Convergence Theorem for Multivalued Almost Type Contractions via Generalized Simulation Functions

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Abstract: -The purpose of this work is to introduce the concepts of generalized multivalued almost type $\mathbb{Z}$-contraction along with $\mathcal{C}$-class functions and generalized Suzuki multivalued almost type $\mathbb{Z}$-contraction along with $\mathcal{C}$-class functions for a pair of mappings, as well as to show that common fixed point theorems for such mappings in complete metric spaces. The results of this study generalize and expand on some established fixed point findings in the literature. We derive several corollaries from our core results and offer examples to support our results.

Key-Words: Multivalued mapping, $\mathcal{C}$-class functions, $\mathbb{Z}$-contraction, Almost type, Suzuki type

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1 Introduction

The origins of fixed point theory can be traced back to the last quarter of the nineteenth century, when repeated approximations were used to establish the existence and uniqueness of solutions to differential equations. It is worth noting that the Banach contraction principle, which was developed by Banach [1]. This solution has been expanded for single and multivalued cases on a metric space in a variety of ways. Nadler [2] developed the concept of multivalued contraction mapping in 1969 and established that it had a fixed point in the entire metric space. Several fixed point theorems were then established by various writers as a generalization of Nadler's theory (see [3], [4], [5], [6], [7], [8]).

Let (Υ, d) be a metric space and $\mathcal{CB}(\Upsilon)$ denote the collection of all nonempty closed and bounded subset of Υ. For $\omega \in \Upsilon$ and $A, B \in \mathcal{CB}(\Upsilon)$, we have

$$d(A,B) = \inf\{d(a,b) : \rho \in A \text{ and } \rho \in B\},$$
$$D(\omega, A) = \inf\{d(\omega,\rho) : \rho \in A\}$$

and

$$\mathcal{H}(A,B) = \max\left\{\sup_{\omega\in A} D(\omega,B), \sup_{\rho\in B} D(\rho,A)\right\}.$$

The function $\mathcal{H}$ is a Hausdorff metric induced by the metric d. It is a metric on $\mathcal{CB}(\Upsilon)$.

Let (Υ, d) be a complete metric space and $\Omega : \Upsilon \to \mathcal{CB}(\Upsilon)$ be a contraction mapping such that

$$\mathcal{H}(\Omega\omega, \Omega\rho) \le \delta d(\omega,\rho)$$

for all $\omega, \rho \in \Upsilon$ and for some $\delta \in [0, 1)]$. It's a typical Banach contraction, [1].

Berinde [9] extended the Zamfirescu fixed point theorem [10] to almost contractions, a class of contractive type mappings, for $\delta \in [0, 1)$ and $\mathcal{L} \ge 0$ such that

$$d(\Omega\omega, \Omega\rho) \le \delta d(\omega,\rho) + \mathcal{L}d(\omega, \Omega\rho) \quad \text{for all } \omega, \rho \in \Upsilon. \tag{1}$$

Khojasteh et al. [11] defined $\mathbb{Z}$-contraction with respect to ζ, which generalizes the Banach contraction principle and integrates various kinds of contraction. Olgun et al. [12] achieved fixed point solutions for generalized $\mathbb{Z}$-contractions.

Later, Chandok et al. [13] expanded the conclusions of [11], [12] by combining the concept of simulation functions with $\mathcal{C}$-class functions and proving the existence and uniqueness of point of coincidence.

Motivated and inspired by almost contractions in (1), Definition 2.3, Definition 2.4 and work of [13], we introduce the notion of extended multivalued almost type $\mathbb{Z}$-contraction with $\mathcal{C}$-class functions and extended multivalued Suzuki almost type $\mathbb{Z}$-contraction with $\mathcal{C}$-class functions for metric space mapping pair.

2 Preliminaries

Definition 2.1. [11] *Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a mapping. Then ζ is called a simulation function if it satisfies the following conditions:*

(ζ_1): $\zeta(0, 0) = 0$;

(ζ_2): $\zeta(v, u) < u - v$ *for all* $u, v > 0$;

(ζ_2): *if* $\{v_n\}, \{u_n\}$ *are sequence in* $(0,\infty)$ *such that* $\lim_{n\to\infty} v_n = \lim_{n\to\infty} u_n > 0$, *then* $\limsup_{n\to\infty} \zeta(v_n, u_n) < 0$.

Argoubi et al. [14] applying innovative the simulation function definition by omitting the condition (ζ_1).

Definition 2.2. [14] *A simulation function is a mapping $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ satisfying the following conditions:*

(ζ_2): $\zeta(v,u) < u - v,\ u,v > 0$;

(ζ_3'): *if $\{v_n\},\{u_n\}$ are sequence in $(0,\infty)$ such that $\lim_{n\to\infty} v_n = \lim_{n\to\infty} u_n > 0$, and $v_n < u_n$ then $\limsup_{n\to\infty} \zeta(v_n,u_n) < 0$.*

Definition 2.3. [12] *Let (Υ, d) be a metric space, $\Omega : \Upsilon \to \Upsilon$ a mapping and $\zeta \in \mathbb{Z}$. Then Ω is called a generalized $\mathbb{Z}$-contraction with respect to ζ if*

$$\zeta(d(\Omega\omega,\Omega\rho),\Theta(\omega,\rho)) \geq 0 \quad \text{for all } \omega,\rho \in \Upsilon,$$

where

$$\Theta(\omega,\rho) = \max\left\{d(\omega,\rho), d(\omega,\Omega\omega), d(\rho,\Omega\rho), \frac{d(\omega,\Omega\rho)+d(\rho,\Omega\omega)}{2}\right\}.$$

Padcharoen et al. [15] on the other hand defined generalized Suzuki type $\mathbb{Z}$-contraction on metric spaces as follows.

Definition 2.4. [15] *Let (Υ, d) be a metric space, $\Omega : \Upsilon \to \Upsilon$ a mapping and $\zeta \in \mathbb{Z}$. Then Ω is called a generalized Suzuki type $\mathbb{Z}$-contraction with respect to ζ if*

$$\frac{1}{2}(d(\omega,\Omega\omega) < d(\omega,\rho) \Rightarrow \zeta(d(\Omega\omega,\Omega\rho),\Theta(\omega,\rho)) \geq 0$$

for all distinct $\omega,\rho \in \Upsilon$, where

$$\Theta(\omega,\rho) = \max\left\{d(\omega,\rho), d(\omega,\Omega\omega), d(\rho,\Omega\rho), \frac{d(\omega,\Omega\rho)+d(\rho,\Omega\omega)}{2}\right\}.$$

Definition 2.5. [16] *A mapping $\mathcal{G} : [0,\infty)^2 \to \mathbb{R}$ has the property $\mathcal{C}_{\mathcal{G}}$, if there exists $\mathcal{C}_{\mathcal{G}} \geq 0$ such that*

($\mathcal{G}_1$): *$\mathcal{G}(u,v) > \mathcal{C}_{\mathcal{G}}$ implies $u > v$;*

($\mathcal{G}_2$): *$\mathcal{G}(u,v) \leq \mathcal{C}_{\mathcal{G}}$ for all $v \in [0,\infty)$.*

Definition 2.6. [17] *A $\mathcal{C}_{\mathcal{G}}$ simulation function is a mapping $\mathcal{G} : [0,\infty)\times[0,\infty) \to \mathbb{R}$ satisfying the following conditions:*

(i): *$\zeta(v,u) < \mathcal{G}(u,v)$ for all $v,u > 0$, where $\mathcal{G} : [0,\infty)[0,\infty)\times[0,\infty) \to \mathbb{R}$ is a $\mathcal{C}$-class function;*

(ii): *if $\{v_n\},\{u_n\}$ are sequence in $(0,\infty)$ such that $\lim_{n\to\infty} v_n = \lim_{n\to\infty} u_n > 0$, and $v_n < u_n$, then $\limsup_{n\to\infty} \zeta(v_n,u_n) < \mathcal{C}_{\mathcal{G}}$.*

Lemma 2.7. [18] *Let (Υ, d) be a metric space and let $\{\omega_n\}$ be a sequence in Υ such that*

$$\lim_{n\to\infty} d(\omega_{2n},\omega_{2n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence in Υ, then there exists $\epsilon > 0$ and two sequence $\omega_{m(k)}$ and $\omega_{n(k)}$ of positive integers such that $\omega_{n(k)} > \omega_{m(k)} > k$ and the following sequence tend to ϵ when $k \to \infty$:

$$d(\omega_{m(k)},\omega_{n(k)}), d(\omega_{m(k)},\omega_{n(k)+1}), d(\omega_{m(k)-1},\omega_{n(k)}), d(\omega_{m(k)-1},\omega_{n(k)+1}), d(\omega_{m(k)+1},\omega_{n(k)+1}).$$

For a non-empty set Υ, let $\mathcal{P}(\Upsilon)$ denotes the power set of Υ. If (Υ, d) is a metric space, then let

$$\mathcal{N}(\Upsilon) = \mathcal{P}(\Upsilon) - \{\emptyset\},$$
$$\mathcal{CB}(\Upsilon) = \{A \in \mathcal{N}(\Upsilon) : A \text{ is closed and bounded}\},$$
$$\mathcal{K}(\Upsilon) = \{A \in \mathcal{N}(\Upsilon) : A \text{ is compact}\}.$$

Definition 2.8. [19] *Let Υ be a non empty set, $\Omega : \Upsilon \to \mathcal{N}(\Upsilon)$ and $\alpha : \Upsilon\times\Upsilon \to [0,\infty)$ be two mappings. Then Ω is said to be an α-admissible whenever for each $\omega \in \Upsilon$ and $\rho \in \Omega\omega$,*

$$\alpha(\omega,\rho) \geq 1 \Rightarrow \alpha(\rho,\eta) \geq 1 \quad \text{for all } \eta \in \Omega\rho.$$

Definition 2.9. [20] *Let Υ be a nonempty set, $\Omega : \Upsilon \to \mathcal{N}(\Upsilon)$ and $\alpha : \Upsilon\times\Upsilon \to [0,\infty)$ be two mappings. Then Ω is said to be triangular α-admissible if Ω is α-admissible and*

$$\alpha(\omega,\rho) \geq 1 \quad \text{and} \quad \alpha(\rho,\eta) \geq 1$$
$$\Rightarrow \alpha(\omega,\eta) \geq 1 \quad \text{for all } \eta \in \Omega\rho.$$

Lemma 2.10. [20] *Let $\Omega : \Upsilon \to \mathcal{N}(\Upsilon)$ be a triangular α-admissible mapping. Assume that there exists $\omega_0 \in \Upsilon$ and $\omega_1 \in \Omega\omega_0$ such that $\alpha(\omega_0,\omega_1) \geq 1$. Then for a sequence $\{\omega_n\}$ such that $\omega_{n+1} \in \Omega\omega_n$, we have $\alpha(\omega_n,\omega_m) \geq 1$ for all $m,n \in \mathbb{N}$ with $n < m$.*

Definition 2.11. [21] *Let (Υ, d) be a metric space, $\alpha : \Upsilon\times\Upsilon \to [0,\infty)$ and $\Omega : \Upsilon \to \mathcal{K}(\Upsilon)$ mappings. Then Ω is said to be an α-continuous multivalued mapping on $(\mathcal{K}(\Upsilon),\mathcal{H})$, if for all sequences $\{\omega_n\}$ with $\omega_n \to \omega \in \Upsilon$ as $n \to \infty$, and $\alpha(\omega_n,\omega_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\Omega\omega_n \to \Omega\omega$ as $n \to \infty$, that is,*

$$\lim_{n\to\infty} d(\omega_n,\omega) = 0 \quad \text{and} \quad \alpha(\omega_n,\omega_{n+1}) \geq 1$$
$$\text{for all } n \in \mathbb{N} \Rightarrow \lim_{n\to\infty} \mathcal{H}(\Omega\omega_n,\Omega\omega) = 0.$$

Definition 2.12. [22] *Let (Υ, d) be a metric space, $\alpha : \Upsilon\times\Upsilon \to [0,\infty)$. The metric space (Υ, d) is said to be α-complete if and only if every Cauchy sequence $\{\omega_n\}$ with $\alpha(\omega_n,\omega_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ converges in Υ.*

3 Main Result

Now we state our main results.

Definition 3.1. *Let (Υ, d) be a metric space and $\Omega, \Lambda : \Upsilon \to \mathcal{K}(\Upsilon)$ and $\alpha : \Upsilon \times \Upsilon \to [0, 1)$ be a function. We say Ω is $\mathbb{Z}_{(\alpha,\mathcal{G})}$ multivalued almost type contraction with respect to ζ such that*

$$\zeta(\alpha(\omega, \rho)\mathcal{H}(\Omega\omega, \Lambda\rho), \beta(\text{₩})\text{₩}) \geq \mathcal{C}_{\mathcal{G}} \tag{2}$$

for all $\omega, \rho \in \Upsilon$ with $\omega \neq \rho$ and $\mathcal{L} \geq 0$, where

$$\text{₩} = \Theta(\omega, \rho) + \mathcal{L}\Psi(\omega, \rho)$$

with

$$\Theta(\omega, \rho) = \max\Big\{d(\omega, \rho), D(\omega, \Omega\omega), D(\rho, \Lambda\rho), \frac{D(\omega, \Lambda\rho) + D(\rho, \Omega\omega)}{2}\Big\}$$

and

$$\Psi(\omega, \rho) = \min\Big\{D(\omega, \Omega\omega), D(\rho, \Lambda\rho), D(\omega, \Lambda\rho), D(\rho, \Omega\omega)\Big\}.$$

Definition 3.2. *Let (Υ, d) be a metric space and $\Omega, \Lambda : \Upsilon \to \mathcal{K}(\Upsilon)$ and $\alpha : \Upsilon \times \Upsilon \to [0, 1)$ be a function. We say Ω is $\mathbb{Z}_{(\alpha,\mathcal{G})}$ Suzuki multivalued almost type contraction with respect to ζ if*

$$\begin{aligned}&\frac{1}{2}\min\{D(\omega, \Omega\omega), D(\rho, \Lambda\rho)\} < d(\omega, \rho)\\ &\Rightarrow \zeta(\text{₤}, \beta(\text{₩})\text{₩}) \geq \mathcal{C}_{\mathcal{G}}\end{aligned} \tag{3}$$

for all $\omega, \rho \in \Upsilon$ with $\Omega\omega \neq \Lambda\rho$ and $\mathcal{L} \geq 0$, where

$$\text{₤} = \alpha(\omega, \rho)\mathcal{H}(\Omega\omega, \Lambda\rho),$$
$$\text{₩} = \Theta(\omega, \rho) + \mathcal{L}\Psi(\omega, \rho)$$

with

$$\Theta(\omega, \rho) = \max\Big\{d(\omega, \rho), D(\omega, \Omega\omega), D(\rho, \Lambda\rho), \frac{D(\omega, \Lambda\rho) + D(\rho, \Omega\omega)}{2}\Big\}$$

and

$$\Psi(\omega, \rho) = \min\Big\{D(\omega, \Omega\omega), D(\rho, \Lambda\rho), D(\omega, \Lambda\rho), D(\rho, \Omega\omega)\Big\}.$$

Theorem 3.3. *Let (Υ, d) be a metric space and $\Omega, \Lambda : \Upsilon \to \mathcal{K}(\Upsilon)$ be $\mathbb{Z}_{(\alpha,\mathcal{G})}$ Suzuki almost type multivalued contraction satisfying:*

(i) *(Υ, d) is an α-complete metric space*;

(ii) *Ω, Λ are triangular α-admissible*;

(iii) *Ω, Λ are an α-continuous multivalued mapping.*

Then Ω and Λ have a common fixed point.

Proof. Let $\omega_0 \in \Upsilon$. Choose $\omega_1 \in \Omega\omega_0$. Then by the definition of Hausdorff metric there exists $\omega_2 \in \Lambda\omega_1$ such that

$$\begin{aligned}0 &< d(\omega_1, \omega_2)\\ &= D(\omega_1, \Lambda\omega_1)\\ &\leq \alpha(\omega_0, \omega_1)\mathcal{H}(\Omega\omega_0, \Lambda\omega_1).\end{aligned} \tag{4}$$

Assume that $D(\omega_0, \Omega\omega_0) > 0$ and $D(\omega_1, \Lambda\omega_1) > 0$ then

$$\frac{1}{2}\min\{D(\omega_0, \Omega\omega_0), D(\omega_1, \Lambda\omega_1)\} < d(\omega_0, \omega_1).$$

Therefore from (3), we have

$$\begin{aligned}&\frac{1}{2}\min\{D(\omega_0, \Omega\omega_0), D(\omega_1, \Lambda\omega_1)\} < d(\omega_0, \omega_1)\\ &\Rightarrow \zeta(\text{₤}_0, \beta(\text{₩}_0)\text{₩}_0) \geq \mathcal{C}_{\mathcal{G}},\end{aligned}$$

where $\text{₤}_0 = \alpha(\omega_0, \omega_1)\mathcal{H}(\Omega\omega_0, \Lambda\omega_1)$ and $\text{₩}_0 = \Theta(\omega_0, \omega_1) + \mathcal{L}\Psi(\omega_0, \omega_1)$.
Consider

$$\begin{aligned}\mathcal{C}_{\mathcal{G}} &\leq \zeta(\text{₤}_0, \beta(\text{₩}_0)\text{₩}_0)\\ &< \mathcal{G}(\beta(\text{₩}_0)\text{₩}_0, \text{₤}_0).\end{aligned} \tag{5}$$

Consequently, we get

$$d(\omega_1, \omega_2) \leq \text{₤}_0 < \beta(\text{₩}_0)\text{₩}_0, \tag{6}$$

where

$$\begin{aligned}&\Theta(\omega_0, \omega_1)\\ &= \max\Big\{d(\omega, \omega_1), D(\omega_0, \Omega\omega_0), D(\omega_1, \Lambda\omega_1), \frac{D(\omega_0, \Lambda\omega_1) + D(\omega_1, \Omega\omega_0)}{2}\Big\}\\ &\leq \max\Big\{d(\omega_0, \omega_1), d(\omega_0, \omega_1), d(\omega_1, \omega_2), \frac{d(\omega_0, \omega_2) + d(\omega_1, \omega_1)}{2}\Big\}\\ &= \max\left\{d(\omega_0, \omega_1), d(\omega_1, \omega_2), \frac{d(\omega_0, \omega_2)}{2}\right\}.\end{aligned}$$

Because

$$\begin{aligned}\frac{d(\omega_0, \omega_2)}{2} &\leq \frac{d(\omega_0, \omega_1) + d(\omega_2, \omega_1)}{2}\\ &\leq \max\{d(\omega_0, \omega_1), d(\omega_1, \omega_2)\}.\end{aligned}$$

Thus,

$$\Theta(\omega_0, \omega_1) \leq \max\{d(\omega_0, \omega_1), d(\omega_1, \omega_2)\}$$

and

$$\begin{aligned}&\Psi(\omega_0,\omega_1)\\&=\min\{D(\omega_0,\Omega\omega_0),D(\omega_1,\Lambda\omega_1),\\&\qquad D(\omega_0,\Lambda\omega_1),D(\omega_1,\Omega\omega_0)\}\\&=\min\{d(\omega_0,\omega_1),d(\omega_1,\omega_2),d(\omega_0,\omega_2),\\&\qquad d(\omega_1,\omega_1)\}\\&=0.\end{aligned}$$

If $\max\{d(\omega_0,\omega_1),d(\omega_1,\omega_2)\} = d(\omega_1,\omega_2)$ and $\Psi(\omega_0,\omega_1)=0$, then (6) becomes

$$\begin{aligned}d(\omega_1,\omega_2)&\le\alpha(\omega_0,\omega_1)\mathcal{H}(\Omega\omega_0,\Lambda\omega_1)\\&<\beta(d(\omega_1,\omega_2))d(\omega_1,\omega_2),\end{aligned}\tag{7}$$

obtain that

$$d(\omega_1,\omega_2)\le\alpha(\omega_0,\omega_1)\mathcal{H}(\Omega\omega_0,\Lambda\omega_1)<d(\omega_1,\omega_2),$$

which is a contradiction. Thus we conclude that

$$\max\{d(\omega_0,\omega_1),d(\omega_1,\omega_2)\}=d(\omega_0,\omega_1).$$

By (6) we get

$$d(\omega_1,\omega_2)<d(\omega_0,\omega_1).$$

Similarly, for $\omega_2\in\Lambda\omega_1$ and $\omega_3\in\Omega\omega_2$ we have

$$d(\omega_2,\omega_3)\le\alpha(\omega_1,\omega_2)\mathcal{H}(\Lambda\omega_1,\Omega\omega_2)<d(\omega_1,\omega_2).$$

This implies

$$d(\omega_2,\omega_3)<d(\omega_1,\omega_2).$$

By continuing in this manner, we construct a sequence $\{\omega_n\}$ in Υ such that $\omega_{2n+1}\in\Omega\omega_{2n}$ and $\omega_{2n+2}\in\Lambda\omega_{2n+1}, n=0,1,2,...$ such that

$$\begin{aligned}0&<d(\omega_{2+1},\omega_{2n+2})\\&=D(\omega_{2n+1},\Lambda\omega_{2n+1})\\&\le\alpha(\omega_{2n},\omega_{2n+1})\mathcal{H}(\Omega\omega_{2n},\Lambda\omega_{2n+1})\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{2}\min\{D(\omega_{2n},\Omega\omega_{2n}),D(\omega_{2n+1},\Lambda\omega_{2n+1})\}\\&<d(\omega_{2n},\omega_{2n+1}).\end{aligned}$$

Hence from (3), we have

$$\begin{aligned}&\frac{1}{2}\min\{D(\omega_{2n},\Omega\omega_{2n}),D(\omega_{2n+1},\Lambda\omega_{2n+1})\}\\&<d(\omega_{2n},\omega_{2n+1})\Rightarrow\zeta(₤_{2n},\beta(₩_{2n})₩_{2n})\ge\mathcal{C}_{\mathcal{G}},\end{aligned}$$

where $₤_{2n}=\alpha(\omega_{2n},\omega_{2n+1})\mathcal{H}(\Omega\omega_{2n},\Lambda\omega_{2n+1})$ and $₩_{2n}=\Theta(\omega_{2n},\omega_{2n+1})+\mathcal{L}\Psi(\omega_{2n},\omega_{2n+1})$.
Consider

$$\begin{aligned}\mathcal{C}_{\mathcal{G}}&\le\zeta(₤_{2n},\beta(₩_{2n})₩_{2n})\\&<\mathcal{G}(\beta(₩_{2n})₩_{2n},₤_{2n}).\end{aligned}\tag{8}$$

Consequently, we get

$$d(\omega_{2n+1},\omega_{2n+2})\le ₤_{2n}<\beta(₩_{2n})₩_{2n},\tag{9}$$

where

$$\begin{aligned}&\Theta(\omega_{2n},\omega_{2n+1})\\&=\max\Big\{d(\omega_{2n},\omega_{2n+1}),D(\omega_{2n},\Omega\omega_{2n}),\\&\qquad D(\omega_{2n+1},\Lambda\omega_{2n+1}),\\&\qquad\frac{D(\omega_{2n},\Lambda\omega_{2n+1})+D(\omega_{2n+1},\Omega\omega_{2n})}{2}\Big\}\\&\le\max\Big\{d(\omega_{2n},\omega_{2n+1}),d(\omega_{2n},\omega_{2n+1}),\\&\qquad d(\omega_{2n+1},\omega_{2n+2}),\\&\qquad\frac{d(\omega_{2n},\omega_{2n+2})+d(\omega_{2n+1},\omega_{2n+1})}{2}\Big\}\\&=\max\Big\{d(\omega_{2n},\omega_{2n+1}),d(\omega_{2n+1},\omega_{2n+2}),\\&\qquad\frac{d(\omega_{2n},\omega_{2n+2})}{2}\Big\}.\end{aligned}$$

Because

$$\begin{aligned}&\frac{d(\omega_{2n},\omega_{2n+2})}{2}\\&\le\frac{d(\omega_{2n},\omega_{2n+1})+d(\omega_{2n+2},\omega_{2n+1})}{2}\\&\le\max\{d(\omega_{2n},\omega_{2n+1}),d(\omega_{2n+1},\omega_{2n+2})\}.\end{aligned}$$

Thus,

$$\begin{aligned}&\Theta(\omega_{2n},\omega_{2n+1})\\&\le\max\{d(\omega_{2n},\omega_{2n+1}),d(\omega_{2n+1},\omega_{2n+2})\}\end{aligned}$$

and

$$\begin{aligned}&\Psi(\omega_{2n},\omega_{2n+1})\\&=\min\Big\{D(\omega_{2n},\Omega\omega_{2n}),D(\omega_{2n+1},\Lambda\omega_{2n+1}),\\&\qquad D(\omega_{2n},\Lambda\omega_{2n+1}),D(\omega_{2n+1},\Omega\omega_{2n})\Big\}\\&=\min\Big\{d(\omega_{2n},\omega_{2n+1}),d(\omega_{2n+1},\omega_{2n+2}),\\&\qquad d(\omega_{2n},\omega_{2n+2}),d(\omega_{2n+1},\omega_{2n+1})\Big\}\\&=0.\end{aligned}$$

If $\max\{d(\omega_{2n},\omega_{2n+1}),d(\omega_{2n+1},\omega_{2n+2})\} = d(\omega_{2n+1},\omega_{2n+2})$ and $\Psi(\omega_{2n},\omega_{2n+1})=0$, then (9) becomes

$$\begin{aligned}&d(\omega_{2n+1},\omega_{2n+2})\\&\le\alpha(\omega_{2n},\omega_{2n+1})\mathcal{H}(\Omega\omega_{2n},\Lambda\omega_{2n+1})\\&<\beta(d(\omega_{2n+1},\omega_{2n+2}))d(\omega_{2n+1},\omega_{2n+2}),\end{aligned}\tag{10}$$

obtain that

$$d(\omega_{2n+1},\omega_{2n+2}) \le \alpha(\omega_{2n},\omega_{2n+1})\mathcal{H}(\Omega\omega_{2n},\Lambda\omega_{2n+1}) < d(\omega_{2n+1},\omega_{2n+2}),$$

which is a contradiction. Thus we conclude that

$$\max\{d(\omega_{2n},\omega_{2n+1}),d(\omega_{2n+1},\omega_{2n+2})\} = d(\omega_{2n},\omega_{2n+1}).$$

By (10) we get

$$d(\omega_{2n+1},\omega_{2n+2}) < d(\omega_{2n},\omega_{2n+1}).$$

Then from (10) we have

$$\begin{aligned} &d(\omega_{2n+2},\omega_{2n+3}) \\ &\le \alpha(\omega_{2n+1},\omega_{2n+2})\mathcal{H}(\Omega\omega_{2n+1},\Lambda\omega_{2n+2}) \\ &< d(\omega_{2n+1},\omega_{2n+2}). \end{aligned}$$

This implies

$$d(\omega_{2n+2},\omega_{2n+3}) < d(\omega_{2n+1},\omega_{2n+2}). \tag{11}$$

Thus $d(\omega_{n+1},\omega_{n+2}) < d(\omega_n,\omega_{n+1})$ for all n. Hence $\{d(\omega_n,\omega_{n+1})\}$ is a strictly decreasing sequence of non-negative real numbers. Thus there exists $Z \ge 0$ such that

$$\lim_{n\to\infty} d(\omega_n,\omega_{n+1}) = Z.$$

Assume that $Z > 0$. So by inequality (8) we obtain,

$$\lim_{n\to\infty} \text{₤}_{2n} = Z \tag{12}$$

and

$$\lim_{n\to\infty} \beta(\text{₩}_{2n})\text{₩}_{2n} = Z. \tag{13}$$

Using (2) and ($\mathcal{G}_2$) of Definition 2.5, get

$$\begin{aligned} \mathcal{C}_{\mathcal{G}} &\le \limsup_{n\to\infty} \zeta(\text{₤}_{2n},\beta(\text{₩}_{2n})\text{₩}_{2n}) \\ &= \limsup_{n\to\infty} \zeta(\text{₤}_{2n},\beta(d(\omega_{2n},\omega_{2n+1}))d(\omega_{2n},\omega_{2n+1})) \\ &< \mathcal{C}_{\mathcal{G}}, \end{aligned}$$

which is a contradiction and nence $z = 0$, i.e.,

$$\lim_{n\to\infty} d(\omega_n,\omega_{n+1}) = 0. \tag{14}$$

We now show that $\{\omega_n\}$ is a Cauchy sequence. Assume, however, that it is not a Cauchy sequence. We suppose that $\epsilon > 0$ exists, as well as two sequences of positive integers, $\{n(k)\}$ and $\{m(k)\}$ such that

$$\begin{aligned} &n(k) > m(k) > k, d(\omega_{n(k)},\omega_{m(k)}) \ge \epsilon, \\ &d(\omega_{n(k)-1},\omega_{m(k)}) < \epsilon. \end{aligned} \tag{15}$$

We obtain using the triangular inequality

$$\begin{aligned} \epsilon &\le d(\omega_{n(k)},\omega_{m(k)}) \\ &\le d(\omega_{n(k)},\omega_{m(k)-1}) + d(\omega_{n(k)-1},\omega_{m(k)}) \\ &< d(\omega_{n(k)},\omega_{n(k)-1}) + \epsilon. \end{aligned}$$

Taking the limit as $k \to \infty$ and applying (14), we get that

$$\lim_{k\to\infty} d(\omega_{n(k)},\omega_{m(k)}) = \epsilon. \tag{16}$$

Using the triangle inequlity, we have

$$\begin{aligned} \epsilon &\le d(\omega_{n(k)},\omega_{m(k)}) \\ &\le d(\omega_{n(k)},\omega_{m(k)+1}) + d(\omega_{n(k)+1},\omega_{m(k)}) \end{aligned}$$

and

$$\begin{aligned} &d(\omega_{n(k)},\omega_{m(k)+1}) \\ &\le d(\omega_{n(k)},\omega_{m(k)}) + d(\omega_{m(k)},\omega_{m(k)+1}). \end{aligned}$$

Again, by taking the limit as $k \to \infty$ and using (11), (12) and (13), we get

$$\lim_{k\to\infty} d(\omega_{n(k)},\omega_{m(k)+1}) = \epsilon. \tag{17}$$

Similarly, we obtain

$$\lim_{k\to\infty} d(\omega_{n(k)+1},\omega_{m(k)}) = \epsilon. \tag{18}$$

Also, we observe that

$$\begin{aligned} &d(\omega_{n(k)+1},\omega_{m(k)+1}) \\ &\le d(\omega_{n(k)+1},\omega_{m(k)}) + d(\omega_{m(k)},\omega_{m(k)+1}) \end{aligned}$$

and

$$\begin{aligned} &d(\omega_{n(k)+1},\omega_{m(k)+1}) \\ &\le d(\omega_{n(k)+1},\omega_{m(k)+1}) + d(\omega_{m(k)},\omega_{m(k)}). \end{aligned}$$

By taking the limit $k \to \infty$ and using (12), (13), (14) and (16), we get

$$\lim_{n\to\infty} d(\omega_{n(k)+1},\omega_{m(k)+1}) = \epsilon. \tag{19}$$

From (14) and (15) we can choose a positive integer $n_0 \ge 1$ such that

$$\begin{aligned} &\frac{1}{2}\left\{D(\omega_{n(k)},\Omega\omega_{n(k)}),D(\omega_{m(k)},\Lambda\omega_{m(k)})\right\} < \frac{\epsilon}{2} \\ &< d(\omega_{n(k)},\omega_{m(k)}) \end{aligned}$$

and consequently,

$$\lim_{k\to\infty} \Theta(\omega_{m(k)},\omega_{n(k)}) = \epsilon. \tag{20}$$

Since $\alpha(\omega_0,\Omega\omega_0) \ge 1$ and Ω,Λ are α-admissile, we get

$$\alpha(\omega_0,\omega_1) = \alpha(\omega_0,\Omega\omega_0) \ge 1.$$

By triangular α-admissile, we get

$$\alpha(\Omega\omega_0,\Lambda\omega_1) = \alpha(\omega_1,\omega_2) \ge 1$$

and

$$\alpha(\Lambda\Omega\omega_0, \Omega\Lambda\omega_1) = \alpha(\omega_2, \omega_3) \geq 1.$$

By proceeding the above process, we conclude that $\alpha(\omega_n, \omega_{n+1}) \geq 1$ for all n Now, we prove that $\alpha(\omega_n, \omega_{n+1}) \geq 1$, for all $m, n \in \mathbb{N}$ with $n < m$. Since

$$\begin{cases} \alpha(\omega_n, \omega_{n+1}) \geq 1, \\ \alpha(\omega_{n+1}, \omega_{n+2}) \geq 1, \end{cases}$$

then, we have

$$\alpha(\omega_n, \omega_{n+2}) \geq 1.$$

Again, since

$$\begin{cases} \alpha(\omega_n, \omega_{n+2}) \geq 1, \\ \alpha(\omega_{n+2}, \omega_{n+3}) \geq 1, \end{cases}$$

we deduce that

$$\alpha(\omega_n, \omega_{n+3}) \geq 1.$$

By proceeding this process, we have

$$\alpha(\omega_n, \omega_m) \geq 1$$

for all $m, n \in \mathbb{N}$ with $m > n$. Let $\omega = \omega_{m(k)}, \rho = \omega_{n(k)}$. from above we obtain $\alpha(\omega_n, \omega_m) \geq 1$. Then by 2.1,

$$\begin{aligned} \mathcal{C}_\mathcal{G} &\leq \zeta(₤_{m(k)}, \beta(₩_{m(k)})₩_{m(k)}) \\ &< \mathcal{G}(\beta(₩_{m(k)})₩_{m(k)}, ₤_{m(k)}), \end{aligned}$$

where $₤_{m(k)} = \alpha(\omega_{m(k)}, \omega_{n(k)})\mathcal{H}(\Omega\omega_{m(k)}, \Lambda\omega_{n(k)})$ and $₩_{m(k)} = \Theta(\omega_{m(k)}, \omega_{n(k)}) + \mathcal{L}\Psi(\omega_{m(k)}, \omega_{n(k)})$. Here $\Theta(\omega_{m(k)}, \omega_{n(k)}) = d(\omega_{m(k)}, \omega_{n(k)})$, by $(\mathcal{G}_1)$, we get

$$\begin{aligned} &d(\omega_{m(k)}, \omega_{n(k)}) \\ &\leq ₤_{m(k)} \\ &< \beta(₩_{m(k)})₩_{m(k)} \\ &< ₩_{m(k)} \\ &= d(\omega_{m(k)}, \omega_{n(k)}) + \mathcal{L}\Psi(\omega_{m(k)}, \omega_{n(k)}). \end{aligned} \tag{21}$$

Using (16), (15) and $\lim_{n\to\infty} \Psi(\omega_{m(k)}, \omega_{n(k)}) = 0$ in (21), we get

$$\lim_{k\to\infty} \alpha(\omega_{m(k)}, \omega_{n(k)})\mathcal{H}(\Omega\omega_{m(k)}, \Lambda\omega_{n(k)}) = \epsilon,$$

and

$$\lim_{k\to\infty} \beta(₩_{m(k)})₩_{m(k)} = \epsilon,$$

where $₩_{m(k)} = \Theta(\omega_{m(k)}, \omega_{n(k)}) + \mathcal{L}\Psi(\omega_{m(k)}, \omega_{n(k)})$. Therefore using (3.1) and (ζ_2) of Definition 2.2, putting $₤_{m(k)} = \alpha(\omega_{m(k)}, \omega_{n(k)})\mathcal{H}(\Omega\omega_{m(k)}, \Lambda\omega_{n(k)})$ and $₩_{m(k)} = \Theta(\omega_{m(k)}, \omega_{n(k)}) + \mathcal{L}\Psi(\omega_{m(k)}, \omega_{n(k)})$, we get

$$\mathcal{C}_\mathcal{G} \leq \zeta(₤_{m(k)}, \beta(₩_{m(k)})₩_{m(k)}) < \mathcal{C}_\mathcal{G},$$

which is a contradiction. As a result, $\{\omega_n\}$ is a Cauchy sequence. Because Υ is complete, we can guarantee that $\{\omega_n\}$ convergence to some $\omega^* \in \Upsilon$, i.e.,

$$\lim_{n\to\infty} d(\omega_n, \omega^*) = 0$$

and so

$$\begin{aligned} \lim_{n\to\infty} d(\omega_n, \omega^*) &= \lim_{n\to\infty} d(\omega_{2n}, \omega^*) \\ &= \lim_{n\to\infty} d(\omega_{2n+1}, \omega^*) = 0. \end{aligned} \tag{22}$$

We now assert that

$$\frac{1}{2} \min \{D(\omega_n, \Omega\omega_n), D(\omega^*, \Lambda\omega^*)\} < d(\omega_n, \omega^*)$$

or

$$\begin{aligned} &\frac{1}{2} \min \{D(\omega^*, \Omega\omega^*), D(\omega_{n+1}, \Lambda\omega_{n+1})\} \\ &< d(\omega^*, \omega_{n+1}) \end{aligned} \tag{23}$$

for all $n \in \mathbb{N}$. Suppose that it is not the case. Then there exist $m \in \mathbb{N}$ such that

$$\frac{1}{2} \min \{D(\omega_m, \Omega\omega_m), D(\omega^*, \Lambda\omega^*)\} \geq d(\omega_m, \omega^*) \tag{24}$$

and

$$\begin{aligned} &\frac{1}{2} \min \{D(\omega^*, \Omega\omega^*), D(\omega_{m+1}, \Lambda\omega_{m+1})\} \\ &\geq d(\omega^*, \omega_{m+1}). \end{aligned} \tag{25}$$

Therefore

$$\begin{aligned} &2d(\omega_m, \omega^*) \\ &\leq \min \{D(\omega_m, \Omega\omega_m), D(\omega^*, \Lambda\omega^*)\} \\ &\leq \min \{d(\omega_m, \omega^*) + D(\omega^*, \Omega\omega_m), D(\omega^*, \Lambda\omega^*)\} \\ &\leq d(\omega_m, \omega^*) + D(\omega^*, \Omega\omega_m) \\ &\leq d(\omega_m, \omega^*) + d(\omega^*, \omega_{m+1}), \end{aligned}$$

which implies that

$$d(\omega_m, \omega^*) \leq d(\omega^*, d\omega_{m+1}). \tag{26}$$

From (23) and (24)

$$\begin{aligned} &d(\omega_m, \omega^*) \\ &\leq d(\omega_{m+1}, \omega^*) \\ &\leq \frac{1}{2} \min \{D(\omega^*, \Omega\omega^*), D(\omega_{m+1}, \Lambda\omega_{m+1})\}. \end{aligned} \tag{27}$$

Since $\frac{1}{2}\min\{D(\omega_m,\Omega\omega_m),D(\omega^*,\Lambda\omega^*)\} < d(\omega_m,\omega_{m+1})$, from (2) we have

$$\begin{aligned}\mathcal{C}_{\mathcal{G}} &\leq \zeta(£_m,\beta(₩_m)₩_m)\\ &< \mathcal{G}(\beta(₩_m)₩_m,£_m),\end{aligned}$$

where $£_m = \alpha(\omega_m,\omega_{m+1})\mathcal{H}(\Omega\omega_m,\Lambda\omega_{m+1})$ and $₩_m = \Theta(\omega_m,\omega_{m+1})+\mathcal{L}\Psi(\omega_m,\omega_{m+1})$.
Consequently, we get

$$d(\omega_{m+1},\omega_{m+2}) \leq £_m < \beta(₩_m)₩_m < ₩_m, \quad (28)$$

where

$$\begin{aligned}&\Theta(\omega_m,\omega_{m+1})\\ &= \max\Big\{d(\omega_m,\omega_{m+1})D(\omega_m,\Omega\omega_m),\\ &\quad D(\omega_{m+1},\Lambda\omega_{m+1}),\\ &\quad \frac{D(\omega_m,\Lambda\omega_{m+1})+D(\omega_{m+1},\Omega\omega_m)}{2}\Big\}\\ &\leq \max\Big\{d(\omega_m,\omega_{m+1}),d(\omega_m,\omega_{m+1}),\\ &\quad d(\omega_{m+1},\omega_{m+2}),\\ &\quad \frac{d(\omega_m,\omega_{m+2})+d(\omega_{m+1},\omega_{m+1})}{2}\Big\}\\ &= \max\Big\{d(\omega_m,\omega_{m+1}),d(\omega_{m+1},\omega_{m+2}),\\ &\quad \frac{d(\omega_m,\omega_{m+2})}{2}\Big\}.\end{aligned}$$

Since

$$\begin{aligned}\frac{d(\omega_m,\omega_{m+2})}{2} &\leq \frac{d(\omega_m,\omega_{m+1})+d(\omega_{m+1},\omega_{m+2})}{2}\\ &\leq \max\{d(\omega_m,\omega_{m+1}),d(\omega_{m+1},\omega_{m+2})\}.\end{aligned}$$

Thus,

$$\Theta(\omega_m,\omega_{m+1}) \leq \max\{d(\omega_m,\omega_{m+1}),d(\omega_{m+1},\omega_{m+2})\}.$$

Also,

$$\Psi(\omega_m,\omega_{m+1})=0.$$

Suppose that $\max\{d(\omega_m,\omega_{m+1}),d(\omega_{m+1},\omega_{m+2})\} = d(\omega_{m+1},\omega_{m+2})$, then from (28) we have

$$d(\omega_{m+1},\omega_{m+2}) < d(\omega_{m+1},\omega_{m+2}),$$

which is a contradiction. Thus we conclude that

$$\max\{d(\omega_m,\omega_{m+1}),d(\omega_{m+1},\omega_{m+2})\} = d(\omega_m,\omega_{m+1}).$$

From (26) we get that

$$d(\omega_{m+1},\omega_{m+2}) < d(\omega_m,\omega_{m+1}). \quad (29)$$

From (27), (28) and (29), we get

$$\begin{aligned}&d(\omega_{m+1},\omega_{m+2})\\ &< d(\omega_m,\omega_{m+1})\\ &\leq d(\omega_m,\omega^*)+d(\omega^*,\omega_{m+1})\\ &\leq \frac{1}{2}\min\{D(\omega^*,\Omega\omega^*),D(\omega_{m+1},\Lambda\omega_{m+1})\}\\ &\quad+\frac{1}{2}\min\{D(\omega^*,\Omega\omega^*),D(\omega_{m+1},\Lambda\omega_{m+1})\}\\ &= \min\{D(\omega^*,\Omega\omega^*),D(\omega_{m+1},\Lambda\omega_{m+1})\}\\ &\leq d(\omega_{m+1},\omega_{m+2}),\end{aligned}$$

which is a contradiction. Hence (25) holds, i.e., for every $n \geq 2$

$$\frac{1}{2}\min\{D(\omega_n,\Omega\omega_n),D(\omega^*,\Lambda\omega^*) < d(\omega_n,\omega^*)\}$$

holds. Hence from (3)

$$\begin{aligned}\mathcal{C}_{\mathcal{G}} &\leq \zeta(£_n,\beta(₩_n)₩_n)\\ &< \mathcal{G}(\beta(₩_n)₩_n,£_n),\end{aligned} \quad (30)$$

where $£_n = \alpha(\omega_n,\omega^*)\mathcal{H}(\Omega\omega_n,\Lambda\omega^*)$ and $₩_n = \Theta(\omega_n,\omega^*)+\mathcal{L}\Psi(\omega_n,\omega^*)$.
Consequently, we get

$$D(\omega_{n+1},\Lambda\omega^*) \leq £_n < ₩_n, \quad (31)$$

where

$$\begin{aligned}&\Theta(\omega_n,\omega^*)\\ &= \max\Big\{d(\omega_m,\omega^*)D(\omega_n,\Omega\omega_n),D(\omega^*,\Lambda\omega^*),\\ &\quad \frac{D(\omega_n,\Lambda\omega^*)+D(\omega^*,\Omega\omega_n)}{2}\Big\}\\ &\leq \max\Big\{d(\omega_n,\omega^*),d(\omega_n,\omega_{n+1}),D(\omega^*,\Lambda\omega^*),\\ &\quad \frac{D(\omega_n,\Lambda\omega^*)+d(\omega^*,\omega_{n+1})}{2}\Big\}\end{aligned}$$

and

$$\begin{aligned}&\Psi(\omega_n,\omega^*)\\ &= \min\{D(\omega_n,\Omega\omega_n),D(\omega^*,\Lambda\omega^*),\\ &\quad D(\omega_n,\Lambda\omega^*),D(\omega^*,\Omega\omega_n)\}\\ &= \min\{d(\omega_n,\omega_{n+1}),D(\omega^*,\Lambda\omega^*),\\ &\quad D(\omega_n,\Lambda\omega^*),D(\omega^*,\Omega\omega_n)\}.\end{aligned}$$

Letting $n\to\infty$ and by using (14) and (22), we obtain

$$\begin{aligned}&\lim_{n\to\infty}\Theta(\omega_n,\omega^*) = D(\omega^*,\Lambda\omega^*),\\ &\lim_{n\to\infty}\Psi(\omega_n,\omega^*) = 0.\end{aligned} \quad (32)$$

Now we show that $\omega^* \in \Lambda\omega^*$. Suppose, on the other hand, that $D(\omega^*,\Lambda\omega^*) > 0$. By allowing $n\to\infty$ in

(31), we obtain

$$\begin{aligned}
&D(\omega^*, \Lambda\omega^*)\\
&= \lim_{n\to\infty} D(\omega_{n+1}, \Lambda\omega^*)\\
&\leq \lim_{n\to\infty} \alpha(\omega_n, \omega^*)\mathcal{H}(\Omega\omega_n, \Lambda\omega^*)\\
&< \lim_{n\to\infty} \Theta(\omega_n, \omega^*) + \mathcal{L} \lim_{n\to\infty} \Psi(\omega_n, \omega^*)\\
&= D(\omega^*, \Lambda\omega^*),
\end{aligned}$$

which is a contradiction. Therefore $\omega^* \in \Lambda\omega^*$. Similarly, we can show that $\omega^* \in \Omega\omega^*$. Thus Ω and Λ have a common fixed point. □

Corollary 3.4. *Let (Υ, d) be a complete metric space and $\Omega : \Upsilon \to \mathcal{CB}(\Upsilon)$ be a generalized multivalued Suzuki type $\mathbb{Z}$-contraction with respect to ζ, i.e.,*

$$\begin{aligned}
&\frac{1}{2}\min\{D(\omega, \Omega\omega), D(\rho, \Lambda\rho)\} < d(\omega, \rho)\\
&\Rightarrow \zeta(\mathcal{H}(\Omega\omega, \Lambda\rho), \Theta(\omega, \rho)) \geq 0 \quad \text{for all } \omega, \rho \in \Upsilon,
\end{aligned}$$

where

$$\Theta(\omega, \rho) = \max\Big\{d(\omega, \rho), D(\omega, \Omega\omega), D(\rho, \Lambda\rho), \frac{D(\omega, \Lambda\rho) + D(\rho, \Omega\omega)}{2}\Big\}.$$

Then Ω and Λ have a common fixed point.

Proof. The proof follows from Theorem 3.3 by taking $\alpha(\omega, \rho) = 1$, $\beta(v) = v$ and $\Psi(\omega, \rho) = 0$. □

Example 3.5. *Let $\Upsilon = \{0, 3, 5\}$ be endowed with the usual metric. Let $\Omega, \Lambda : \Upsilon \to \mathcal{CB}(\Upsilon)$ be defined by*

$$\Omega\omega = \begin{cases} \{\frac{\omega}{7}\} & if \quad \omega \in \{0, 5\} \\ \{0, \frac{1}{7}\} & if \quad \omega = 3, \end{cases}$$

and $\Lambda\omega = \{\frac{\omega}{5}\}$ for all $\omega \in \Upsilon$.
We now define $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\zeta(v, u) = \frac{6}{7}u - v$ for all $u, v \in [0, \infty)]$. We can now confirm the inequality (2) for all $\omega, \rho \in \Upsilon$ with $\Omega\omega \neq \Lambda\rho$. Note that for all $\omega, \rho \in \Upsilon$ with $\Omega\omega \neq \Lambda\rho$ the inequality $\frac{1}{2}\min\{D(\omega, \Omega\omega), D(\omega, \Lambda\omega)\} < d(\omega, \rho)$ gives

$$(\omega, \rho) \in \{(0, 3), (3, 0), (0, 5), (5, 0), (3, 5), (5, 3)\}.$$

Then from (2), we have

$$\zeta(\mathcal{H}(\Omega\omega, \Lambda\rho), \Theta(\omega, \rho)) = \frac{6}{7}\Theta(\omega, \rho) - \mathcal{H}(\Omega\omega, \Lambda\rho) \geq 0.$$

That implies that

$$\mathcal{H}(\Omega\omega, \Lambda\rho) \leq \frac{6}{7}\Theta(\omega, \rho).$$

Case (i) for $\omega = 0, \rho = 3$;

$$\mathcal{H}(\Omega 0, \Lambda 3) = \mathcal{H}(\{0\}, \Big\{\frac{3}{5}\Big\}) = \frac{3}{5} \leq \frac{6}{7}\Theta(0, 3).$$

Case (ii) for $\omega = 3, \rho = 0$;

$$\mathcal{H}(\Omega 3, \Lambda 0) = \mathcal{H}(\Big\{0, \frac{1}{7}\Big\}, \{0\}) = \frac{1}{7} \leq \frac{6}{7}\Theta(3, 0).$$

Case (iii) for $\omega = 0, \rho = 5$;

$$\mathcal{H}(\Omega 0, \Lambda 5) = \mathcal{H}(\{0\}, \{1\}) = 1 \leq \frac{6}{7}\Theta(0, 5).$$

Case (iv) for $\omega = 5, \rho = 0$;

$$\mathcal{H}(\Omega 5, \Lambda 0) = \mathcal{H}(\Big\{\frac{5}{7}\Big\}, \{0\}) = \frac{5}{7} \leq \frac{6}{7}\Theta(5, 0).$$

Case (v) for $\omega = 3, \rho = 5$;

$$\mathcal{H}(\Omega 3, \Lambda 5) = \mathcal{H}(\Big\{0, \frac{1}{7}\Big\}, \{1\}) = 1 \leq \frac{6}{7}\Theta(3, 5).$$

Case (v) for $\omega = 5, \rho = 3$;

$$\mathcal{H}(\Omega 5, \Lambda 3) = \mathcal{H}(\Big\{\frac{5}{7}\Big\}, \Big\{\frac{3}{5}\Big\}) = \frac{5}{7} \leq \frac{6}{7}\Theta(5, 3).$$

That all of the hypotheses in Corollary 3.4 are met. As a result, 0 is a common fixed point owned by Ω and Λ.

Corollary 3.6. *Let (Υ, d) be a complete metric space and $\Omega : \Upsilon \to \mathcal{CB}(\Upsilon)$ be a generalized multivalued Suzuki type $\mathbb{Z}$-contraction with respect to ζ, i.e.,*

$$\begin{aligned}
&\frac{1}{2}D(\omega, \Omega\omega) < d(\omega, \rho)\\
&\Rightarrow \zeta(\mathcal{H}(\Omega\omega, \Omega\rho), \Theta(\omega, \rho)) \geq 0,
\end{aligned} \tag{33}$$

for all $\omega, \rho \in \Upsilon$ with $\omega \neq \rho$, where

$$\Theta(\omega, \rho) = \max\Big\{d(\omega, \rho), D(\omega, \Omega\omega), D(\rho, \Omega\rho), \frac{D(\omega, \Omega\rho) + D(\rho, \Omega\omega)}{2}\Big\}.$$

Then Ω has a fixed point $\omega^* \in \Upsilon$ and for $\omega \in \Upsilon$ the sequence $\{\Omega^n\omega\}$ convergences to ω^*.

Proof. The proof follows from Theorem 3.3 by taking $\Omega = \Lambda$. □

4 Conclusion

Despite its novel applications, the search for fixed point theorems involving contraction type conditions has received much interest in recent decades. In this context, we analyzed convergence point results for such mappings and illustrative for support theorem based on the new idea of Suzuki type $\mathbb{Z}$-contraction mappings obeying an admissibility type condition in generalized metric spaces via the concept of $\mathcal{C}$-functions.

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