# On Ruled Surfaces of Coordinate Finite Type

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*Abstract:* - This article. in the introduction, gives a brief historic description on surfaces of finite Chen-type and of coordinate finite Chen-type according to the first, second and third fundamental form of a surface in the Euclidean space $E^3$. Then, an important class of surfaces was introduced, namely, the ruled surfaces were classified according to its coordinate finite Chen type with respect to the second fundamental form.

*Key-Words:* - Ruled surfaces, Surfaces in the Euclidean 3-space, Surfaces of coordinate finite Chen-type, Laplace-Beltrami operator.

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## 1 Introduction

Euclidean immersions of finite type were defined by B.-Y. Chen about forty years ago and since then research concerning this topic has become active by many differential geometers. Many results on this subject have been collected in the book [7]. Let $M^n$ be an $n$-dimensional submanifold of an arbitrary dimensional Euclidean space $E^m$. Denote by $\Delta^I$ the Beltrami- Laplace operator on $M^n$ with respect to the first fundamental form $I$ of $M^n$. A submanifold $M^n$ is said to be of finite type with respect to the first fundamental form $I$, if the vector field $\boldsymbol{x}$ of $M^n$ can be written as a finite sum of nonconstant ei-genvectors of the Laplacian $\Delta^I$, that is,

$$\boldsymbol{x} = \boldsymbol{c} + \sum_{i=1}^{k} \boldsymbol{x}_i \,, \tag{1}$$

where $\Delta^I \boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i$, $i = 1,\dots, k$, $\boldsymbol{c}$ is a constant vector and $\lambda_1, \lambda_2, \dots, \lambda_k$ are eigenvalues of $\Delta^I$. Moreover, if there are exactly $k$ nonconstant eigenvectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k$ appearing in (1) which all belong to different eigenvalues $\lambda_1, \dots, \lambda_k$, then $M^n$ is said to be of *I-type* $k$. However, if $\lambda_i = 0$ for some $i = 1, \dots, k$, then $M^n$ is said to be of *null I-type k* , otherwise $M^n$ is said to be of infinite type.

The class of finite type submanifolds in an arbitrary dimensional Euclidean space is very large, meanwhile results about surfaces of finite type in the Euclidean 3-space with respect to the first fundamental form is very little known. Actually, so far, minimal surfaces, the circular cylinders, and the spheres are the only known surfaces of finite type in the Euclidean 3-space. So in [8] B.-Y. Chen mentions the following problem

**Problem1.** Determine all surfaces of finite type in $E^3$.

Important families of surfaces were studied by different authors by proving that finite type ruled surfaces, [10], finite type quadrics, [9], finite type tubes [6], finite type cyclides of Dupin, [11], [12], finite type cones, [13], and finite type spiral surfaces [5] are surfaces of the only known examples in $E^3$. However, for surfaces of revolution, translation surfaces as well as helicoidal surfaces, the classification of its finite type surfaces is not known yet.

In this area, S. Stamatakis and H. Al-Zoubi studied the notion of surfaces of finite type with respect to the second or third fundamental forms. Based on this view, we raise the following questions:

**Problem 2.** Classify all surfaces of finite *II*-type in $E^3$.

**Problem 3.** Classify all surfaces of finite *III*-type in $E^3$.

According to problem 2, ruled surfaces [1] and tubes are the only families that were studied according to their finite type classification. However, for all other classical families of surfaces, the classification of its finite type surfaces is not known yet.

This type of study can be also extended to any smooth map, not necessary for the position vector of the surface, for example, the Gauss map of a surface. Here again, we give the following other two problems

**Problem 4.** Classify all surfaces of finite *II*-type Gauss map in $E^3$.

**Problem 5.** Classify all surfaces of finite *III*-type Gauss map in $E^3$.

On one hand, an interesting theme within this context is to study surfaces in $E^3$ for which the position vector $\boldsymbol{x}$ satisfies the condition $\Delta^J\boldsymbol{x} = A\boldsymbol{x}$, $J = I, II$, and A is a square matrix of order 3. Surfaces satisfying this condition are said of coordinate finite type. So we are led to the following problems

**Problem 6.** Classify all surfaces of coordinate finite *II*-type in $E^3$.

**Problem 7.** Classify all surfaces of coordinate finite *III*-type in $E^3$.

On the other hand, the last two problems mentioned above can be applied for the Gauss map of a surface, that is

**Problem 8.** Classify all surfaces of coordinate finite *II*-type Gauss map in $E^3$.

**Problem 9.** Classify all surfaces of coordinate finite *III*-type Gauss map in $E^3$.

Here also some results concerning the last two problems can be found in [2] and [3].

In [4] the authors classified surfaces of revolution in the Lorentz-Minkowski space, while in [15] translation surfaces in $Sol_3$ were studied.

## 2 Fundamentals

Let $\boldsymbol{x} = \boldsymbol{x}(u^1, u^2)$ be a parametric representation of a surface $S$ in the Euclidean space $E^3$ with non vanishing Gauss curvature. Let $I = g_{ij}du^idu^j$, $II = b_{ij}du^idu^j$ and $III = e_{ij}du^idu^j$ be the thee well-known fundamental forms of $S$. For sufficient differentiable functions $f(u^1, u^2)$ and $g(u^1, u^2)$ on $S$, the first differential parameter of Beltrami with respect to the fundamental form $J = I, II, III$ is defined by

$$\nabla^J(f,g) := a^{ij} f_{/i}\, g_{/j}, \tag{2}$$

where $f_{/i} := \dfrac{\partial f}{\partial u^i}$ and $(a^{ij})$ denotes the inverse tensor of $(g_{ij})$, $(b_{ij})$ and $(e_{ij})$ for $J = I, II$ and $III$ respectively.

The second differential parameter of Beltrami with respect to the fundamental form $J = I, II, III$ of $M$ is defined by

$$\Delta^J f := -a^{ij}\, \nabla^J_i f_{/j}, \tag{3}$$

where $f$ is a sufficiently differentiable function, $\nabla^J_i$ is the covariant derivative in the $u^i$ direction with respect to the fundamental form $J$ and $(\alpha^{ij})$ stands, as in definition (2), for the inverse tensor of $(g_{ij})$, $(b_{ij})$ and $(e_{ij})$ for $J = I$, $II$ and $III$ respectively. Applying (3) for the position vector $\boldsymbol{x}$ of $S$ we have

$$\Delta^{II}\boldsymbol{x} = -\frac{1}{2K}\ \mathrm{grad}^{III}K - 2\boldsymbol{n}. \tag{4}$$

From (4) we obtain the following result:

**Theorem 1** A surface $S$ in $E^3$ is of *II*-type 1 if and only if $S$ is part of a sphere.

Interesting research also one can follow the idea in [14] by defining the first and second Laplace operator using the definition of the fractional vector operators.

Up to now, the only known surfaces of finite *II*-type in $E^3$ are parts of spheres. In this paper we will pay attention to surfaces of finite *II*-type. Firstly, we will establish a formula for $\Delta^{II}\boldsymbol{x}$. Further, we continue our study by proving coordinate finite type ruled surfaces in the Euclidean 3-space, that is, their position vector $\boldsymbol{x}$ satisfies:

$$\Delta^{II}\boldsymbol{x} = \Lambda\boldsymbol{x} \tag{5}$$

## 3 Main Result

In the three-dimensional Euclidean space $E^3$ let $S$ be a ruled $C^r$-surface, $r \geq 3$, of nonvanishing Gaussian curvature defined by an injective $C^r$-immersion $\boldsymbol{x} = \boldsymbol{x}(s, t)$ on a region $U := I \times \mathbb{R}$ ($I \subset \mathbb{R}$ open interval) of $\mathbb{R}^2$. The surface $S$ can be expressed in terms of a directrix curve $\Gamma$: $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ and a unit vector field $\boldsymbol{\beta}(s)$ pointing along the rulings as follows

$$S: \boldsymbol{x}(s,t) = \boldsymbol{\alpha}(s) + t\boldsymbol{\beta}(s), \quad s \in J, \quad -\infty < t < \infty \tag{6}$$

Moreover, we can take the parameter $s$ to be the arc length along the spherical curve $\boldsymbol{\beta}(s)$. Thus for the curves $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ we have

$$<\boldsymbol{\alpha}', \boldsymbol{\beta}> = 0, \quad <\boldsymbol{\beta}, \boldsymbol{\beta}> = 1, \quad <\boldsymbol{\beta}', \boldsymbol{\beta}'> = 1,$$

where the differentiation with respect to $s$ is denoted by a prime and $< , >$ denotes the standard scalar product in $E^3$. It can be easily verified that the first and the second fundamental forms of $S$ are given by

$$I = q ds^2 + dt^2,$$
$$II = \frac{p}{\sqrt{q}} ds^2 + \frac{2A}{\sqrt{q}} dtds,$$

where

$$q = <\boldsymbol{\alpha}', \boldsymbol{\alpha}'> + 2<\boldsymbol{\alpha}', \boldsymbol{\beta}'>t + t^2,$$
$$p = (\boldsymbol{\alpha}', \boldsymbol{\beta}, \boldsymbol{\alpha}'') + [(\boldsymbol{\alpha}', \boldsymbol{\beta}, \boldsymbol{\beta}'') + (\boldsymbol{\beta}', \boldsymbol{\beta}, \boldsymbol{\alpha}'')]t + (\boldsymbol{\beta}', \boldsymbol{\beta}, \boldsymbol{\beta}'')t^2,$$
$$A = (\boldsymbol{\alpha}', \boldsymbol{\beta}, \boldsymbol{\beta}').$$

If, for simplicity, we put

$$\kappa := <\boldsymbol{\alpha}', \boldsymbol{\alpha}'>, \qquad \lambda := <\boldsymbol{\alpha}', \boldsymbol{\beta}'>,$$
$$\mu := (\boldsymbol{\beta}', \boldsymbol{\beta}, \boldsymbol{\beta}''), \qquad \nu := (\boldsymbol{\alpha}', \boldsymbol{\beta}, \boldsymbol{\beta}'') + (\boldsymbol{\beta}', \boldsymbol{\beta}, \boldsymbol{\alpha}''),$$
$$\rho := (\boldsymbol{\alpha}', \boldsymbol{\beta}, \boldsymbol{\alpha}''),$$

then we obtain the following relations

$$q = t^2 + 2\lambda t + \kappa, \qquad p = \mu t^2 + \nu t + \rho.$$

Furthermore, the Gaussian curvature $K$ of $S$ is given by

$$K = -\frac{A^2}{q^2}.$$

Since $S$ does not contain parabolic points, therefore

$$A \neq 0, \ \forall s \in J.$$

The Beltrami operator with respect to the second fundamental form can be expressed as follows

$$\Delta^{II} = \frac{\sqrt{q}}{A}\left(2\frac{\partial^2}{\partial s \partial t} - \frac{p}{A}\frac{\partial^2}{\partial t^2} - \frac{p_t}{A}\frac{\partial}{\partial t}\right), \tag{7}$$

where $p_t := \frac{\partial p}{\partial t}$.

Applying (7) for the position vector $\boldsymbol{x}$ we find

$$\Delta^{II}\boldsymbol{x} = \frac{1}{\sqrt{q}}\left(\frac{2q}{A}\boldsymbol{\beta}' - \frac{qp_t}{A^2}\boldsymbol{\beta}\right) \tag{8}$$

Let $(x_1, x_2, x_3)$ be the component functions of $\boldsymbol{x}$. Then it is well-known that

$$\Delta^{II}\boldsymbol{x} = (\Delta^{II}x_1, \Delta^{II}x_2, \Delta^{II}x_3). \tag{9}$$

Let $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ be the coordinate functions of the vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ respectively. From (8) we have

$$\Delta^{II}x_i = \frac{1}{\sqrt{q}}\left(\frac{2q}{A}\beta_i' - \frac{qp_t}{A^2}\beta_i\right), \ i = 1, 2, 3$$

Denote by $\lambda_{ij}$ the entries of the matrix $\Lambda$, i, j = 1, 2, 3, where all entries are real numbers. By using (6), and (8) condition (5) is found to be equivalent to the following system

$$\frac{2\sqrt{q}}{A}\beta_i' - \frac{\sqrt{q}\,p_t}{A^2}\beta_i = (\lambda_{i1}\alpha_1 + \lambda_{i2}\alpha_2 + \lambda_{i3}\alpha_3) + (\lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3)t, \tag{10}$$
$$i = 1, 2, 3.$$

We put

$$X_i := \lambda_{i1}\alpha_1 + \lambda_{i2}\alpha_2 + \lambda_{i3}\alpha_3, \qquad i = 1, 2, 3,$$
$$Y_i := \lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3, \qquad i = 1, 2, 3.$$

Then equations (10) reduce to

$$\frac{2\sqrt{q}}{A}\beta_i' - \frac{\sqrt{q}\,p_t}{A^2}\beta_i = X_i + tY_i$$

We raise the last ratio to the square and we get

$$q\left(4\frac{\beta_i'^2}{A^2} + \frac{p_t^2}{A^4}\beta_i^2 - 4\frac{p_t}{A^3}\beta_i'\beta_i\right) = X_i^2 + 2tX_iY_i + t^2Y_i^2,$$
$$i = 1, 2, 3.$$

or

$$4A^2(t^2 + 2\lambda t + \kappa)\beta_i'^2 + (t^2 + 2\lambda t + \kappa)(4\mu^2t^2 + 2\mu\nu t + \nu^2)\beta_i^2 - 4A(t^2 + 2\lambda t + \kappa)(2\mu t + \nu)\beta_i\beta_i' = A^4(X_i^2 + 2tX_iY_i + t^2Y_i^2),$$
$$i = 1, 2, 3,$$

which can be written analytically as

$$\begin{aligned}&4\mu^2\beta_i^2t^4 + (2\mu\nu\beta_i^2 + 8\lambda\mu^2\beta_i^2 - 8A\mu\beta_i\beta_i')t^3 \\ &+ (4A^2\beta_i'^2 + \nu^2\beta_i^2 + 4\kappa\mu^2\beta_i^2 + 4\lambda\mu\nu\beta_i^2 - 4A\nu\beta_i\beta_i' - 16A\lambda\mu\beta_i\beta_i' - A^4Y_i^2)t^2 \\ &+ (8A^2\lambda\beta_i'^2 + 2\lambda\nu^2\beta_i^2 + 2\kappa\mu\nu\beta_i^2 - 8A\kappa\mu\beta_i\beta_i' - 8\lambda\nu A\beta_i\beta_i' - 2A^4X_iY_i)t \\ &+ 4A^2\kappa\beta_i'^2 + \kappa\nu^2\beta_i^2 - 4A\kappa\nu\beta_i\beta_i' - A^4X_i^2 = 0,\end{aligned} \tag{11}$$
$$i = 1, 2, 3.$$

It's easily verified that (11) are polynomials in $t$ with functions in s as coefficients for $i$ = 1, 2, 3. This means that the coefficients of the powers of $t$ in (11) must be zeros, and so we have the following equations

$$4\mu^2\beta_i^2 = 0, \tag{12}$$

$$2\mu\nu\beta_i^2 + 8\lambda\mu^2\beta_i^2 - 8A\mu\beta_i\beta_i' = 0, \tag{13}$$

$$4A^2\beta_i'^2 + \nu^2\beta_i^2 + 4\kappa\mu^2\beta_i^2 + 4\lambda\mu\nu\beta_i^2 - 4A\nu\beta_i\beta_i' - 16A\lambda\mu\beta_i\beta_i' - A^4Y_i^2 = 0, \tag{14}$$

$$8A^2\lambda\beta_i'^2 + 2\lambda\nu^2\beta_i^2 + 2\kappa\mu\nu\beta_i^2 - 8A\kappa\mu\beta_i\beta_i' - 8\lambda\nu A\beta_i\beta_i' - 2A^4X_iY_i = 0, \tag{15}$$

$$4A^2\kappa\beta_i'^2 + \kappa\nu^2\beta_i^2 - 4A\kappa\nu\beta_i\beta_i' - A^4X_i^2 = 0, \tag{16}$$
$$i = 1, 2, 3.$$

Since (12) holds true for each $i$ = 1, 2, 3 we conclude

$$\mu = (\boldsymbol{\beta}', \boldsymbol{\beta}, \boldsymbol{\beta}'') = 0.$$

This means that $\boldsymbol{\beta}'$, $\boldsymbol{\beta}$, $\boldsymbol{\beta}''$are linearly dependent vectors, so there exist two functions $\sigma_1 = \sigma_1(s)$ and $\sigma_2 = \sigma_2(s)$ such that

$$\boldsymbol{\beta}'' = \sigma_1\boldsymbol{\beta} + \sigma_2\boldsymbol{\beta}'. \tag{17}$$

Differentiating the relation $<\boldsymbol{\beta}', \boldsymbol{\beta}'> = 1$, we get

$$<\boldsymbol{\beta}', \boldsymbol{\beta}''> = 0. \tag{18}$$

From (17) and (18) we obtain

$$\boldsymbol{\beta}'' = \sigma_1\boldsymbol{\beta}.$$

Hence $v = (\boldsymbol{\beta}', \boldsymbol{\beta}, \boldsymbol{\alpha}'')$. Relations (14), (15) and (16) become

$$4A^2\beta_i'^2 + v^2\beta_i^2 - 4Av\beta_i\beta_i' - A^4\mathrm{Y}_i^2 = 0, \tag{20}$$

$$8A^2\lambda\beta_i'^2 + 2\lambda v^2\beta_i^2 - 8\lambda vA\beta_i\beta_i' - 2A^4\mathrm{X}_i\mathrm{Y}_i = 0, \tag{21}$$
$$4A^2\kappa\beta_i'^2 + \kappa v^2\beta_i^2 - 4A\kappa v\beta_i\beta_i' - A^4\mathrm{X}_i^2 = 0,$$
$$i = 1, 2, 3.$$

Multiplying (20) by $2\lambda$, and from (21) one finds

$$\mathrm{X}_i = \lambda\mathrm{Y}_i,\ i = 1, 2, 3.$$

Or in vector notation

$$\mathbf{X} = \lambda\mathbf{Y}$$

Which can be written $\Lambda\boldsymbol{\alpha} = \lambda\Lambda\boldsymbol{\beta}$. Now we have the following two cases:

Case I. $\Lambda$ is the zero matrix. Then from (8) and taking into account $p_t = v$, it can be easily verified that $2A\boldsymbol{\beta}' - v\boldsymbol{\beta} = \mathbf{0}$, which is a contradiction since it yields that $A = 0$.

Case II. $\boldsymbol{\alpha} = \lambda\boldsymbol{\beta}$. Differentiating this equation with respect to $s$ we find

$$\boldsymbol{\alpha}' = \lambda'\boldsymbol{\beta} + \lambda\boldsymbol{\beta}'$$

Taking the inner product of both sides of the above equation with respect to $\boldsymbol{\beta}$ we find that $\lambda' = 0$, that is, $\lambda$ is constant. Hence we will get $\boldsymbol{\alpha}' = \lambda\boldsymbol{\beta}'$ and this leads us to that $A = 0$ a case which has been excluded. Thus we have proved the following

**Theorem 2.** There are no ruled surfaces in the Euclidean 3-space that satisfy the relation (5).

# 4 Conclusion

Firstly, we introduce the class of ruled surfaces in the Euclidean 3-space. Then, we define a formula for the Laplace operator regarding the second fundamental form *II*. Finally, we classify the ruled surfaces satisfying the relation $\Delta^{II}\boldsymbol{x} = \Lambda\boldsymbol{x}$, for a real square matrix $\Lambda$ of order 3. We proved that there are no ruled surfaces in the Euclidean 3-space that satisfy the relation $\Delta^{II}\boldsymbol{x} = \Lambda\boldsymbol{x}$. An interesting research one can follow, if this type of study can be applied to other families of surfaces that have not been investigated yet such as quadric surfaces, tubular surfaces, or spiral surfaces.

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**Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)**

**Hassan Al-Zoubi** carried out the introduction and the main result of the article.

**Amjed Zraiqat**, **Hamza Alzaareer** and **Tareq Hamadneh** have improved section 2.

**Waseem Al-Mashaleh** has reviewed and checked the calculations of this paper.

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