 0 \right);$$

3) if gravity is not considered, i.e. $g=0$ or liquids have identical density $\rho_1 = \rho_2$. The assumptions 2) and 3) provide the condition fulfillment $\frac{\partial \vec{f}}{\partial s} = 0.$

3 Formulation of the Stationary Problem

Stationary boundary value problem for $(s(x),p(x))$ will be as follows:

$$\begin{cases} div(K_0 a \nabla s - bV + F) \equiv -div\vec{V}_1 = 0 & \text{in } \Omega, \ (15) \\ s = s_0(x) & \text{on } \partial\Omega_2, \ (16) \\ \vec{V}_1 \cdot \vec{n} = b \cdot R & \text{on } \partial\Omega_1, \ (17) \\ div(K \cdot \nabla p + \vec{f}) \equiv -div\vec{V} = 0 & \text{in } \Omega, (18) \\ p = p_0 & \text{on } \partial\Omega_2, (19) \\ \vec{V} \cdot \vec{n} = R & \text{on } \partial\Omega_1, (20) \end{cases}$$

If coefficients $K = K_0(x)k(s)$ and $\vec{f}(x,s)$ are independent of s, it follows from (18) that

$$div(K \cdot \nabla p + \vec{f}) = div(C_1 K_0(x) \cdot \nabla p(x) + \vec{f}(x)) = \\ = C_1 div(K_0(x) \cdot \nabla p(x)) + h(x) = 0, \quad (21)$$

where $C_1 = k(s) = const, \quad h(x) = div\vec{f}(x).$

Further we suppose that we have homogeneous and isotropic earth. Then from (21) we receive $K_0(x) = C_2 = const$ and $C_3 \Delta_3(x) = -h(x)$ *or*

$\Delta p(x) = -h_1(x),$ where $C_3 = C_1 C_2$ and $h_1(x) = \frac{h(x)}{C_3}.$

$$V = C_1 K_0(x) \nabla p(x) + \vec{f}(x) = C_3 \nabla p(x) + \vec{f}(x) \quad (22)$$

if the medium is homogeneous and isotropic. Therefore, we obtain the problem:

$$\begin{cases} \Delta p(x) = -h_1(x), & \text{in } \Omega, & (23) \\ p(x) = p_0, & \text{on } \Omega_2, & (24) \\ \vec{V} \cdot \vec{n} = R & \text{on } \partial\Omega_1, & (25) \end{cases}$$

(23) - (25) is Dirichlet problem for Poisson equation [8-11]. Further we consider the case, when $\partial\Omega \equiv \partial\Omega_1$. Then from the conditions (13) we receive $\int_\Omega p(x)dx = \int_{\partial\Omega} R(x)dx = 0.$

As $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ then $\partial\Omega = \partial\Omega_2,$ we further suppose that $\partial\Omega_2 \equiv \partial\Omega.$ Then from (23) – (25) we obtain Dirichlet problem for Poisson equation:

$$\begin{cases} \Delta p(x) = -h_1(x), & \text{in } \Omega, & (26) \\ p(x) = p_0(0), & \text{on } \Omega. & (27) \end{cases}$$

Solution, the first and second derivatives from solutions of this problem, in other words $p(x), \nabla p(x)$ and $\Delta p(x)$ are estimated by the Monte-Carlo method.

4 Evaluation of the Solution and Derivatives of the Solution by Methods Monte Carlo

We formulate algorithms of Monte-Carlo methods for evaluating the solution and derivatives of the solution of the problem (26) – (27).

Using a special Fredholm integral equation of the 2nd kind with a degenerate kernel, the form of which is determined by the "ball" Green function of problem (26), (27) we construct an algorithm for solving the Dirichlet problem for the Poisson equation. It is known that this Green function will take the form

$$G(r,d) = \frac{1}{4\pi}\left(\frac{1}{|r-r'|} - \frac{1}{d}\right), |r-r'| \le d(r) \equiv d.$$

Then, for a ball of radius d_0 centered at a point x_0, the solution of the problem (26), (27) at a point $x_0$can be represented as

$$p(x_0)=\int\limits_{S(x_0)}-\frac{\partial G(r,x_0)}{\partial\vec{n}}\Bigg|_{|r|=d_0}p(y)dy+\int\limits_{|r-x_0|\le d_0}G(r,x_0)\cdot h_1(r)dr, \quad (28)$$

where $d_0=d(x_0)$. The first integral in (28) is the integral over the surface of the sphere. $S(x_0)$, the second one is all over the sphere $|r-x_0|\le d_0$.

Relation (28) can be considered as a conjugate (according to the terminology accepted in the theory of Monte-Carlo methods) Fredholm integral equation of the 2nd kind with a generalized kernel representing a uniform probability distribution on the sphere $S(x_0)$; after the introduction of this kernel, the first integral in (28) becomes three-dimensional. Let's return to equation (28). It is not difficult to understand that the standard Monte-Carlo algorithms apply to such integral equations if the kernel features are included in the transition density of the modeled Markov chain. In this case, from this point x_0 we should move to the surface of the sphere. $S(x_0)$; we call such a chain "random walk by spheres". The relation (28) must be supplemented with the following equality:

$$p(x_0)=p_0(x);\quad x\in\partial\Omega, \quad (29)$$

which means that the kernel of the integral equation vanishes if the first argument $x\in\partial\Omega$. Thus, after reaching the border, the chain should be cut off, adding the value to the estimate $p_0(x)$ with the appropriate weight.

These considerations lead to an unbiased probabilistic evaluation of the solution at the point. x_0, which is unrealizable, since with probability 1, "random walk by spheres" do not reach the boundary in a finite number of steps. This is due to the fact that the norm of the integral operator in the space we are considering L_1 equal to 1.

Next, we will construct a "biased", but realizable estimate of the solution of problem (28) and estimate the amount of bias.

Suppose that the solution of the Dirichlet problem is known at every point of the set $\partial\Omega_\varepsilon-\varepsilon-$ surrounding area of the border $\partial\Omega$.

Then for the function *p(r)* we can write the following integral equation:

$$p(r)=\int\limits_{\Omega}k(r,r')\,p(r')dr'+\varphi(r), \quad (30)$$

where $k(r,r')=\begin{cases}\delta_r(r'), \text{ if } r\notin\partial\Omega_\varepsilon\\ 0, \text{ if } r\in\partial\Omega_\varepsilon\end{cases}$

$$\varphi(r)=\begin{cases}\dfrac{1}{4\pi}\displaystyle\int\limits_{|r-r'|\le d}\left(\frac{1}{|r-r'|}-\frac{1}{d}\right)h(r')dr', \text{ if } r\notin\partial\Omega_\varepsilon,\\ p(r), \qquad\qquad \text{if } r\in\partial\Omega_0\end{cases} \quad (31)$$

Here $d=d(r),\delta_r(r')-$ generalized density corresponds to a uniform probability distribution on the sphere *S(r)*. It is obvious that the Neumann series for equation (30) converges, since the norm of the integral operator. *K* in a natural space for this task L_1 will be less than 1. Really,

$$\iint\limits_{\Omega}k(r,r')k(r',r'')dr'dr''\le$$

$$\le\int\limits_{\Omega-\partial\Omega_\varepsilon}\delta_r(r')\left(\int\limits_{\Omega}\delta_{r'}(r'')dr''\right)dr'=$$

$$=\int\limits_{\Omega-\partial\Omega_\varepsilon}\delta_r(r')dr'\le 1-\upsilon(\varepsilon).$$

Here $\upsilon(\varepsilon)=\dfrac{\varepsilon^2}{4d_{\max}^2}$, where $d_{\max}$ - the exact upper bound of the radii of spheres lying entirely in Ω. From here we obtain $\left\|K^2\right\|_{L_1(\Omega)}\le 1-\upsilon(\varepsilon)<1$. Consequently, this relation ensures the convergence of the Neumann series and, thereby, the possibility of using Monte-Carlo methods for equation (30). Equation (30) has the form of a conjugate integral equation, so to estimate $p(x_0)$ we can apply the ratio

$$p(x_0)=M\xi,\ \xi=\varphi(x_0)+\sum_{i=1}^{N}\varphi(x_i), \quad (32)$$

where $\varphi(x_i)-$ determined from (31). Here $\{x_i\}$ -

Markov chain, which we determine as follows: $q(r)=\delta(r-x_0)$ – the density of the initial distribution; $q(r,r')=\delta_r(r')$ – transition density from r in r'; $g(r)$ – the probability of a chain termination, determined by the expression:

$$g(r)=\begin{cases}0, & r\notin\partial\Omega_\varepsilon,\\ 1, & r\in\partial\Omega_\varepsilon.\end{cases}$$

Such a chain is called "random walk by spheres".

We assumed that the solution $p(r)$ known in $\partial\Omega_\varepsilon$. This allows us to get ε - biased evaluation of the solution. Indeed, instead of exact values $p(r)$ in $\partial\Omega_\varepsilon$ we can use approximate values, for example, taking them from the nearest border points, i.e., assuming:

$$p(r)\approx p_0(r^*),\ r\in\partial\Omega_\varepsilon,$$
$$r^*\in\partial\Omega,\left|r-r^*\right|=d(r)\le\varepsilon.$$

Then $\left|p-p_\varepsilon\right|=\left|M(\varphi(r_N)-\varphi(r^*))\right|\le c\cdot\varepsilon,$ where c – is some constant that is finite due to the limitation in the area $\overline{\Omega}$ derivatives of the solution.

It turns out that the integral expressing $\varphi(r)$ by $r\notin\partial\Omega_\varepsilon,$ it can be estimated by the Monte-Carlo method by one random "node" distributed with density

$$g_{\rho,\theta,\phi}(x)=\frac{1}{2\pi}\cdot\frac{\sin\theta}{2}\cdot\frac{6x(1-x/d)}{d^2},\ 0\le x\le d.$$

Appropriate random evaluation $\varphi(r)$ has the form $\varphi_1(\vec{r},\rho,\vec{\omega})=\dfrac{d^2}{6}\cdot h_1(\vec{r}+\rho\cdot\vec{\omega}),$ with $M\varphi_1(\vec{r},\rho,\vec{\omega})=\varphi(\vec{r}).$

We construct an algorithm of Monte-Carlo methods for estimating the solution of problem (26), (27) at a given point x_0:

1) from the point $x_0\in\Omega$ modeling the chain $\{x_n\}$ of Markov before the first hit in the ε is the neighborhood of the boundary $\partial\Omega_\varepsilon,$ we determine the point $x_N^*\in\partial\Omega.$ Here x_N^* is the point of the boundary closest to the last state x_N point of the boundary, N is the random number of the last state of the Markov chain;

2) respectively, the density

$$g_{\rho,\theta,\phi}(x)=\frac{1}{2\pi}\cdot\frac{\sin\theta}{2}\cdot\frac{6x(1-x/d)}{d^2},\ 0\le x\le d$$

in each random sphere $S(x)$, entirely lying in the area Ω, the value of the function is calculated $\Phi(\vec{r},\rho,\vec{\omega})=\dfrac{d^2}{6}\cdot h_1(\vec{r}+\rho\cdot\vec{\omega}).$ Here $\vec{r}$ – radius vector of a point $x,\rho=\left|\vec{r}-\vec{r}'\right|$ – random distance between the centers of the spheres, $\vec{\omega}$ – isotropic random unit vector, $\left|\vec{r}-\vec{r}'\right|\le d(\vec{r})\equiv d,$ the density $g_{\rho,\theta,\phi}(x)$ is given in the polar coordinate system.

3) We calculate a random variable ξ_ε by the formula

$$\xi_\varepsilon=\Phi(x_0,\rho_0,\omega_0)+\sum_{n=1}^{N-1}\Phi(x_n,\rho_n,\omega_n)+p_0(x_N^*).$$

The desired estimate of the solution $p(x_0)$ at a given point x_0 is obtained by averaging over all trajectories of the quantity $\xi_\varepsilon,$ that is

$$p_\varepsilon(x_0)=\frac{1}{L}\sum_{i=1}^{L}(\xi_\varepsilon)_i,\qquad(33)$$

where L is the number of trajectories outgoing from the point x_0.

The theorem. Variance of a random variable ξ_ε uniformly bounded by ε, therefore, $D\xi_\varepsilon<const<\infty.$

As proof, due to the limited initial pressure and initial saturation, i.e. due to the limited function $p_0(x)$ and $s_0(x)$ it is enough to assume that $p_0(x)=0.$ We obtain (it is sufficient to consider the case of the Dirichlet problem for the Poisson equation, i.e. in the Helmholtz equation we use c=0).

$$\varphi(r)=\begin{cases}\dfrac{1}{4\pi}\displaystyle\int_{|r-r'|\le d}\left(\frac{1}{|r-r'|}-\frac{1}{d}\right)h(r')dr', & \text{if } r\notin\partial\Omega_\varepsilon,\\ p(r), & \text{if } r\in\partial\Omega_0\end{cases}$$

Because of $Q_n\le 1$ in $\varphi\big|_{c\ne0}\le\varphi\big|_{c=0},$ it is enough to consider the case c=0, in which $Q_n\equiv 1.$ In this case, the algorithm under study is a direct

modeling for the integral equation (30) [9].

The corresponding variance is expressed by the following Ermakov - Zolotukhin formula [10]:

$$D\xi_{\varepsilon} = \left(f_{\varepsilon}, \varphi[2f_{\varepsilon}^{*} - \varphi]\right), \tag{34}$$

where f_{ε} – density of the centers of spheres, and f_{ε}^{*} – solution of the problem for a given value ε. It can be shown that $f_{\varepsilon}(d) \le \frac{c_1}{d}$. (see, for example, §3.4, [11]). At the same time $f_{\varepsilon}^{*} < c_2$, $\varphi(d) \le c_3 d^2$, by virtue of the determination $\varphi(x)$, i.e.(31).

From here we obtain the statement of the theorem by integrating the variance (34) along a fairly narrow "border layer".

Now we estimate the derivative with respect to the x_i $(i = 1,2,3)$, solution of problem (26) - (27) at that point x_0. For this purpose, using $x_{0,x}$, we denote a point $\left(x_0^1 + x, x_0^2, x_0^3\right)$ by (temporarily, the upper index will correspond to the number of the independent variable, and the lower one - to a fixed point). Then the solution to problem (26), (27) at the point $x_{0,x}$ for a sphere of radius $d(x_0)$ has the form

$$p(x_{0,x}) = \int_{S(x_0)} w(\omega, x)p(y)dy + \int_{|r-x_0|\le d} G_x(r, d_0)h_1(r)dr. \tag{35}$$

The spherical Green's function $G_x(r, d_0)$ and its normal derivative $w(\omega, x)$ are expressed by the formulas:

$$G_x(r, d_0) = \frac{1}{4\pi}\left(\frac{1}{|r - x_{0,x}|} - \frac{d_0}{\sqrt{(x^1)^2 \cdot r^2 + d_0^2 - 2d_0 \cdot x^1 \cdot x_p}}\right),$$

$$w(\omega, x) = \frac{1}{4\pi}\frac{d_0(d_0^2 - (x^1)^2)}{\left(d_0^2 + (x^1)^2 - 2d_0 \cdot x^1 \cdot a\right)^{\frac{3}{2}}},$$

where ω – the unit vector of the direction from x_0 to x_1, and $a = a(x, \omega)$ – the cosine of the angle between ω and the axis x, x_p – is the magnitude of the projection of the vector $r - x_0$ onto the axis x, $x < d(x_0) = d_0$. We differentiate relation (35) with respect to x, taking into account the expressions for the definition $G_x(r, d_0)$ and $w(\omega, x)$, by setting x=0, we obtain:

$$\frac{\partial p_{\varepsilon}(x_0)}{\partial x_i} = \frac{1}{4\pi}\int_{|\vec{r}-x_0|<d_0} \frac{x_i(d_0^3 - |\vec{r} - x_0|^3)}{d_0^3|\vec{r} - x_0|^3} \times \times h_1(\vec{r})d\vec{r} + M\left\{\frac{3l_i}{d_0}\left[\sum_{j=1}^{N}\frac{1}{4\pi d_j} \cdot \times \right.\right. \tag{36}$$

$$\left.\left. \times \int_{|\vec{r}-x_0|<d_j} \frac{d_j - |\vec{r} - x_j|}{|\vec{r} - x_j|} \cdot h_1(\vec{r})d\vec{r} + p_0(x_N^*)\right]\right\}$$

Here $l_i = l(x_i, \vec{\omega})$ is the cosine of the angles between $\vec{\omega}$ and coordinate axes $x_i, i = 1,2,3$, M is the expectation operator. The first integrals in expressions (36) can be estimated by the Monte-Carlo method from one random "node" with density

$$f_{x_1}(\vec{r}) = \frac{x_1(d_0^3 - |\vec{r} - x_0|^3)}{d_0^3|\vec{r} - x_0|^3}, |\vec{r} - x_0| < d_0 \tag{37}$$

To evaluate all three derivatives by x_i it is advisable to use density

$$\left(\sum_{i=1}^{3} f_x^2(\vec{r})\right)^{\frac{1}{2}} \equiv \tilde{f}_{x_i}(\vec{r}) = \frac{d_0^3 - |\vec{r} - x_0|^3}{d_0^2|\vec{r} - x_0|^2}, \tag{38}$$

where $|\vec{r} - x_0| < d_0$, $(x_1 + x_2 + x_3)^{\frac{1}{2}} = |\vec{r} - x_0|$.

In this case, the quality of the algorithm is estimated by the sum of the variance of the estimates of the corresponding integrals.

Integrals standing under the sign of mathematical expectation in (36) can also be evaluated by one random "node" distributed with density

$$f_{\eta}(\vec{r}) = \frac{d^3 - |\vec{r} - \vec{x}|^3}{d^2|\vec{r} - \vec{x}|^2}, \quad |\vec{r} - \vec{x}| < d. \tag{39}$$

In work [8] algorithms for modeling random variables with given distribution laws, namely

with densities, are given:

1) $g_{\rho,\theta,\phi}(x)=\frac{1}{2\pi}\cdot\frac{\sin\theta}{2}\cdot\frac{6x(1-x/d)}{d^2}, 0\le x\le d.$

2) $f_\eta(\vec{r})=\frac{d^3-|\vec{r}-\vec{x}|^3}{d^3|\vec{r}-\vec{x}|^2}, \quad |\vec{r}-\vec{x}|<d.$

Random variables with such densities are encountered when evaluating the solution $p(x)$ and derivatives of the solution $\frac{\partial p(x)}{\partial x_i}, (i=1,2,3).$ Dirichlet problems for the Poisson equation.

First, let's consider the algorithm for constructing a random variable with density $g_\rho(x)=\frac{6x(1-x/d)}{d^3}, 0\le x\le d.$ Replacement $y=\frac{x}{d}$ leads to

$$f_\rho(y)=6y(1-y), \quad 0\le y\le 1 \tag{40}$$

This expression corresponds to the distribution of the second ordinal statistic of three sample values of a random variable evenly distributed in the interval (0, 1). And the guiding cosines $l_i, (i=1,2,3),$ i.e. the cosines of the angles between the unit vector $\vec{\omega}(\theta,\varphi)$ and coordinate axes x_1, x_2, x_3 are modeled using the algorithm:

1) $\mu=1-2\alpha_1; \xi=1-2\alpha_2; \eta=1-2\alpha_3;$

2) $q=\xi^2+\eta^2,$ if $q<1,$ that

3) $l_1=\mu; \; l_2=\xi\cdot\sqrt{\frac{1-\mu^2}{q}}; \; l_3=\eta\cdot\sqrt{\frac{1-\mu^2}{q}};$

otherwise, paragraph 2 is done. Here $\alpha_i\in(0,1)$ for $i=1,2,3-$ sample values of a random variable α, evenly distributed in (0,1). Now let's consider the algorithm for modeling a random variable η, distributed with density

$$f_\eta(\vec{r})=\frac{d^3-|\vec{r}-\vec{x}|^3}{d^3\cdot|\vec{r}-\vec{x}|^2}, \; |\vec{r}-\vec{x}|<d.$$

Moving on to the polar coordinate system with the center $\vec{x}$ and, making a replacement $y=\frac{|\vec{r}-\vec{x}|}{d}$, we find the density $f_n(y)=\frac{4(1-y^3)}{3}, 0\le y\le 1.$ Hence, the modeling formula is obtained

$$\eta=\alpha_1\cdot\sqrt[4]{\alpha_2} \tag{41}$$

Note that formula (36) makes it possible to evaluate and $\nabla p_\varepsilon(x)$.

5 Assessment of $\vec{V}(x)$.

Using the estimates $p_\varepsilon(x)\frac{\partial p_\varepsilon(x)}{\partial x_i}, (i=1,2,3)$ from formula (36), we find $\nabla p_\varepsilon(x)$. Further, formula (22) for a homogeneous isotropic medium gives us

$$\vec{V}_\varepsilon(x)=C_3\nabla p_\varepsilon(x)+\vec{f}(x). \tag{42}$$

To determine the saturation s(x), we consider equations (15) - (17).

Let us denote by the vector $\vec{W}$

$$\vec{W}=-b\vec{V}+\vec{F}. \tag{43}$$

We remind that $b=b(s)=K_1\cdot K^{-1}$,

$\vec{F}=\vec{F}(x,s)=-k_{01}\cdot k_{02}\cdot k^{-1}\cdot K_0.$

$\left[\nabla_x\cdot p_c+(\rho_2-\rho_1)\cdot\vec{g}\right]=\vec{f}_0(x,s)-b(s)\cdot\vec{f}(x,s).$

from (14) and $K_1=k\cdot K_0,$ we assumed that $k=const$ and $K_0=C_2=const.$ Then $b=C_4=const.$ We also assumed that $\vec{f}(x,s)$ does not depend on s, i.e. $\vec{f}(x,s)\equiv\vec{f}(x).$ If the condition is satisfied 2) with respect to the filtration tensor for a homogeneous liquid

$K_0(x),$ that is, if $\frac{1}{\overline{m}(x)}\cdot\det K_0(x)=const,$

then $\frac{\partial p_c}{\partial x_i}=0, \; (i=1,2,3).$

From here and from (4) it follows $\vec{f}_0=K\int_s^1\nabla_x\cdot\frac{\partial p_c}{\partial s}\cdot\frac{k_{02}}{k}\cdot d\xi=\vec{f}_0(x).$

Therefore, under some of the above assumptions, we obtain the estimate

$$\vec{W}_\varepsilon = \vec{W}_\varepsilon(x) = -b\vec{V}_\varepsilon + \vec{F}. \quad (44)$$

Now, substituting (44) into (15) and taking into account $K_0 = C_2 = const$, we obtain

$$\begin{cases} dif(C_2 a\nabla s + \vec{W}\varepsilon) = 0, & \text{in } \Omega \quad (45) \\ s = s_0(x), & \text{on } \partial\Omega \quad (46) \end{cases}$$

6 Conclusion

In this paper, the issues of applying Monte-Carlo algorithms to filtration problems are investigated. We were able to apply the algorithms of "random walk by spheres" and "random walk by boundary" Monte-Carlo methods to solve the stationary problem of filtration of two immiscible inhomogeneous incompressible liquids in a porous medium. We have constructed a ε −biased estimate of the solution and derivatives of the solution (the first derivative of saturation, the first and second derivatives of pressure) of the stationary problem of two-phase filtration of incompressible immiscible liquids in a porous medium. Using the same Monte-Carlo algorithms, it is possible to solve the filtration problem taking into account temperature (i.e. the energy equation is added to the filtration equations). The scientific novelty of the research consists in the fact that, theoretically and practically, the method of statistical testing (Monte-Carlo) for solving boundary value problems of stationary and non-stationary filtration in areas of arbitrary configuration, which has positive possibilities of implementation on modern computers, is being developed.

References:

[1] Temam R, *Navier-Stokes equations. Theory and numerical analysis.* (Mir, Moscow, 1981), pp. 408.

[2] Ladyzhenskaya O.A.,*Mathematical problems in the dynamics of a viscous incompressible fluid. (*Nauka, Moscow, 1970), pp. 288.

[3] Belonosov S.M. and Chernous K.A.,*Boundary value problems for the Navier- Stokes equations.* (Nauka, Moscow,1985), pp. 311.

[4] Ermakov S.M., Mikhailov G.A. Statistical modeling. 3rd ed., supplemented. (M.: Fizmatlit, 2000), pp. 298.

[5] Mikhailov G.A. Optimization of Monte Carlo weight methods (Nauka, Moscow, Gl. ed. phys.-mat. lit., 2007), pp. 280.

[6] Ermakov S.M. Monte Carlo method and related issues (Nauka, Moscow, 2015), pp. 492.

[7] Antontsev S.N., Kazhihov A.V., Monahov V.N., *Boundary value problems in the mechanics of inhomogeneous fluids.* (Nauka, Novosibirsk, 1983), pp. 319.

[8] Mishustin B.A., On the solution of the Dirichlet problem for the Laplace equation by the method of statistical tests. *ZhVMiMF*, 1967, Vol. 7, 5, pp. 1178-1187.

[9] Smagulov Sh.C. and Shakenov K.K.,*"*On the solution of the stationary problem of two-phase filtration by the Monte Carlo method" in Materials of the International conference "*Actual problems of mathematics and mathematical modeling of ecological systems*" dedicated to the 60th anniversary of academician Sultangazin U.M. ("Gylym-Almaty", 1996.), pp. 55.

[10] Shakenov K.K., Sultanova M.S., Tastanov M.G. "Numerical solution the one Musket – Leverett's model by MonteCarlo methods and probability difference method" in Materials of International conference *"Informational technology and mathematical modeling in science, technology and education"* dedicated to 75th anniversary of academician A. Djaynakov (Bishkek, Kyrgyzstan, 2016), pp. 220-226.

[11] Tastanov M.G., Utemissova A.A., Maier F.F., Ysmagul R.S. Solution of problems for the relaxation model proceeding under linear Darsy's law using Monte- Carlo and probabilistic-difference method. *Global and Stochastic Analysis*, 2020, Vol. 7 No. 1,pp. 87-98.

[12] Ermakov S.M., *Monte Carlo method and related issues.*Second edition,expanded.(Nauka, Moskow, 1975), pp. 346-363.

[13] Ermakov S.M. and Zolotukhin V.G., *Application of the Monte Carlo method for calculating protection against nuclear radiation.* Questions of physics of reactor protection. (Atomizdat, Moscow, 1963),

pp.171-182.
[14] Yelepov B.S., Kronberg A.A., Mikhailov G.A., Sabelfeld K.K., *Solving boundary value problems by the Monte Carlo method.* (Nauka, Novosibirsk, 1980), pp.176
[15] N. Shimcin. Monte-Carlo Methods – Lecture Notes, Spring, 2015.