Upper and lower $\alpha(\Lambda, sp)$-continuous multifunctions

JEERANUNT KHAMPAKDEE
Mathematics and Applied Mathematics Research Unit
Department of Mathematics, Faculty of Science, Mahasarakham University
Maha Sarakham, 44150
THAILAND

CHAWALIT BOONPOK
Mathematics and Applied Mathematics Research Unit
Department of Mathematics, Faculty of Science, Mahasarakham University
Maha Sarakham, 44150
THAILAND

Abstract: Our main purpose is to introduce the concepts of upper and lower $\alpha(\Lambda, sp)$-continuous multifunctions. In particular, some characterizations of upper and lower $\alpha(\Lambda, sp)$-continuous multifunctions are established.

Key–Words: $\alpha(\Lambda, sp)$-open set, upper $\alpha(\Lambda, sp)$-continuous multifunction, lower $\alpha(\Lambda, sp)$-continuous multifunction

Received: August 18, 2021. Revised: July 19, 2022. Accepted: August 21, 2022. Published: September 20, 2022.

1 Introduction

It is well known that various types of continuity for functions and multifunctions play a significant role in the theory of classical point set topology. Stronger and weaker forms of open sets play an important role in the researching of generalizations of continuity in topological spaces. Using different forms of open sets, many authors have introduced and studied various types of continuity for functions and multifunctions. In 1983, Mashhour et al. [8] introduced the concept of α-continuous functions. In 1986, Neubrunn [9] extended the concept of α-continuous functions to multifunctions and introduced the notions of upper and lower α-continuous multifunctions. In 1993, Popa and Noiri [11] investigated some characterizations of upper and lower α-continuous multifunctions. In [7], the present author introduced and studied the notions of upper and lower $\star$-continuous multifunctions in ideal topological spaces. Some characterizations of upper and lower $\alpha(\star)$-continuous multifunctions are investigated in [6]. In 2020, Viriyapong and Boonpok [14] introduced and investigated the notions of upper and lower $(\tau_1, \tau_2)\alpha$-continuous multifunctions in bitopological spaces. The concept of β-open sets was first introduced by Abd El-Monsef et al. [1]. Noiri and Hatir [10] introduced and investigated the notions of Λ_{sp}-sets, Λ_{sp}-closed sets and spg-closed sets. In [5] by considering the concept of Λ_{sp}-sets, introduced and investigated the notions of (Λ, sp)-closed sets, (Λ, sp)-open sets and (Λ, sp)-closure operators. Moreover, some characterizations of upper and lower (Λ, sp)-continuous multifunctions are established in [5]. In [3], the present authors introduced and studied the concept of weakly (Λ, sp)-continuous multifunctions. The purpose of the present paper is to introduce the concepts of upper and lower $\alpha(\Lambda, sp)$-continuous multifunctions. In particular, several characterizations of upper and lower $\alpha(\Lambda, sp)$-continuous multifunctions are discussed.

2 Preliminaries

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be *β-open* [1] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. The complement of a β-open set is called *β-closed*. The family of all β-open sets of a topological space (X, τ) is denoted by $\beta(X, \tau)$.

Definition 1. [10] *Let A be a subset of a topological space (X, τ). A subset $\Lambda_{sp}(A)$ is defined as follows:* $\Lambda_{sp}(A) = \cap\{U \mid A \subseteq U, U \in \beta(X, \tau)\}$.

Lemma 2. [10] *For subsets A, B and $A_\alpha(\alpha \in \nabla)$ of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq \Lambda_{sp}(A)$.

(2) If $A \subseteq B$, *then* $\Lambda_{sp}(A) \subseteq \Lambda_{sp}(B)$.

(3) $\Lambda_{sp}(\Lambda_{sp}(A)) = \Lambda_{sp}(A)$.

(4) If $U \in \beta(X, \tau)$, *then* $\Lambda_{sp}(U) = U$.

(5) $\Lambda_{sp}(\cap\{A_\alpha | \alpha \in \nabla\}) \subseteq \cap\{\Lambda_{sp}(A_\alpha) | \alpha \in \nabla\}$.

(6) $\Lambda_{sp}(\cup\{A_\alpha | \alpha \in \nabla\}) = \cup\{\Lambda_{sp}(A_\alpha) | \alpha \in \nabla\}$.

Definition 3. [10] *A subset A of a topological space (X, τ) is called a Λ_{sp}-set if $A = \Lambda_{sp}(A)$.*

Lemma 4. [10] *For subsets A and $A_\alpha(\alpha \in \nabla)$ of a topological space (X, τ), the following properties hold:*

(1) $\Lambda_{sp}(A)$ *is a* Λ_{sp}*-set.*

(2) If A is β-open, then A is a Λ_{sp}-set.

(3) If A_α is a Λ_{sp}-set for each $\alpha \in \nabla$, then $\cap_{\alpha\in\nabla} A_\alpha$ is a Λ_{sp}-set.

(4) If A_α is a Λ_{sp}-set for each $\alpha \in \nabla$, then $\cup_{\alpha\in\nabla} A_\alpha$ is a Λ_{sp}-set.

Definition 5. [5] *A subset A of a topological space (X, τ) is called (Λ, sp)-closed if $A = T \cap C$, where T is a Λ_{sp}-set and C is a β-closed set. The complement of a (Λ, sp)-closed set is called (Λ, sp)-open.*

Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a (Λ, sp)-*cluster point* [5] of A if $A \cap U \neq \emptyset$ for every (Λ, sp)-open set U of X containing x. The set of all (Λ, sp)-cluster points of A is called the (Λ, sp)-*closure* [5] of A and is denoted by $A^{(\Lambda,sp)}$.

Lemma 6. [5] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-closure, the following properties hold:*

(1) $A \subseteq A^{(\Lambda,sp)}$ *and* $[A^{(\Lambda,sp)}]^{(\Lambda,sp)} = A^{(\Lambda,sp)}$.

(2) If $A \subseteq B$, *then* $A^{(\Lambda,sp)} \subseteq B^{(\Lambda,sp)}$.

(3) $A^{(\Lambda,sp)}$ *is* (Λ, sp)*-closed.*

(4) A is (Λ, sp)-closed if and only if $A = A^{(\Lambda,sp)}$.

Definition 7. [5] *Let A be a subset of a topological space (X, τ). The union of all (Λ, sp)-open sets contained in A is called the (Λ, sp)-interior of A and is denoted by $A_{(\Lambda,sp)}$.*

Lemma 8. [5] *Let A and B be subsets of a topological space (X, τ). For the (Λ, sp)-interior, the following properties hold:*

(1) $A_{(\Lambda,sp)} \subseteq A$ *and* $[A_{(\Lambda,sp)}]_{(\Lambda,sp)} = A_{(\Lambda,sp)}$.

(2) If $A \subseteq B$, *then* $A_{(\Lambda,sp)} \subseteq B_{(\Lambda,sp)}$.

(3) $A_{(\Lambda,sp)}$ *is* (Λ, sp)*-open.*

(4) A is (Λ, sp)-open if and only if $A_{(\Lambda,sp)} = A$.

(5) $[X - A]^{(\Lambda,sp)} = X - A_{(\Lambda,sp)}$.

(6) $[X - A]_{(\Lambda,sp)} = X - A^{(\Lambda,sp)}$.

Definition 9. [5] *A subset A of a topological space (X, τ) is said to be $\alpha(\Lambda, sp)$-open (resp. $s(\Lambda, sp)$-open) if $A \subseteq [[A_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$ (resp. $A \subseteq [A_{(\Lambda,sp)}]^{(\Lambda,sp)}$).*

The complement of an $\alpha(\Lambda, sp)$-open (resp. $s(\Lambda, sp)$-open) set is called $\alpha(\Lambda, sp)$-*closed* (resp. $s(\Lambda, sp)$-*closed*). The family of all $\alpha(\Lambda, sp)$-open (resp. $s(\Lambda, sp)$-open) sets in a topological space (X, τ) is denoted by $\alpha\Lambda_{sp}O(X, \tau)$ (resp. $s\Lambda_{sp}O(X, \tau)$).

Let A be a subset of a topological space (X, τ). The intersection of all $\alpha(\Lambda, sp)$-closed (resp. $s(\Lambda, sp)$-closed) sets of X containing A is called the $\alpha(\Lambda, sp)$-*closure* [4] (resp. $s(\Lambda, sp)$-*closure* [13]) of A and is denoted by $A^{\alpha(\Lambda,sp)}$ (resp. $A^{s(\Lambda,sp)}$). The union of all $\alpha(\Lambda, sp)$-open (resp. $s(\Lambda, sp)$-open) sets of X contained in A is called the $\alpha(\Lambda, sp)$-*interior* (resp. $s(\Lambda, sp)$-*interior*) of A and is denoted by $A_{\alpha(\Lambda,sp)}$ (resp. $A_{s(\Lambda,sp)}$).

A subset N_x of a topological space (X, τ) is said to be a (Λ, sp)-*neighbourhood* of a point $x \in X$ if there exists a (Λ, sp)-open set U such that $x \in U \subseteq N_x$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [2] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X \mid F(x) \subseteq B\}$$

and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A} F(x)$. Then, F is said to be a *surjection* if $F(X) = Y$, or equivalently, if for each $y \in Y$, there exists an $x \in X$ such that $y \in F(x)$.

3 Characterizations of upper and lower upper and lower $\alpha(\Lambda, sp)$-continuous multifunctions

In this section, we introduce the notions of upper and lower $\alpha(\Lambda, sp)$-continuous multifunctions. Moreover, several characterizations of upper and lower $\alpha(\Lambda, sp)$-continuous multifunctions are discussed.

Lemma 10. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) $A \in \alpha\Lambda_{sp}O(X, \tau)$;

(2) $U \subseteq A \subseteq [U^{(\Lambda,sp)}]_{(\Lambda,sp)}$ for some (Λ, sp)-open set U;

(3) $U \subseteq A \subseteq U^{s(\Lambda,sp)}$ for some (Λ, sp)-open set U;

(4) $A \subseteq [A_{(\Lambda,sp)}]^{s(\Lambda,sp)}$.

Lemma 11. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is $\alpha(\Lambda, sp)$-closed in (X, τ) if and only if $[A^{(\Lambda,sp)}]_{s(\Lambda,sp)} \subseteq A$;

(2) $[A^{(\Lambda,sp)}]_{s(\Lambda,sp)} = [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$;

(3) $A^{\alpha(\Lambda,sp)} = A \cup [[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}$.

Definition 12. *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:*

(1) upper $\alpha(\Lambda, sp)$-continuous at $x \in X$ if, for each (Λ, sp)-open set V of Y such that $F(x) \subseteq V$, there exists an $\alpha(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq V$;

(2) lower $\alpha(\Lambda, sp)$-continuous at $x \in X$ if, for each (Λ, sp)-open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists an $\alpha(\Lambda, sp)$-open set U of X containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$;

(3) upper (lower) $\alpha(\Lambda, sp)$-continuous if F has this property at each point of X.

Theorem 13. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper $\alpha(\Lambda, sp)$-continuous at a point x of X;

(2) $x \in [[F^+(V)]_{(\Lambda,sp)}]^{s(\Lambda,sp)}$ for each (Λ, sp)-open set V of Y containing $F(x)$;

(3) for each $U \in s\Lambda_{sp}O(X, \tau)$ containing x and each (Λ, sp)-open set V of Y containing $F(x)$, there exists a nonempty (Λ, sp)-open set U_V of X such that $U_V \subseteq U$ and $F(U_V) \subseteq V$.

Proof. (1) ⇒ (2): Let V be any (Λ, sp)-open set of Y such that $F(x) \subseteq V$. Then, there exists an $\alpha(\Lambda, sp)$-open set U of X containing x such that $F(U) \subseteq V$; hence $x \in U \subseteq F^+(V)$. Since U is (Λ, sp)-open, by Lemma 10,

$$x \in U \subseteq [U_{(\Lambda,sp)}]^{s(\Lambda,sp)} \subseteq [[F^+(V)]_{(\Lambda,sp)}]^{s(\Lambda,sp)}.$$

(2) ⇒ (3): Let V be any (Λ, sp)-open set of Y such that $F(x) \subseteq V$. Then,

$$x \in [[F^+(V)]_{(\Lambda,sp)}]^{s(\Lambda,sp)}.$$

Let U be any $s(\Lambda, sp)$-open set containing x. Then, $U \cap [F^+(V)]_{(\Lambda,sp)} \neq \emptyset$ and $U \cap [F^+(V)]_{(\Lambda,sp)}$ is $s(\Lambda, sp)$-open in X. Put

$$U_V = [U \cap [F^+(V)]_{(\Lambda,sp)}]_{(\Lambda,sp)},$$

then U_V is a nonempty (Λ, sp)-open set of Y, $U_V \subseteq U$ and $F(U_V) \subseteq V$.

(3) ⇒ (1): Let $s\Lambda_{sp}O(X, x)$ be the family of all $s(\Lambda, sp)$-open sets of X containing x. Let V be any (Λ, sp)-open set of Y such that $F(x) \subseteq V$. For each $U \in s\Lambda_{sp}O(X, x)$, there exists a nonempty (Λ, sp)-open set U_V such that $U_V \subseteq U$ and $F(U_V) \subseteq V$. Let $W = \cup\{U_V \mid U \in s\Lambda_{sp}O(X, x)\}$. Then, W is (Λ, sp)-open in X, $x \in W^{s(\Lambda,sp)}$ and $F(W) \subseteq U$. Put $S = W \cup \{x\}$, then $W \subseteq S \subseteq W^{s(\Lambda,sp)}$. Thus, by Lemma 10, $x \in S \in \alpha\Lambda_{sp}O(X, \tau)$ and $F(S) \subseteq V$. This shows that F is upper $\alpha(\Lambda, sp)$-continuous at x. □

Theorem 14. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower $\alpha(\Lambda, sp)$-continuous at a point x of X;

(2) $x \in [[F^-(V)]_{(\Lambda,sp)}]^{s(\Lambda,sp)}$ for each (Λ, sp)-open set V of Y such that $F(x) \cap V \neq \emptyset$;

(3) for each $U \in s\Lambda_{sp}O(X, \tau)$ containing x and each (Λ, sp)-open set V of Y such that

$$F(x) \cap V \neq \emptyset,$$

there exists a nonempty (Λ, sp)-open set U_V of X such that $F(z) \cap V \neq \emptyset$ for every $z \in U_V$ and $U_V \subseteq U$.

Proof. The proof is similar to that of Theorem 13. □

A subset N_x of a topological space (X, τ) is called an $\alpha(\Lambda, sp)$-*neighbourhood* of a point $x \in X$ if there exists an $\alpha(\Lambda, sp)$-open set U such that

$$x \in U \subseteq N_x.$$

Theorem 15. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper $\alpha(\Lambda, sp)$-continuous;

(2) $F^+(V)$ is $\alpha(\Lambda, sp)$-open in X for every (Λ, sp)-open set V of Y;

(3) $F^-(V)$ is $\alpha(\Lambda, sp)$-closed in X for every (Λ, sp)-closed set V of Y;

(4) $[[F^-(B)]^{(\Lambda,sp)}]_{s(\Lambda,sp)} \subseteq F^-(B^{(\Lambda,sp)})$ for every subset B of Y;

(5) $[F^-(B)]^{\alpha(\Lambda,sp)} \subseteq F^-(B^{(\Lambda,sp)})$ for every subset B of Y;

(6) for each point x of X and each (Λ, sp)-neighbourhood V of $F(x)$, $F^+(V)$ is an $\alpha(\Lambda, sp)$-neighbourhood of x;

(7) for each point x of X and each (Λ, sp)-neighbourhood V of $F(x)$, there exists an $\alpha(\Lambda, sp)$-neighbourhood U of x such that $F(U) \subseteq V$.

Proof. (1) $\Rightarrow$ (2): Let V be any (Λ, sp)-open set V of Y and let $x \in F^+(V)$. By Theorem 13, we have $x \in [[F^+(V)]_{(\Lambda,sp)}]^{s(\Lambda,sp)}$ and hence

$$F^+(V) \subseteq [[F^+(V)]_{(\Lambda,sp)}]^{s(\Lambda,sp)}.$$

It follows from Lemma 10 that $F^+(V)$ is $\alpha(\Lambda, sp)$-open in X.

(2) $\Leftrightarrow$ (3): This follows from the fact that $F^+(Y - B) = X - F^-(B)$ for any subset B of Y.

(3) $\Rightarrow$ (4): Let B be any subset of Y. Then, $F^-(B^{(\Lambda,sp)})$ is $\alpha(\Lambda, sp)$-closed in X, by Lemma 11, we have $[[F^-(B)]^{(\Lambda,sp)}]_{s(\Lambda,sp)} \subseteq [[F^-(B^{(\Lambda,sp)})]^{(\Lambda,sp)}]_{s(\Lambda,sp)} \subseteq F^-(B^{(\Lambda,sp)})$.

(4) $\Rightarrow$ (5): Let B be any subset of Y. By Lemma 11, we have

$$\begin{aligned}[F^-(B)]^{\alpha(\Lambda,sp)} &= F^-(B) \cup [[F^-(B)]^{(\Lambda,sp)}]_{s(\Lambda,sp)} \\ &\subseteq F^-(B^{(\Lambda,sp)}).\end{aligned}$$

(5) $\Rightarrow$ (3): Let V be any $\alpha(\Lambda, sp)$-closed set of Y. Then, $[F^-(V)]^{\alpha(\Lambda,sp)} \subseteq F^-(V^{(\Lambda,sp)}) = F^-(V)$. This shows that $F^-(V)$ is $\alpha(\Lambda, sp)$-closed in X.

(2) $\Rightarrow$ (6): Let $x \in X$ and let V be a (Λ, sp)-neighbourhood of $F(x)$. Then, there exists a (Λ, sp)-open set G of Y such that $F(x) \subseteq G \subseteq V$. Thus, $x \in F^+(G) \subseteq F^+(V)$. Since $F^+(G)$ is $\alpha(\Lambda, sp)$-open in X, $F^+(V)$ is an $\alpha(\Lambda, sp)$-neighbourhood of x.

(6) $\Rightarrow$ (7): Let $x \in X$ and let V be a (Λ, sp)-neighbourhood of $F(x)$. Put $U = F^+(V)$, then U is an $\alpha(\Lambda, sp)$-neighbourhood of x and $F(U) \subseteq V$.

(7) $\Rightarrow$ (1): Let $x \in X$ and let V be any (Λ, sp)-open set of Y such that $F(x) \subseteq V$. Then, V is a (Λ, sp)-neighbourhood of $F(x)$. There exists an $\alpha(\Lambda, sp)$-neighbourhood U of x such that $F(U) \subseteq V$. Therefore, there exists an $\alpha(\Lambda, sp)$-open set G of X such that $x \in G \subseteq U$; hence $F(G) \subseteq V$. ☐

Theorem 16. *For a multifunction $F : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower $\alpha(\Lambda, sp)$-continuous;

(2) $F^-(V)$ is $\alpha(\Lambda, sp)$-open in X for every (Λ, sp)-open set V of Y;

(3) $F^+(V)$ is $\alpha(\Lambda, sp)$-closed in X for every (Λ, sp)-closed set V of Y;

(4) $[[F^+(B)]^{(\Lambda,sp)}]_{s(\Lambda,sp)} \subseteq F^+(B^{(\Lambda,sp)})$ for every subset B of Y;

(5) $[F^+(B)]^{\alpha(\Lambda,sp)} \subseteq F^+(B^{(\Lambda,sp)})$ for every subset B of Y;

(6) $F(A^{\alpha(\Lambda,sp)}) \subseteq [F(A)]^{(\Lambda,sp)}$ for every subset A of X;

(7) $F([A^{(\Lambda,sp)}]_{s(\Lambda,sp)}) \subseteq [F(A)]^{(\Lambda,sp)}$ for every subset A of X;

(8) $F([[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \subseteq [F(A)]^{(\Lambda,sp)}$ for every subset A of X.

Proof. The proofs except for the following are similar to those of Theorem 15 and are thus omitted.

(5) $\Rightarrow$ (6): Let A be any subset of X. Since $A \subseteq F^+(F(A))$, we have $A^{\alpha(\Lambda,sp)} \subseteq [F^+(F(A))]^{\alpha(\Lambda,sp)} \subseteq F^+([F(A)]^{(\Lambda,sp)})$ and $F(A^{\alpha(\Lambda,sp)}) \subseteq [F(A)]^{(\Lambda,sp)}$.

(6) $\Rightarrow$ (7): This follows immediately from Lemma 11.

(7) $\Rightarrow$ (8): This is obvious by Lemma 11.

(8) $\Rightarrow$ (1): Let $x \in X$ and let V be any (Λ, sp)-open set of Y such that $F(x) \cap V \neq \emptyset$. Then,

$$x \in F^-(V).$$

We shall show that $F^-(V)$ is $\alpha(\Lambda, sp)$-open in X. By the hypothesis, we have

$$F([[[F^+(Y-V)]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \subseteq [F(F^+(Y-V))]^{(\Lambda,sp)} \subseteq Y - V$$

and hence $[[[F^+(Y-V)]^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)} \subseteq F^+(Y-V) = X - F^-(V)$. Therefore, $F^-(V) \subseteq [[[F^-(V)]_{(\Lambda,sp)}]^{(\Lambda,sp)}]_{(\Lambda,sp)}$. This shows that $F^-(V)$ is $\alpha(\Lambda, sp)$-open in X. Put $U = F^-(V)$. Then, $x \in U \in \alpha\Lambda_{sp}O(X,\tau)$ and $F(z) \cap V \neq \emptyset$ for every $z \in U$. Thus, F is lower $\alpha(\Lambda, sp)$-continuous. □

Definition 17. *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be $\alpha(\Lambda, sp)$-continuous if, for every (Λ, sp)-open set V of Y, $f^{-1}(V)$ is $\alpha(\Lambda, sp)$-open in X.*

Corollary 18. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

(1) f is $\alpha(\Lambda, sp)$-continuous;

(2) $f^-(F)$ is $\alpha(\Lambda, sp)$-closed in X for every (Λ, sp)-closed set F of Y;

(3) $[[f^{-1}(B)]^{(\Lambda,sp)}]_{s(\Lambda,sp)} \subseteq f^{-1}(B^{(\Lambda,sp)})$ for every subset B of Y;

(4) $[f^{-1}(B)]^{\alpha(\Lambda,sp)} \subseteq f^{-1}(B^{(\Lambda,sp)})$ for every subset B of Y;

(5) for each $x \in X$ and each (Λ, sp)-neighbourhood V of $f(x)$, $f^{-1}(V)$ is an $\alpha(\Lambda, sp)$-neighbourhood of x;

(6) for each $x \in X$ and each (Λ, sp)-neighbourhood V of $f(x)$, there exists an $\alpha(\Lambda, sp)$-neighbourhood U of x such that $f(U) \subseteq V$;

(7) $f(A^{\alpha(\Lambda,sp)}) \subseteq [f(A)]^{(\Lambda,sp)}$ for every subset A of X;

(8) $f([A^{(\Lambda,sp)}]_{s(\Lambda,sp)}) \subseteq [f(A)]^{(\Lambda,sp)}$ for every subset A of X;

(9) $f([[A^{(\Lambda,sp)}]_{(\Lambda,sp)}]^{(\Lambda,sp)}) \subseteq [f(A)]^{(\Lambda,sp)}$ for every subset A of X.

Definition 19. *A collection $\mathcal{U}$ of subsets of a topological space (X,τ) is called (Λ, sp)-locally finite if every $x \in X$ has a (Λ, sp)-neighbourhood which intersects only finitely many elements of $\mathcal{U}$.*

Definition 20. *A subset A of a topological space (X,τ) is said to be:*

(i) (Λ, sp)-paracompact if every cover of A by (Λ, sp)-open sets of X is refined by a cover of A which consists of (Λ, sp)-open sets of X and is locally finite in X;

(ii) (Λ, sp)-regular if, for each $x \in A$ and each (Λ, sp)-open set U of X containing x, there exists a (Λ, sp)-open set V of X such that $x \in V \subseteq V^{(\Lambda,sp)} \subseteq U$.

Lemma 21. *If A is a (Λ, sp)-regular (Λ, sp)-paracompact subset of a topological space (X,τ) and U is a (Λ, sp)-open neighbourhood of A, then there exists a (Λ, sp)-open set V of X such that $A \subseteq V \subseteq V^{(\Lambda,sp)} \subseteq U$.*

A multifunction $F : (X,\tau) \to (Y,\sigma)$ is said to be *punctually (Λ, sp)-paracompact* (resp. *punctually (Λ, sp)-regular*) if for each $x \in X$, $F(x)$ is (Λ, sp)-paracompact (resp. (Λ, sp)-regular). By $F^{\alpha(\Lambda,sp)} : (X,\tau) \to (Y,\sigma)$, we shall denote a multifunction defined as follows: $F^{\alpha(\Lambda,sp)}(x) = [F(x)]^{\alpha(\Lambda,sp)}$ for each point $x \in X$.

Lemma 22. *If $F : (X,\tau) \to (Y,\sigma)$ is punctually (Λ, sp)-regular and punctually (Λ, sp)-paracompact, then $[F^{\alpha(\Lambda,sp)}]^+(V) = F^+(V)$ for every (Λ, sp)-open set V of Y.*

Proof. Let V be any (Λ, sp)-open set of Y and let $x \in [F^{\alpha(\Lambda,sp)}]^+(V)$. Then, $[F(x)]^{\alpha(\Lambda,sp)} \subseteq V$ and hence $F(x) \subseteq V$. Thus, $x \in F^+(V)$. This shows that $[F^{\alpha(\Lambda,sp)}]^+(V) \subseteq F^+(V)$. Let V be any (Λ, sp)-open set of Y and let $x \in F^+(V)$. Then, $F(x) \subseteq V$. Since $F(x)$ is (Λ, sp)-regular and (Λ, sp)-paracompact, by Lemma 21, there exists a (Λ, sp)-open set G such that $F(x) \subseteq G \subseteq G^{(\Lambda,sp)} \subseteq V$; hence $[F(x)]^{\alpha(\Lambda,sp)} \subseteq G^{(\Lambda,sp)} \subseteq V$. This shows that $x \in [F^{\alpha(\Lambda,sp)}]^+(V)$ and hence $F^+(V) \subseteq [F^{\alpha(\Lambda,sp)}]^+(V)$. Thus, $[F^{\alpha(\Lambda,sp)}]^+(V) = F^+(V)$. □

Theorem 23. *Let $F : (X,\tau) \to (Y,\sigma)$ is punctually (Λ, sp)-regular and punctually (Λ, sp)-paracompact. Then, F is upper $\alpha(\Lambda, sp)$-continuous if and only if $F^{\alpha(\Lambda,sp)} : (X,\tau) \to (Y,\sigma)$ is upper $\alpha(\Lambda, sp)$-continuous.*

Proof. Suppose that F is upper $\alpha(\Lambda, sp)$-continuous. Let $x \in X$ and let V be any (Λ, sp)-open set of Y such that $F^{\alpha(\Lambda,sp)}(x) \subseteq V$. By Lemma 22, $x \in [F^{\alpha(\Lambda,sp)}]^+(V) = F^+(V)$. Since F is upper $\alpha(\Lambda, sp)$-continuous, there exists $U \in \alpha\Lambda_{sp}O(X,\tau)$ containing x such that $F(U) \subseteq V$. Since $F(z)$ is (Λ, sp)-regular and (Λ, sp)-paracompact for each $z \in U$, by Lemma 21, there exists a (Λ, sp)-open set H such that $F(z) \subseteq H \subseteq H^{(\Lambda,sp)} \subseteq V$. Therefore, we have $[F(z)]^{\alpha(\Lambda,sp)} \subseteq H^{(\Lambda,sp)} \subseteq V$ for each $x \in U$ and hence $F^{\alpha(\Lambda,sp)}(U) \subseteq V$. This shows that $F^{\alpha(\Lambda,sp)}$ is upper $\alpha(\Lambda, sp)$-continuous.

Conversely, suppose that $F^{\alpha(\Lambda,sp)} : (X,\tau) \to (Y,\sigma)$ is upper $\alpha(\Lambda,sp)$-continuous. Let $x \in X$ and let V be any (Λ,sp)-open set of Y such that $F(x) \subseteq V$. By Lemma 22, $x \in F^+(V) = [F^{\alpha(\Lambda,sp)}]^+(V)$ and hence $F^{\alpha(\Lambda,sp)}(x) \subseteq V$. Since $F^{\alpha(\Lambda,sp)}$ is upper $\alpha(\Lambda,sp)$-continuous, there exists $U \in \alpha\Lambda_{sp}O(X,\tau)$ containing x such that $F^{\alpha(\Lambda,sp)}(U) \subseteq V$; hence $F(U) \subseteq V$. This shows that F is upper $\alpha(\Lambda,sp)$-continuous. □

Lemma 24. *For a multifunction $F : (X,\tau) \to (Y,\sigma)$, it follows that for each $\alpha(\Lambda,sp)$-open set V of Y $[F^{\alpha(\Lambda,sp)}]^-(V) = F^-(V)$.*

Proof. Let V be any $\alpha(\Lambda,sp)$-open set of Y. Let $x \in [F^{\alpha(\Lambda,sp)}]^-(V)$. Then, $[F(x)]^{\alpha(\Lambda,sp)} \cap V \neq \emptyset$ and hence $F(x) \cap V \neq \emptyset$. Thus, $x \in F^-(V)$. This shows that $[F^{\alpha(\Lambda,sp)}]^-(V) \subseteq F^-(V)$. Let $x \in F^-(V)$. Then, we have $\emptyset \neq F(x) \cap V \subseteq [F(x)]^{\alpha(\Lambda,sp)} \cap V$. Thus, $x \in [F^{\alpha(\Lambda,sp)}]^-(V)$ and hence $F^-(V) \subseteq [F^{\alpha(\Lambda,sp)}]^-(V)$. Therefore, $[F^{\alpha(\Lambda,sp)}]^-(V) = F^-(V)$. □

Theorem 25. *A multifunction $F : (X,\tau) \to (Y,\sigma)$ is lower $\alpha(\Lambda,sp)$-continuous if and only if $F^{\alpha(\Lambda,sp)} : (X,\tau) \to (Y,\sigma)$ is lower $\alpha(\Lambda,sp)$-continuous.*

Proof. By utilizing Lemma 24, this can be proved similarly to that of Theorem 23. □

A topological space (X,τ) is called Λ_{sp}*-compact* [12] if every cover of X by (Λ,sp)-open sets of X has a finite subcover.

Definition 26. *A topological space (X,τ) is said to be $\alpha\Lambda_{sp}$-compact if every $\alpha(\Lambda,sp)$-open cover of X has a finite subcover.*

Theorem 27. *Let $F : (X,\tau) \to (Y,\sigma)$ be an upper $\alpha(\Lambda,sp)$-continuous surjective multifunction such that $F(x)$ is Λ_{sp}-compact for each $x \in X$. If (X,τ) is $\alpha\Lambda_{sp}$-compact, then (Y,σ) is Λ_{sp}-compact.*

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a (Λ,sp)-open cover of Y. For each $x \in X$, $F(x)$ is Λ_{sp}-compact and there exists a finite subset $\nabla(x)$ of ∇ such that

$$F(x) \subseteq \cup\{V_\alpha \mid \alpha \in \nabla(x)\}.$$

Put $V(x) = \cup\{V_\alpha \mid \alpha \in \nabla(x)\}$. Since F is upper $\alpha(\Lambda,sp)$-continuous, there exists an $\alpha(\Lambda,sp)$-open $U(x)$ of X containing x such that $F(U(x)) \subseteq V(x)$. The family $\{U(x) \mid x \in X\}$ is an $\alpha(\Lambda,sp)$-open cover of X and there exists a finite number of points, say, $x_1, x_2, ..., x_n$ in X such that

$$X = \cup\{U(x_i) \mid 1 \leq i \leq n\}.$$

Thus, $Y = F(X) = F(\overset{n}{\underset{i=1}{\cup}} U(x_i)) = \overset{n}{\underset{i=1}{\cup}} F(U(x_i)) \subseteq \overset{n}{\underset{i=1}{\cup}} V(x_i) = \overset{n}{\underset{i=1}{\cup}} [\cup_{\alpha\in\nabla(x_i)} V_\alpha]$. This shows that (Y,σ) is Λ_{sp}-compact. □

4 Conclusion

The field of the mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. This paper is concerned with the concepts of of upper and lower $\alpha(\Lambda,sp)$-continuous multifunctions. Several characterizations of upper and lower $\alpha(\Lambda,sp)$-continuous multifunctions are obtained. The ideas and results of this paper may motivate further research.

Acknowledgements

This research project was financially supported by Mahasarakham University.

References:

[1] M.–E. Abd El-Monsef, S.–N. El-Deeb and R.–A. Mahmoud, β-open sets and β-continuous mappings, *Bull. Fac. Assiut Univ.* 12, 1983, pp. 77–90.

[2] C. Berge, Espaces topologiques fonctions multivoques, Dunod, Paris, 1959.

[3] C. Boonpok and C. Viriyapong, Weakly (Λ,sp)-continuous multifunctions, *Eur. J. Pure Appl. Math.* 15(2), 2022, pp. 528–536.

[4] C. Boonpok and J. Khampakdee, On almost $\alpha(\Lambda,sp)$-continuous multifunctions, *Eur. J. Pure Appl. Math.* 15(2), 2022, pp. 626–634.

[5] C. Boonpok, (Λ,sp)-closed sets and related topics in topological spaces, *WSEAS Tran. Math.* 19, 2020, pp. 312–322.

[6] C. Boonpok, On some types of continuity for multifunctions in ideal topological spaces, *Adv. Math. Sci. J.* 9(3), 2020, pp. 859–886.

[7] C. Boonpok, On continuous multifunctions in ideal topological spaces, *Lobachevskii. J. Math.* 40(1), 2019, pp. 24–35.

[8] A.–S. Mashhous, I.–A. Hasanein and S.–N. El-Deeb, α-continuous and α-open mappings, *Acta Math. Hungar.* 41, 1983, pp. 213–218.

[9] T. Neubrunn, Strongly quasi-continuous multivalued mappings, General Topology and its Relations to Modern Analysis and Algebra VI (Prague 1986) Heldermann, Berlin, 1988, pp. 351–359.

[10] T. Noiri and E. Hatir, Λ_{sp}-sets and some weak separation axions, *Acta Math. Hungar.* 103, 2004, pp. 225–232.

[11] V. Popa and T. Noiri, On upper and lower α-continuous multifunctions, *Math. Slovaca* 43(4), 1993, pp. 477–491.

[12] C. Viriyapong and C. Boonpok, (Λ, sp)-continuous functions, *WSEAS Tran. Math.* 21, 2022, pp. 380–385.

[13] C. Viriyapong and C. Boonpok, Weak quasi (Λ, sp)-continuity for multifunctions, *Int. J. Math. Comput. Sci.* 17, 2022, pp. 1201–1209.

[14] C. Viriyapong and C. Boonpok, $(\tau_1, \tau_2)\alpha$-continuity for multifunctions, *J. Math.* 2020, 2020, 6285763.