Nonlenear Integro-differential Equations and Splines of the Fifth Order of Approximation

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Abstract: - In this paper, we consider the solution of nonlinear Volterra–Fredholm integro-differential equation, which contains the first derivative of the function. Our method transforms the nonlinear Volterra-Fredholm integro-differential equations into a system of nonlinear algebraic equations. The method based on the application of the local polynomial splines of the fifth order of approximation is proposed.
Theorems about the errors of the approximation of a function and its first derivative by these splines are given. With the help of the proposed splines, the function and the derivative are replaced by the corresponding approximation. Note that at the beginning, in the middle and at the end of the interval of the definition of the integro-differential equation, the corresponding types of splines are used: the left, the right or the middle splines of the fifth order of approximation. When using the spline approximations, we also obtain the corresponding formulas for numerical differentiation. which we also apply for the solution of integro-differential equations. The formulas for approximation of the function and its derivative are presented. The results of the numerical solution of several integro-differential equations are presented. The proposed method is shown that it can be applied to solve integro-differential equations containing the second derivative of the solution.

Key-Words: - Nonlinear Volterra–Fredholm integro-differential equations, polynomial splines, fifth order of approximation

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1 Introduction

As is known, one of the creators of the theory of integral and integro-differential equations is V. Volterra. His works are relevant to this day. The theory of integro-differential equations is most fully discussed in the works of Volterra himself [1]. Volterra first began to study integral equations in 1884 (see [2]). This work is devoted to the distribution of electric charge on a spherical segment. Volterra showed that this problem leads (in modern terms) to the solution of an integral equation of the first kind with a symmetric kernel. Volterra's first work on integro-differential equations was a work on the theory of elasticity. As is known, integro-differential equations connect an unknown function and its (private) derivatives. Integro-differential equations arise in various branches of mathematical physics. For example, under certain conditions, the electric or magnetic polarization depends not only on the electromagnetic field at a given moment, but also on the history of the electromagnetic field of the substance at all previous moments (hysteresis)[3]. Methods of solving of integral equations is considered in books [4], [5].

As noted in paper [6], "integral equations have been one of the principal tools in various areas of applied mathematics, physics and engineering. Scientists have investigated the topic of integro-differential equations through their work in many scientific applications such as heat transfer, the diffusion process in general, and neutron diffusion and biological species coexist together with increasing and decreasing rates of generating". The nonlinear Volterra–Fredholm integro-differential equations arise in neurosciences. Paper [7] extends the results of the synaptically generated wave propagation through a network of connected excitatory neurons to a continuous model, defined by a Volterra-Fredholm integro-differential equation, which includes memory effects of the past in the propagation. In paper [8], an effective direct method to determine the numerical solution of the specific nonlinear Volterra–Fredholm integro-differential equations is proposed. The method is based on new vector forms for the representation of triangular functions and its operational matrix. In paper [9], the new schemes are developed derived on the hybrid of the three-point half-sweep linear rational finite difference approaches with the half-sweep composite trapezoidal approach.In paper [10], the numerical solution of periodic Fredholm–Volterra integro–differential equations of first-order is discussed in a reproducing kernel Hilbert space. A new O(n) time complexity numerical method for computing the solutions of Basset integro-differential equations is presented in paper [11]. A new class of two-step collocation methods for the numerical solution of Volterra integro-differential equations is proposed in [12]. The approach, proposed in paper [13], is based on Galerkin formulation and Legendre polynomials. In paper [14], the Chebyshev pseudo-spectral method to solve the pattern nonlinear second order systems of Fredholm integro-differential equations is used.

Polynomial local splines of the fifth order of approximation have proven themselves well in solving interpolation problems, solving boundary value problems and solving Fredholm and Volterra integral equations [15]. In this paper, we will consider the solution of integro-differential equations from papers [6] and [9] using polynomial local splines of the fifth order of approximation. This method transforms the nonlinear Volterra-Fredholm integro-differential equations into a system of nonlinear algebraic equations.

In this paper, we will consider the nonlenear integro-differential equations of the form

$$u'(x)+\alpha_1\int_0^x K_1(x,s)F\big(u(s),u'(s)\big)ds$$

$$+\alpha_2\int_0^1 K_2(x,s)G(u(s),u'(s))ds = f(x),$$

$|\alpha_1| + |\alpha_2| \neq 0,\ x \in [0,1].$

In section 2, we consider the properties of splines of the fifth order of approximation. In section 3, we consider the solution of integro-differential equations using splines of the fifth order of approximation

2 Approximation with the Local Splines of the Fifth Order of Approximation

The general theory of constructing local interpolation splines is considered in the monograph by prof. Yu.K. Dem'yanovich and I. G. Burova. Let a, b be real and n be an integer. Let the values of the function $u(x)$ be known at the nodes of the grid $\{t_i\}$: $a = t_0 < t_1 < \cdots < t_n = b$. Approximation with the local splines of the fifth order of approximation is built separately on each grid interval $[t_i, t_{i+1}]$.

Denote $u_i = u(t_i)$. At the beginning of the interval $[a, b]$, we apply the approximation with the right splines:

$$U_{R4}^i(x) = \sum_{j=i}^{i+4} u_j w_j(x),\ x \in [t_i, t_{i+1}],$$

where u_j, $j = 0, \ldots, n$, are the values of the function in nodes t_j the basis splines $w_i(x)$ are the next:

$$w_i(x) = \frac{(x-t_{i+1})(x-t_{i+2})(x-t_{i+3})(x-t_{i+4})}{(t_i-t_{i+1})(t_i-t_{i+2})(t_i-t_{i+3})(t_i-t_{i+4})},$$

$$w_{i+1}(x) = \frac{(x-t_i)(x-t_{i+2})(x-t_{i+3})(x-t_{i+4})}{(t_{i+1}-t_i)(t_{i+1}-t_{i+2})(t_{i+1}-t_{i+3})(t_{i+1}-t_{i+4})},$$

$$w_{i+2}(x) = \frac{(x-t_i)(x-t_{i+1})(x-t_{i+3})(x-t_{i+4})}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})(t_{i+2}-t_{i+3})(t_{i+2}-t_{i+4})},$$

$$w_{i+3}(x) = \frac{(x-t_i)(x-t_{i+1})(x-t_{i+2})(x-t_{i+4})}{(t_{i+3}-t_i)(t_{i+3}-t_{i+1})(t_{i+3}-t_{i+2})(t_{i+3}-t_{i+4})},$$

$$w_{i+4}(x) = \frac{(x-t_i)(x-t_{i+1})(x-t_{i+2})(x-t_{i+3})}{(t_{i+4}-t_i)(t_{i+4}-t_{i+1})(t_{i+4}-t_{i+2})(t_{i+4}-t_{i+3})}.$$

In the middle of the interval $[a, b]$, we apply the approximation with the middle splines:

$U_{S4}^i(x) = \sum_{j=i-2}^{i+2} u_j w_j^s(x),\ x \in [t_i, t_{i+1}],$

where

$$w_{i-2}^s(x) = \frac{(x-t_{i-1})(x-t_i)(x-t_{i+1})(x-t_{i+2})}{(t_{i-2}-t_{i-1})(t_{i-2}-t_i)(t_{i-2}-t_{i+1})(t_{i-2}-t_{i+2})},$$

$$w_{i-1}^s(x) = \frac{(x-t_{i-2})(x-t_i)(x-t_{i+1})(x-t_{i+2})}{(t_{i-1}-t_{i-2})(t_{i-1}-t_i)(t_{i-1}-t_{i+1})(t_{i-1}-t_{i+2})},$$

$$w_i^s(x) = \frac{(x-t_{i-2})(x-t_{i-1})(x-t_{i+1})(x-t_{i+2})}{(t_i-t_{i-2})(t_i-t_{i-1})(t_i-t_{i+1})(t_i-t_{i+2})},$$

$$w_{i+1}^s(x) = \frac{(x-t_{i-2})(x-t_{i-1})(x-t_i)(x-t_{i+2})}{(t_{i+1}-t_{i-2})(t_{i+1}-t_{i-1})(t_{i+1}-t_i)(t_i-t_{i+2})},$$

$$w_{i+2}^s(x) = \frac{(x-t_{i-2})(x-t_{i-1})(x-t_i)(x-t_{i+1})}{(t_{i+2}-t_{i-2})(t_{i+2}-t_{i-1})(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}.$$

At the end of the interval $[a,b]$, we apply the approximation with the right splines:

$$U_{L4}^i(x) = \sum_{j=i-3}^{i+1} u_j w_j(t),\ t \in [t_i, t_{i+1}],$$

where the basis splines are the following:

$$w_{i-3}(x) = \frac{(x-t_{i-2})(x-t_{i-1})(x-t_i)(x-t_{i+1})}{(t_{i-3}-t_{i-2})(t_{i-3}-t_{i-1})(t_{i-3}-t_i)(t_{i-3}-t_{i+1})},$$

$$w_{i-2}(x) = \frac{(x-t_{i-3})(x-t_{i-1})(x-t_i)(x-t_{i+1})}{(t_{i-2}-t_{i-3})(t_{i-2}-t_{i-1})(t_{i-2}-t_i)(t_{i-2}-t_{i+1})},$$

$$w_{i-1}(x) = \frac{(x-t_{i-3})(x-t_{i-2})(x-t_i)(x-t_{i+1})}{(t_{i-1}-t_{i-3})(t_{i-1}-t_{i-2})(t_{i-1}-t_i)(t_{i-1}-t_{i+1})},$$

$$w_i(x) = \frac{(x-t_{i-3})(x-t_{i-2})(x-t_{i-1})(x-t_{i+1})}{(t_i-t_{i-3})(t_i-t_{i-2})(t_i-t_{i-1})(t_i-t_{i+1})},$$

$$w_{i+1}(x) = \frac{(x-t_{i-3})(x-t_{i-2})(x-t_{i-1})(x-t_i)}{(t_{i+1}-t_{i-3})(t_{i+1}-t_{i-2})(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}.$$

Applying these formulas, it is possible to approximate the first derivatives of a function $u(x)$. In this case we use the same values of the function at the grid nodes and derivatives from the basic splines. On a uniform grid of nodes with step h we construct the approximation of the first derivative of function u in the form:

$$(U_{R4}^i(x))' = \sum_{j=i}^{i+4} u_j w_j{}'(x),\ x \in [t_i, t_{i+1}],$$

where $t \in [0,1]$,

$$w'_j(x_j + th) = \frac{2t^3 - 15t^2 + 35t - 25}{12\,h},$$

$$w'_{j+1}(x_j + th) = \frac{27t^2 - 4t^3 - 52t + 24}{6\,h},$$

$$w'_{j+2}(x_j + th) = \frac{2t^3 - 12t^2 + 19t - 6}{2\,h},$$

$$w'_{j+3}(x_j + th) = \frac{-4t^3 + 21t^2 - 28t + 8}{2\,h},$$

$$w'_{j+4}(x_j + th) = \frac{2t^3 - 9t^2 + 11t - 3}{12\,h}.$$

First of all, we formulate and prove an approximation theorem, which is necessary to determine the error in the solution of the considered integro-differential equation.

Denote

$x_0 = t_i$, $x_1 = t_{i+1}$, $x_2 = t_{i+2}$, $x_3 = t_{i+3}$, $x_4 = t_{i+4}$.

It is known that for the fourth-degree interpolation polynomial $P_4(x)$ constructed from the nodes $x_0, x_1, \dots, x_4$ the next relation is valid:

$$u(x) - P_4(x) = u[x, x_0, x_1, \dots, x_4](x-x_0)\dots(x-x_4)\ . \quad (1)$$

Here we use the standard notation for the fifth-order divided difference for the function $u(x)$.

The divided difference $u[x, x_0, x_1, \dots, x_4]$ has the form:

$$u[x, x_0, x_1, \dots, x_4] = \int_0^1 \int_0^{z_0} \dots \int_0^{z_3} u^{(5)}[x_0 + (x_1 - x_0)z_0 + \dots + (x_4 - x_3)z_3 + (x - x_4)z_4] z_4 dz_0 \dots dz_3 dz_4\ . \quad (2)$$

Next, consider the question of estimating the difference of derivatives $u'(x) - P'_4(x)$.

Theorem 1. The next inequality is valid:

$$|u'(x) - P'_4(x)| \le \{2\,|(x-x_0)\dots(x-x_4)| + |\{(x-x_0)\dots(x-x_4)\}'|\} \max_{\tau\in[x_0,x_4]} |u^{(5)}(\tau)|/5!\ (3)$$

Proof. Differentiating identity (1) we have the relation:

$$u'[x, x_0, x_1, \dots, x_4] = \\ = \int_0^1 \int_0^{z_0} \dots \int_0^{z_3} u^{(6)}[\, x_0 + (x_1 - x_0)z_0 + \dots + (x_4 - x_3)z_3 \\ +(x - x_4)z_4]z_4 dz_0 \dots dz_3 dz_4 \; . \quad (4)$$

Since the absolute value of the integral does not exceed the integral of the absolute value of the integrand, we obtain the inequality:

$$|u'[x, x_0, x_1, \dots, x_4]| \le \max_{\tau \in L\{x, x_0, \dots, x_4\}} |u^{(6)}(\tau)| \; . \; (5)$$

Here $L\{x, x_0, \dots x_4\}$ means the smallest segment containing the points $x, x_0, \dots x_4$. Let us calculate the integral on the right side of relation (4) over z_3. Assuming to integrate by parts, we first write down the obvious equality

$$J_4 = \int_0^{z_3} u^{(6)}\,[x_0 + (x_1 - x_0)z_0 + \\ \dots + (x_4 - x_3)z_3 + (x - x_4)z_4]z_4 dz_4 \\ = \int_0^{z_3} z_4 du^{(5)}\,[x_0 + (x_1 - x_0)z_0 + \cdots \\ (x_4 - x_3)z_3 + (x - x_4)z_4](x - x_4)^{-1} \, . \, (6)$$

By integrating in parts, we get the equality

$$J_4 = \{z_3 u^{(5)}[x_0 + (x_1 - x_0)z_0 + \\ \dots + (x_4 - x_3)z_3 + (x - x_4)z_4] \Big|_{z_4 = 0}^{z_4 = z_3} \\ - \int_0^{z_3} u^{(5)}\,[x_0 + (x_1 - x_0)z_0 + \\ \dots + (x_4 - x_3)z_3 + (x - x_4)z_4]dz_4\}(x - x_4)^{-1} \\ = \left\{z_3 u^{(5)}[x_0 + (x_1 - x_0)z_0 + \cdots + (x - x_3)z_3\right] - \\ - \int_0^{z_3} u^{(5)}\,[x_0 + (x_1 - x_0)z_0 + \cdots \\ +(x - x_4)z_4]dz_4\}(x - x_4)^{-1} \, . \, (7)$$

By relations (4) and (7) we deduce the formula

$$u'[x, x_0, x_1, \dots, x_4] = \\ \{\int_0^1 \int_0^{z_0} \dots \int_0^{z_2} z_3\, u^{(5)}[\, x_0 + (x_1 - x_0)z_0 + \\ \dots + (x - x_3)z_3]dz_0 \dots dz_3 \\ - \int_0^1 \int_0^{z_0} \dots \int_0^{z_3} u^{(5)}[\, x_0 + (x_1 - x_0)z_0 + \\ +(x - x_4)z_4]dz_0 \dots dz_4\}(x - x_4)^{-1}. \quad (8)$$

We assume that the nodes are ordered:

$$x_0 \; < \; x_1 \; < \; x_0 \; < \dots < x_4, x \; \in \; [x_0, x_4). \, (9)$$

Note that the following equality is true

$$\int_0^1 \int_0^{z_0} \dots \int_0^{z_2} z_3 dz_0 \dots dz_3 = \\ = \int_0^1 \int_0^{z_0} \dots \int_0^{z_3} dz_0 \dots dz_3 dz_4 = 1/5! \; (10)$$

Let us estimate the first term of relation (8). We take out the maximum of the absolute value of the function $u^{(5)}(\tau)$ from the expression under the integrals and then use equality (10). As a result, we get the inequality:

$$|\int_0^1 \int_0^{z_0} \dots \int_0^{z_2} z_3\, u^{(5)}[\, x_0 + (x_1 - x_0)z_0 + \\ \dots + (x - x_3)z_3]dz_0 \dots dz_3| \le 1/5! \max_{\tau \in [x_0, x_4]} \left|u^{(5)}(\tau)\right|.$$

A similar estimate is obtained for the second term in (8).

$$|\int_0^1 \int_0^{z_0} \dots \int_0^{z_3} u^{(5)}[\, x_0 + (x_1 - x_0)z_0 + \\ +(x - x_4)z_4]dz_0 \dots dz_4| \le 1/5! \max_{\tau \in [x_0, x_4]} \left|u^{(5)}(\tau)\right|.$$

Finally, we deduce the next inequality from formula (8) using condition (9):

$$|u'[x, x_0, x_1, \dots, x_4]|$$

$$\leq 2/5! \max_{\tau\in[x_0,x_4]} |u^{(5)}(\tau)||(x - x_4)^{-1}|. \quad (11)$$

After differentiating (1), we have the next equality:

$$u'(x) - P'_4(x) =$$

$$= u'[x, x_0, x_1, \dots, x_4](x - x_0) \dots (x - x_4)$$

$$+u[x, x_0, x_1, \dots, x_4]\{(x - x_0) \dots (x - x_4)\}'. \quad (12)$$

Now let us take into account the well-known relation. Namely, for some point ξ ∈ $[x_0, x_4]$, the next equality is valid:

$$u[x, x_0, x_1, \dots, x_4] = 1/5!\, u^{(5)}(\tau)\,,\ \tau \in [x_0, x_4].$$

Since by assumption $x \in [x_0, x_4]$, this implies the inequality:

$$|u[x, x_0, x_1, \dots, x_4]| \leq 1/5! \max_{\tau\in[x_0,x_4]} |u^{(5)}(\tau)|\,. \quad (13)$$

From formula (12) with the help of relations (11) and (13) we obtain the estimate:

$$|u'(x) - P'_4(x)| \leq$$

$$\leq |u'[x, x_0, x_1, \dots, x_4]||(x - x_0) \dots (x - x_4)|$$

$$+|u[x, x_0, x_1, \dots, x_4]||\{(x - x_0) \dots (x - x_4)\}'| \leq$$

$$\leq 2/5! \max_{\tau\in[x_0,x_4]} |u^{(5)}(\tau)|\,|\{(x - x_0) \dots (x - x_4)\}|$$

$$+1/5! \times$$

$$\times \max_{\tau\in[x_0,x_4]} |u^{(5)}(\tau)|\,|\{(x - x_0) \dots (x - x_4)\}'| \quad (14).$$

From inequality (14) we have relation (3).

The proof of Theorem 2 is complete.

Remark. Theorem 2 is proved when there is a non-uniform grid of nodes.

On a more detailed note, we have

$$u(x) - P_4(x) =$$

$$= u[x, x_0, x_1, \dots, x_4](x - x_0) \dots (x - x_4).$$

Differentiating this equality, we get

$$u'(x) - (P_4(x))'$$

$$= u'[x, x_0, x_1, \dots, x_4](x - x_0) \dots$$

$$(x - x_4) + u[x, x_0, x_1, \dots, x_4]Q(x)\,,$$

$$Q(x) = (x - x_1) \dots (x - x_4) +$$

$$(x - x_0)(x - x_2) \dots (x - x_4) +$$

$$(x - x_0)(x - x_1)(x - x_3)(x - x_4) +$$

$$(x - x_0)(x - x_1)(x - x_2)(x - x_4)$$

$$+(x - x_0) \dots (x - x_3).$$

Now, consider the approximation by right splines on a uniform grid of knots. In the case of a uniform grid, we have $x_k = x_0 + kh$. Replacing $x = x_0 + th$, $t \in [0,1]$, in the expression Q we have:

$$Q(x) = h^4(5t^4 - 40t^3 + 105t^2 - 100t + 24)\,.$$

Now it is easy to obtain error estimates for the right splines when $x \in [t_i, t_{i+1}]$:

$$\left|u(x) - U^i_{R4}(x)\right| \leq 3.63\frac{h^5}{5!} \max_{\tau\in[t_i,t_{i+4}]} \left|u^{(5)}(\tau)\right|\,,$$

$$\left|u'(x) - (U^i_{R4})'(x)\right| \leq 24\frac{h^4}{5!} \max_{\tau\in[t_i,t_{i+4}]} \left|u^{(5)}(\tau)\right|\,.$$

Similarly, we can obtain error estimates for the middle splines. Now it is easy to obtain error estimates for the middle splines:

$$\left|u(x) - U^i_{S4}(x)\right| \leq 1.42\ \frac{h^5}{5!} \max_{\tau\in[t_{i-2},t_{i+2}]} \left|u^{(5)}(\tau)\right|\,,$$

$$\left|u'(x) - (U^i_{S4})'(x)\right| \leq 6\ \frac{h^4}{5!} \max_{\tau\in[t_{i-2},t_{i+2}]} \left|u^{(5)}(\tau)\right|.$$

Note that the inequalities turn into equalities on function $u = x^5$.

Table 1 shows the actual errors of approximation of functions and the first derivative of the functions when $h = 0.1$, $[a, b] = [0,1]$. Table 2 shows the theoretical errors of approximation of functions and the first derivative of the functions when $h = 0.1$, $[a, b] = [0,1]$.

The data presented in the Tables are consistent with the theoretical results formulated in the Theorems.

Table 1. The Actual Errors of Approximations

$u(x)$	Right splines		Middle splines	
	Errors of appr.of func.	Errors of appr.of deriv. func.	Errors of appr. of func.	Errors of appr. of deriv. func.
x^5	$0.363 \cdot 10^{-4}$	$0.240 \cdot 10^{-2}$	$0.142 \cdot 10^{-4}$	$0.590 \cdot 10^{-3}$
$\sin(x)$	$0.291 \cdot 10^{-6}$	$0.193 \cdot 10^{-4}$	$0.117 \cdot 10^{-6}$	$0.488 \cdot 10^{-5}$
$\cos(x) - \sin(x)$	$0.427 \cdot 10^{-6}$	$0.282 \cdot 10^{-4}$	$0.1669 \cdot 10^{-6}$	$0.694 \cdot 10^{-5}$

Table 2. The Theoretical Errors of Approximations

$u(x)$	Right splines		Middle splines	
	Errors of appr.of func.	Errors of appr. of deriv.	Errors of appr. of func.	Errors of appr. of func.
x^5	$0.363 \cdot 10^{-4}$	$0.240 \cdot 10^{-2}$	$0.142 \cdot 10^{-4}$	$0.590 \cdot 10^{-3}$
$\sin(x)$	$0.3025 \cdot 10^{-6}$	$0.20 \cdot 10^{-4}$	$0.1183 \cdot 10^{-6}$	$0.50 \cdot 10^{-5}$
$\cos(x) - \sin(x)$	$0.428 \cdot 10^{-6}$	$0.283 \cdot 10^{-4}$	$0.1673 \cdot 10^{-6}$	$0.707 \cdot 10^{-5}$

When solving the integro-differential equation

$$u'(x) + \alpha_1 \int_0^x F\big(u(s), u'(s)\big)ds$$

$$+\alpha_2 \int_0^1 G(u(s), u'(s))ds \ = f(x),$$

we replace the function $u(x)$ and its first derivative $u'(x)$ with approximations constructed with the splines of the fifth order of approximation. Next, we present the results of solving several integro-differential equations. The value of the first derivative at the node we approximate with the formulas of numerical differentiation obtained with the help of the splines of the fifth order of approximation.

3 Problem Solution

Below are the results of the numerical solution of several integro-differential equations. To solve the equations, a uniform grid of nodes was constructed with step of $h = 0.1$. After replacing the unknown function with a fifth-order approximation with some coefficients, we have to solve a system of nonlinear equations. Then we can visualize the solution by connecting the obtained points with splines of the fifth order of approximation. In addition, it is possible to obtain a piecewise given expression not only for the desired function, but also for the first derivative of the desired function. In the figures, the numbers of grid nodes are marked along the abscissa axis.

Example 1 (Example 4.1. from paper [6]). Consider the nonlinear Volterra–Fredholm integro-differential equation, as follows:

$$u'(x) + u(x)$$

$$+\frac{1}{2}\int_0^x xu^2(s)ds - \frac{1}{4}\int_0^1 su^3(s)ds = f(x)\,,$$

where

$$f(x) = 2x + x^2 + x^6 - \frac{1}{32}, u(0) = 0.$$

The exact solution is $u(x) = x^2$.

Table 3 presents the results of calculations. The first column represents the grid nodes with step $h = 0.1$. The second column presents the values of the solution at the grid nodes, obtained using splines of the fifth order of approximation. The third column gives the solution presented in paper [6] (at $n = 8, m = 8$). The fourth column contains the solution from paper [8].

Table 3. The results of calculations (*Example* 1)

t_i	Example 1		
	Splines of the fifth order of approximation	Paper [6] $n = 8$, $m = 8$	Paper [8] $m = 16$
0	0	0.	0
0.1	0.010	0.010031	0.010978
0.2	0.040	0.040075	0.040702
0.3	0.090	0.0901	0.090736
0.4	0.160	0.160094	0.161077
0.5	0.250	0.250228	0.250164
0.6	0.360	0.360502	0.361120
0.7	0.490	0.490583	0.490819
0.8	0.64	0.640374	0.640819
0.9	0.81	0.810047	0.811118
1	1.0	0.999986	1.000149

Figure 1 shows the errors in the solution of problem 1, found using splines of the fifth order of approximation. Figure 2 shows the errors of problem 1 found in paper [6]. Figure 3 shows the solutions to problem 1 found using paper [8]'s .

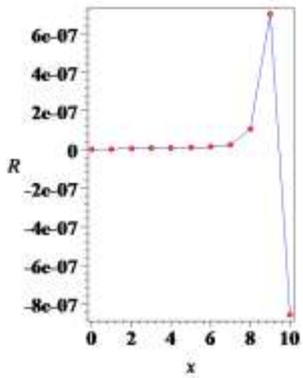

Fig. 1: The plot of the errors in the solution of problem 1, found using splines of the fifth order of approximation

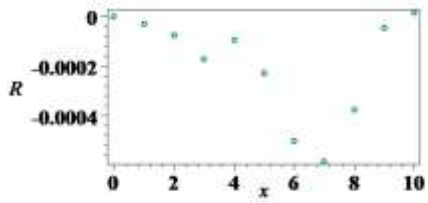

Fig. 2: The plot of the errors in the solution of problem 1, found in paper [6]

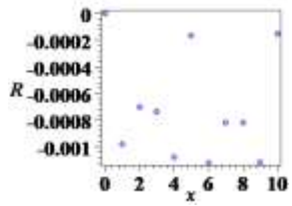

Fig. 3: The plot of the errors in the solution of problem 1, found in paper [8]

Figures 4 and 5 show the graphs of the solution and the graph of the first derivative of the solution restored using splines of the fifth order of approximation.

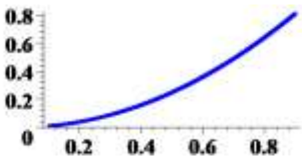

Fig. 4: The plot of the solution of problem 1, found using splines of the fifth order of approximation

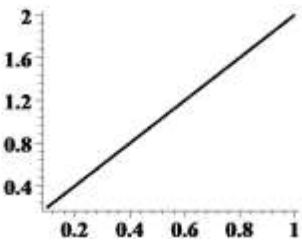

Fig. 5: The plot of the errors of the first derivative of the solution of problem 1, found using splines of the fifth order of approximation

Example 2 (Example 4.2. from paper [6]). Consider the nonlinear Volterra integro-differential equation, as follows:

$$u'(x) - \int_0^x \cos(x-s)\, u^2(s) ds = -2\sin(x) - \frac{1}{3}\cos(x) - \frac{2}{3}\cos(2x),$$

and the exact solution $u(x) = cos(x) - sin(x)$.

Table 4 presents the results of calculations. The first column represents the grid nodes with step $h = 0.1$. The second column presents the values of the solution at the grid nodes, obtained using splines of the fifth order of approximation. The third column gives the solution presented in paper [6] (at $n = 8, m = 8$).

Table 4. The results of calculations (*Example* 2)

t_i	Example 2	
	Splines of the fifth order of approximation	Paper [6] $\boldsymbol{n = 8}$, $\boldsymbol{m = 8}$
0	1	0.999999
0.1	0.895169	0.895186
0.2	0.781394	0.781653
0.3	0.659813	0.659732
0.4	0.531639	0.530699
0.5	0.398153	0.398169
0.6	0.260688	0.260969
0.7	0.120619	0.120671
0.8	-0.020655	-0.020638
0.9	-0.161719	-0.161638
1	-0.301186	-0.301983

Figure 6 shows the errors in absolute values of the solution of problem 2, found using splines of the fifth order of approximation. Figure 7 shows the errors of problem 2 found in paper [6].

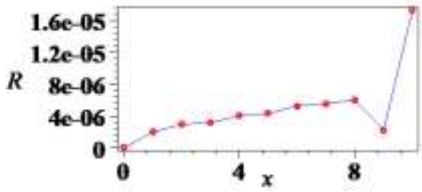

Fig. 6: The plot of the errors in the solution of problem 2, found using splines of the fifth order of approximation

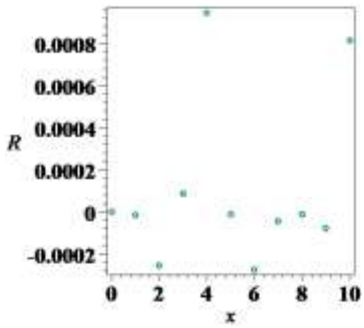

Fig. 7: The plot of the errors in the solution of problem 2, found in paper [6]

Figures 8 and 9 show the graphs of the solution and the graph of the first derivative of the solution, restored using splines of the fifth order of approximation.

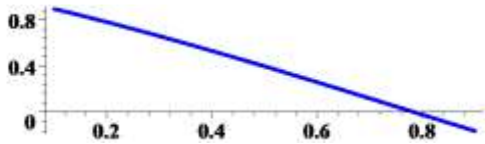

Fig. 8: The plot of the solution of problem 2, found using splines of the fifth order of approximation

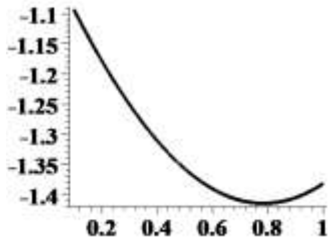

Fig. 9: The plot of the first derivative of the solution of problem 2, found using splines of the fifth order of approximation

Example 3 (Example 4.3. from paper [6]). Consider the nonlinear Volterra–Fredholm integro-differential equation, as follows:

$$u'(x) + x^2 u(x) - \int_0^x (x-s)u^2(s)ds + \int_0^1 e^s u(s)ds = f(x)$$

$$f(x) = 1 + e + \frac{x^2}{2} + \frac{2x^3}{3} - \frac{x^4}{12}$$

where $u(0) = 1$. The exact solution is $u(x) = (x+1)$.

Table 5 presents the results of calculations. The first column represents the grid nodes with step $h = 0.1$. The second column presents the values of the solution at the grid nodes, obtained using splines of the fifth order of approximation. The third column gives the solution presented in paper [6] (at $n = 8, m = 8$).

Table 5. The results of calculations (*Example* 3)

t_i	Example 3	
	Splines of the fifth order of approximation	Paper [6] $n = 8$, $m = 8$
0	1	0.999999
0.1	1.100001	1.100625
0.2	1.200001	1.200373
0.3	1.300002	1.300626
0.4	1.400003	1.400681
0.5	1.500003	1.500599
0.6	1.600004	1.601830
0.7	1.700005	1.702132
0.8	1.800008	1.806721
0.9	1.900036	1.913578
1	1.999757	2.009838

Figure 10 shows the errors of the solution obtained using splines of the fifth order of approximation, Figure 11 shows the errors obtained using the method of paper [6]. In these two figures, along the abscissa axis, grid nodes from the interval [0,1] are marked.

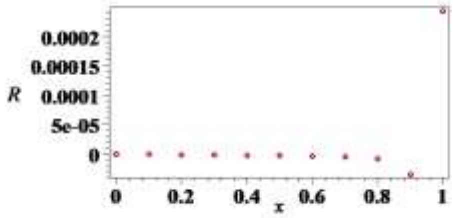

Fig. 10: The plot of the errors in the solution of problem 3, found using splines of the fifth order of approximation

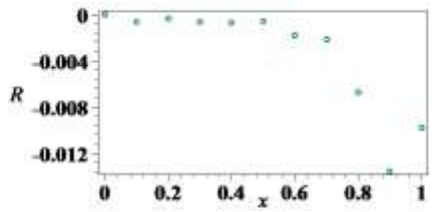

Fig. 11: The plot of the errors in the solution of problem 3, found in paper [6]

Example 4. Finally, consider an integro-differential equation containing a second derivative (see [9]).

$$u''(x) = 32x + \int_{-1}^{1} (1 - xs)u(s)ds,$$

$$-1 \le x \le 1, \qquad u(-1) = -\frac{5}{2}, u(1) = 15/2.$$

The exact solution to this problem is the next: $u(x) = 5x^3 + \frac{3}{2}x^2 + 1$.

For the approximation of the second derivative, we obtain the formula in the same way, namely by twice differentiating the spline approximation of the function.

In paper [9] with the number of nodes 32, the error of the solution was approximately 10^{-4}. In our case, with 8 nodes in the interval $-1 \leq x \leq 1$, the error was 10^{-18}. Fig. 12 shows a plot of the solution error obtained with splines of the 5th order of approximation. The node numbers are plotted along the abscissa axis.

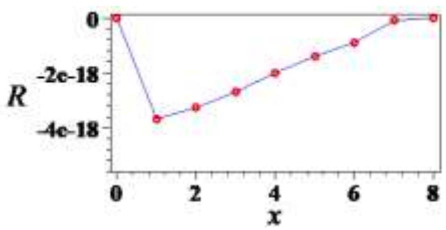

Fig. 12: The plot of the errors in the solution of Problem 4 obtained with splines of the 5th order of approximation.

Such a high accuracy of the solution is explained by the fact that spline approximations are exact on polynomials up to the fourth degree. In other words, the approximation error is zero for polynomials up to the fourth degree.

4 Conclusion

This paper considers the solution of nonlinear integro-differential equations with the first derivative of the unknown function using a method based on the application of local polynomial splines of the fifth order of approximation. As a result of solving the system of nonlinear equations, we obtain the values of the solution at the grid nodes. Further, applying these splines of the fifth order of approximation, we can connect the solution values at the grid nodes with the line. In addition, we can find and visualize the first derivative of the solution on a given interval.

Thus, with the help of splines of the fifth order of approximation, we are able to obtain a solution at any point in the interval, as well as the derivative of the solution. Theorems about the errors of approximations of functions and the first derivative with the local polynomial splines of the fifth order of approximation are given.

One example of solution of the integro-differential equation with the second derivative of the unknown is given.

Note that it is assumed that the integral of the product of the kernel and the basis function is calculated without error. In this case, to obtain a solution, it is required that the solution be five times continuously differentiable and the kernel a continuous function. Otherwise, the corresponding quadrature formulas can be used to calculate the integral from the product of the kernel and the basis functions.

Next, we will consider in details the solutions of integro-differential equations containing the second derivative. In addition, cases of using a non-uniform grid, as well as non-polynomial approximations, will be considered.

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