Exact solutions of the nonlinear loaded Benjamin-Ono equation

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Abstract: In this paper, we investigate the non-linear loaded two-dimensional Benjamin-Ono equation by the functional variable method. The advantage of this method is reliability and efficiency. Using this method we obtained exact solitary and periodic wave solutions. The solving procedure is very simple and the traveling wave solutions of this equation are demonstrated by hyperbolic and trigonometric functions. The graphical representations of some obtained solutions are demonstrated to better understand their physical features.

Key-Words: non-linear loaded Benjamin-Ono equation, solitary wave solutions, functional variable method, periodic wave solutions, trigonometric function, hyperbolic function.

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1 Introduction

Non-linear partial differential equations are important equations used in modeling many phenomena in science and engineering applications. One of the most important nonlinear evolution equation is the Benjamin-Ono(BO) equation. In 1967, the general theoretical treatment of a new class of long stationary waves with finite amplitude was studied by Benjamin and Ono developed Benjamin's theory in which was taken a class of non-linear evolution equations. The BO equation describes internal waves between two stratified homogenous fluids with different densities, where one of the layers is infinitely deep[1, 2, 3, 4]. This equation is expressed in the following basic form

$$u_{tt} + \alpha \left(u^2\right)_{xx} + \beta u_{xxxx} + \gamma u_{yyyy} = 0. \qquad (1)$$

where α, β and γ are non zero constants.

The literature is rich in different studies to find special solutions of the nonlinear BO equation, such as the existence of multi-soliton solution by several authors[1, 5], existence of periodic solutions by Ambrose and Wilkening[6], and certain discrete solutions by Tutiya[7]. We can indicate several methods that can be used to obtain exact special solutions to the non-linear BO equation, such as, numerical method[8, 9], Hirota bilinear method[10], extended truncated expansion method[11], generalization of the homoclinic breather method[12], tanh expansion method[13], constructive method[14] and others[15, 16, 17, 18, 19, 20, 21] are used to derive accurate solutions.

In this paper, we consider the following the non-linear loaded two-dimensional BO equation

$$u_{tt} - \alpha \left(u^2\right)_{xx} - \beta u_{xxxx} - \gamma u_{yyyy} + \\ +\varphi(t)u(0,0,t)u_{xx} = 0, \qquad (2)$$

where $u(x, y, t)$ is unknown function, $x \in R$, $y \in R$, $t \geq 0$, α, β and γ are any constants, and $\varphi(t)$ are the given real continuous functions.

We investigate the non-linear loaded two-dimensional BO equation by the functional variable method(FVM). The advantage of this method is reliability and efficiency. Using this method we obtained exact solitary and periodic wave solutions. The solving procedure is very simple and the traveling wave solutions of this equation are demonstrated by hyperbolic and trigonometric functions. The graphical representations of some obtained solutions are demonstrated to better understand their physical features.

In recent years, due to the intensive study of the problems of optimal management of the agroecosystem for instance, the problem of long-term forecasting and regulation of groundwater levels and soil moisture interest in loaded equations has increased significantly. Among the works devoted to loaded equations, one should especially note the works of A. Kneser[22], L. Lichtenstein[23], A. M. Nakhushev[24], and others. A complete explanation of solutions of the nonlinear loaded PDEs and their uses can be found in papers[25, 26, 27, 28, 29, 30].

The article is organized as follows. In Section 2, we present some basic information about the description of the functional variable method. Section 3 is devoted to solutions of the non-linear loaded two-dimensional BO equation. In Section 4, we present the graphical representation of the non-linear loaded two-dimensional BO equation. Finally, conclusions are presented in Section 5.

2 Explanation of the functional variable method

We consider NPDE of the form

$$P(u, u_x, u_y, u_t, u_{xx}, u_{tt}, u_{yy}, u_{xy}, ...) = 0, \quad (3)$$

where P is a polynomial in $u = u(x, y, t)$ and its partial derivatives.

Step 1. It is used the following transformation for a travelling wave solution of eq.3:

$$u(x, y, t) = u(\xi), \quad (4)$$

with

$$\frac{\partial u}{\partial x} = p\frac{du}{d\xi}, \frac{\partial u}{\partial y} = q\frac{du}{d\xi}, \frac{\partial u}{\partial t} = -k\frac{du}{d\xi}, ..., \quad (5)$$

where

$$\xi = px + qy - kt, \qquad p = const, \ q = const \quad (6)$$

and k is the speed of the traveling wave.

Substituting eq.4 and eq.5 into NPDE eq.3 we get the following ODE of the form

$$F(u, \ u', u'', u''', ...) = 0, \qquad u' = \frac{du}{d\xi} \quad (7)$$

here F is a polynomial in $u(\xi)$, $u'(\xi)$, $u''(\xi)$, $u'''(\xi), ...$

Step 2. Let

$$u' = F(u). \quad (8)$$

It follows that

$$\int \frac{du}{F(u)} = \xi + \xi_0, \quad (9)$$

we suppose $\xi_0 = 0$ for convenience. Now we can calculate higher order derivatives of u:

$$\begin{aligned} u'' &= \tfrac{1}{2}\tfrac{d(F^2(u))}{du} \\ u''' &= \tfrac{1}{2}\tfrac{d^2(F^2(u))}{du^2}\sqrt{F^2(u)} \\ u'''' &= \tfrac{1}{2}\left[\tfrac{d^3(F^2(u))}{du^3}F^2(u) + \tfrac{d^2(F^2(u))}{du^2}\tfrac{d(F^2(u))}{du}\right] \end{aligned} \quad (10)$$

Step 3. Putting eq.10 into eq.7, we obtain

$$H(u, \frac{dF(u)}{du}, \frac{d^2F(u)}{du^2}, \frac{d^3F(u)}{du^3}, ...) = 0. \quad (11)$$

The key idea of this particular form eq.11 is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, eq.11 provides the expression of F and this, together with eq.8, give solutions to the original problem.

3 Solutions of the non-linear loaded two-dimensional Benjamin-Ono equation

We will find the exact solution of the non-linear loaded two-dimensional BO equation by the functional variable method. For doing this, in eq.2, let use the following transformation.

$$u(x, y, t) = u(\xi), \ \xi = px + qy - kt. \quad (12)$$

where $p = const, \ q = const$ and k is the speed of the traveling wave.

It is easy to show that after transformation (12), the nonlinear partial differential eq.2 can be transformed into an ordinary differential equation of the form

$$u'' = \frac{k^2 + \varphi(t)u(0, 0, t)p^2}{\beta p^4 + \gamma q^4}u - \frac{\alpha p^2}{\beta p^4 + \gamma q^4}u^2. \quad (13)$$

According to (10), eq.13 can be written as follows

$$\frac{1}{2}\frac{d(F^2(u))}{du} = \frac{k^2+\varphi(t)u(0,0,t)p^2}{\beta p^4+\gamma q^4}u- -\frac{\alpha p^2}{\beta p^4 + \gamma q^4}u^2. \quad (14)$$

Integrating eq.14 and after simple simplification, we get

$$F(u) = u\sqrt{\mu(t) - \eta u}. \quad (15)$$

where $\mu(t) = \frac{k^2+\varphi(t)u(0,0,t)p^2}{\beta p^4+\gamma q^4}$, $\eta = \frac{2\alpha p^2}{3(\beta p^4+\gamma q^4)}$.

From eq.8 and eq.15 we deduce that

$$\frac{du}{u\sqrt{\mu(t) - \eta u}} = d\xi. \quad (16)$$

After integrating eq.16, we have

$$u(x, y, t) = -\frac{6(k^2+\varphi(t)u(0,0,t)p^2)}{\alpha p^2}\times \times\left(\frac{e^{\sqrt{\frac{k^2+\varphi(t)u(0,0,t)p^2}{\beta p^4+\gamma q^4}}(px+qy-kt)}}{\left(1-e^{\sqrt{\frac{k^2+\varphi(t)u(0,0,t)p^2}{\beta p^4+\gamma q^4}}(px+qy-kt)}\right)^2}\right). \quad (17)$$

The function $u(0, 0, t)$ can be easily obtained based on expression eq.17.

We get two types of solutions of the non-linear loaded two-dimensional BO equation eq.2 as follows:

1) When $\frac{k^2+\varphi(t)u(0,0,t)p^2}{\beta p^4+\gamma q^4} > 0$, we get the solitary solution

$$u(x,y,t) = -\frac{6(k^2+\varphi(t)u(0,0,t)p^2)}{\alpha p^2}\times$$

$$\times\left(cth^2\left(\sqrt{\mu(t)}\frac{px+qy-kt}{2}\right)-1\right). \quad (18)$$

where $\mu(t) = \frac{k^2+\varphi(t)u(0,0,t)p^2}{\beta p^4+\gamma q^4}$.

2) When $\frac{k^2+\varphi(t)u(0,0,t)p^2}{\beta p^4+\gamma q^4} < 0$, we get the periodic solution

$$u(x,y,t) = \frac{6(k^2+\varphi(t)u(0,0,t)p^2)}{\alpha p^2}\times$$

$$\times\left(ctg^2\left(\sqrt{\mu(t)}\frac{px+qy-kt}{2}\right)+1\right). \quad (19)$$

where $\mu(t) = \frac{k^2+\varphi(t)u(0,0,t)p^2}{\beta p^4+\gamma q^4}$.

The graphs of solutions of the non-linear loaded two-dimensional BO equation by using distinct values of random parameter will be demonstrated.

If $k=-1,\ \alpha=-6,\ \beta=1,\ \gamma=1,\ \varphi(t)=t,\ p=1,\ q=1$ then we have

$$u(x,y,t) = (1+tu(0,0,t))\times$$

$$\times\left(cth^2\left(\sqrt{\frac{1+tu(0,0,t)}{2}}\frac{x+y+t}{2}\right)-1\right). \quad (20)$$

If $k=-1,\ \alpha=6,\ \beta=-1,\ \gamma=-1,\ \ \varphi(t)=t^2,\ p=1,\ q=1$, then we have

$$u(x,y,t) = (1+t^2u(0,0,t))\times$$

$$\times\left(ctg^2\left(\sqrt{\frac{(1+t^2u(0,0,t))}{2}}\frac{x+y+t}{2}\right)+1\right). \quad (21)$$

4 Graphical representation of the non-linear loaded two-dimensional Benjamin-Ono equation

After visualizing the graphs of the soliton solutions (Figure 1) and the periodic wave solutions (Figure 2), the use of distinct values of random parameters is demonstrated to better understand their physical features. It is known that the parameters included in the solutions have a deep connection with the amplitudes and velocities. A soliton or solitary wave in the concept of mathematical physics defined as a self-reinforcing wave package that retains its shape. It propagates at a constant amplitude and velocity. Solitons are solutions of a common class of nonlinearly partially differential equations with weak linearity describing physical systems. The existence of periodic travelling waves usually depends on the parameter values in a mathematical equation. If there is a periodic travelling wave solution, then there is typically a family of such solutions, with different wave speeds. In this regard, we can explore some of the non-linear phenomena that take place in physics, applied mathematics and technology.

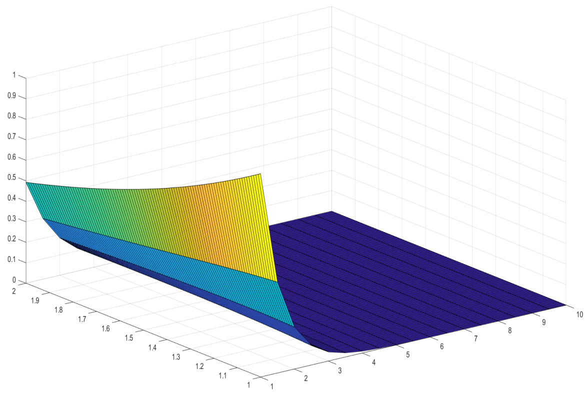

Figure 1: Solitary wave solution of the non-linear loaded two-dimensional BO equation for $k=-1$, $\alpha=-6,\ \beta=1,\ \gamma=1,\ \varphi(t)=t,\ p=1,\ q=1$.

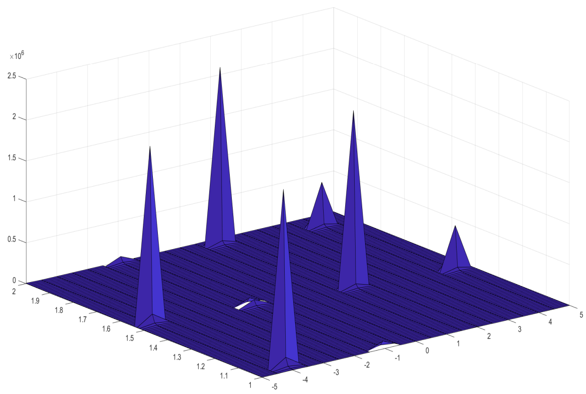

Figure 2: Periodic wave solution of the non-linear loaded two-dimensional BO equation for $k=-1$, $\alpha=6,\ \beta=-1,\ \gamma=-1,\ \ \varphi(t)=t^2,\ p=1,$ $q=1$.

5 Conclusion

The functional variable method has been successfully used to obtain the soliton solutions and the periodic solutions of the non-linear loaded two-dimensional BO equation. We have shown that this method can provide a useful way to efficiently find the exact structures of solutions to a variety of non-linear wave equations. After visualizing the graphs of the soliton solutions and the periodic wave solutions, the use of distinct values of random parameters is demonstrated to better understand their physical features. It is known that the parameters included in the solutions have a deep connection with the amplitudes and velocities. In this regard, we can explore some of the non-linear phenomena that take place in physics, applied mathematics and technology. We conclude that when revealing the internal mechanisms of physical phenomena, it will be necessary to find an exact solution to the problem.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Bazar Babajanov and Fakhriddin Abdikarimov conceived of the presented idea. Bazar Babajanov developed the theory and performed the computations. Fakhriddin Abdikarimov verified the analytical methods. Both authors discussed the results and contributed to the final manuscript. Both authors contributed to the article and approved the submitted version.
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