Euclidean Jordan algebras and some new inequalities over the parameters of a strongly regular graph

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Abstract: Let's consider a primitive strongly regular graph G and it's adjacency matrix A. Next we consider the Euclidean subalgebra $\mathcal{A}$ of the Euclidean Jordan algebra of real symmetric matrices of order n, with the Jordan product and with the inner product of two matrices as being the usual trace of two matrices. Finally, we make a spectral analysis of an Hadamard series of an element of $\mathcal{A}$ to establish some new conditions over the spectrum and the parameters of the primitive strongly regular graph G.

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1 Introduction

For a precise description of Euclidean Jordan algebras one must cite the monograph book, Analysis on Symmetric cones, of Jacques Faraut and Adam Korányi, see [1].

The Euclidean jordan algebras become a good theoretical environment to develop may applications in many branches of research of mathematics, see for instance [2–12] but our main goal is recurring to this theory to develop some properties over the spectrum of some discrete structures like the strongly regular graphs and the association schemes, see for instance [13–18].

This paper is organized as follows. In the section 2 we present some notes about Euclidean Jordan algebras, namely the more relevant notions about finite dimensional real Euclidean Jordan algebras. In the following section we present some notes about strongly regular graphs necessary for a clear exposition of this paper. Finally, in the last section we present two new inequalities over the parameters and the spectrum of a primitive strongly regular graph in the environment of Euclidean Jordan algebras. On one new inequality we establish a new relation between the parameters and one eigenvalue of a strongly regular graph, see inequality (29), and in the other new inequality we established a relation between only the parameters of a regular graph, see the inequality (30).

2 Some Notes on Euclidean Jordan Algebras

In this section we present the more relevant definitions and results of the theory of Euclidean Jordan algebras relevant for this paper.

For good monographs about Jordan algebras we must cite the Book, "A taste of Jordan Algebras" written by Kevin McCrimmon, see [19], and "Statistical Applications of Jordan Algebras" written by James. D. Malley, see [20].

A real finite dimensional Jordan algebra $\mathcal{A}$ is an algebra with an operation of multiplication of vectors $\star$ such that for any of its elements x and y we have:

$$\begin{aligned} x \star y &= y \star x, \\ x^{2\star} \star (x \star y) &= x \star (x^{2\star} \star y), \end{aligned}$$

where $x^{2\star} = x \star x$. And for any natural number k the powers of order k, are defined in the following way:

$$\begin{aligned} x^{0\star} &= \mathbf{e}, x^{1\star} = x, \\ x^{k\star} &= x \star x^{(k-1)\star}, k \geq 2. \end{aligned}$$

An element $\mathbf{e}$ of a real finite dimensional Euclidean Jordan algebra $\mathcal{A}$ is an unit element of $\mathcal{A}$ if $\mathbf{e} \star x = x \star \mathbf{e} = x$ for any element x in $\mathcal{A}$.

Example 1 *Let's consider the finite dimensional algebra $\mathcal{A}$ over $\mathbb{R}$ of real symmetric matrices of order n with the usual operations of addiction of matrices and of multiplication of a matrix by a real number. Then,* *considering the operation $\star$, instead of the usual operation of multiplication of matrices, defined for any x and y in $\mathcal{A}$ by $x \star y = \frac{xy+yx}{2}$, then $\mathcal{A}$ is a Jordan algebra. Indeed, let x and y be elements of $\mathcal{A}$, then we have the following calculations:*

$$x \star y = \frac{xy+yx}{2} = \frac{yx+xy}{2} = y \star x.$$

Firstly, we must say that for any element x of $\mathcal{A}$ we have $x^{2\star} = x^2$ where x^2 represent the usual square of a symmetric matrix of order n. Indeed, $x^{2\star} = \frac{xx+xx}{2} = \frac{x^2+x^2}{2} = x^2$.

Next, we will show that $x^{2\star}\star(x\star y) = x\star(x^{2\star}\star y)$. Since, we have:

$$\begin{aligned} x^{2\star} \star (x \star y) &= \frac{x^{2\star}(x \star y) + (x \star y)x^{2\star}}{2} \\ &= \frac{x^2\left(\frac{xy+yx}{2}\right) + \left(\frac{xy+yx}{2}\right)x^2}{2} \\ &= \frac{x^2(xy+yx) + (xy+yx)x^2}{4} \\ &= \frac{x^2xy + x^2yx + xyx^2 + yxx^2}{4} \\ &= \frac{x^3y + x^2yx + xyx^2 + yx^3}{4}, \end{aligned}$$

and since

$$\begin{aligned} x \star (x^{2\star} \star y) &= \frac{x(x^{2\star} \star y) + (x^{2\star} \star y)x}{2} \\ &= \frac{x\left(\frac{x^2y+yx^2}{2}\right) + \left(\frac{x^2y+yx^2}{2}\right)x}{2} \\ &= \frac{x(x^2y+yx^2) + (x^2y+yx^2)x}{4} \\ &= \frac{x^3y + xyx^2 + x^2yx + yx^3}{4} \\ &= \frac{x^3y + x^2yx + xyx^2 + yx^3}{4}. \end{aligned}$$

So, we have proved that $x^{2\star}\star(x\star y) = x\star(x^{2\star}\star x)$. for any x and y of $\mathcal{A}$. And, therefore we conclude that $\mathcal{A}$ is a Jordan real. We will denote sometimes this Euclidean Jordan algebra $\mathcal{A}$ by the notation $Sym(n, \mathbb{R})$.

A real finite dimensional Euclidean Jordan algebra is a real finite dimensional Jordan algebra equipped with the multiplication of vectors $\star$, and provided with an inner product $\bullet|\bullet$ such that for any three of it's elements x,y, and z the equality (1) is verified.

$$(x \star y)|z = y|(x \star z). \qquad (1)$$

Example 2 *Let's consider the Jordan algebra $\mathcal{A} = Sym(n, \mathbb{R})$, equipped with the vector operation $\star$ such that $x\star y = \frac{xy+yx}{2}$ for any x and y of $\mathcal{A}$, and provided with the inner product $\bullet|\bullet$ such that $x|y = trace(x\star y)$ for any elements x and y of $\mathcal{A}$. Then $\mathcal{A}$ is an Euclidean Jordan algebra, before showing that we will prove that $trace(x \star y) = trace(xy)$ for any two of it's elements x and y.*

Indeed, we have

$$\begin{aligned} trace(x \star y) &= trace(\frac{xy+yx}{2}) \\ &= \frac{1}{2}trace(xy+yx) \\ &= \frac{1}{2}(trace(xy) + trace(yx)) \\ &= \frac{1}{2}(trace(xy) + trace(xy)) \\ &= \frac{1}{2}(2trace(xy)) \\ &= trace(xy). \end{aligned}$$

Now, we consider a natural number k, and x, y and z elements of $\mathcal{A}$. The powers of order k of the element x, $x^{k\star}$ are defined in following way.

$$\begin{aligned} x^{0\star} &= e, \\ x^{1\star} &= x, \\ x^{k\star} &= x \star x^{(k-1)\star}, k \geq 2. \end{aligned}$$

Next, we will show that $(x \star y)|z = y|(x \star z)$. So, we have the following calculations.

$$\begin{aligned} (x \star y)|z &= trace\left(\frac{(x \star y)z + z(x \star y)}{2}\right) \\ &= trace((x \star y)z) \\ &= trace\left(\left(\frac{xy+yx}{2}\right)z\right) \\ &= trace\left(\frac{(xy)z + (yx)z}{2}\right) \\ &= trace\left(\frac{x(yz)}{2}\right) + trace\left(\frac{(yx)z}{2}\right) \\ &= trace\left(\frac{(yz)x}{2}\right) + trace\left(\frac{y(xz)}{2}\right) \\ &= trace\left(\frac{y(zx)}{2}\right) + trace\left(\frac{(xz)y}{2}\right) \\ &= trace\left(\frac{y(zx)}{2}\right) + trace\left(\frac{y(xz)}{2}\right) \\ &= trace\left(\frac{y(xz)}{2}\right) + trace\left(\frac{y(zx)}{2}\right) \\ &= trace\left(y\frac{xz+zx}{2}\right) \\ &= trace\left(\frac{\left(y\frac{xz+zx}{2}\right) + \frac{xz+zx}{2}y}{2}\right) \end{aligned}$$

$$
\begin{aligned}
&= \textit{trace}\left(y \star \left(\frac{xz+zx}{2}\right)\right) \\
&= \textit{trace}\,(y \star (x \star z)) \\
&= y|(x \star z)
\end{aligned}
$$

.

The unit of this Euclidean Euclidean Jordan algebra is the identity matrix $\mathbf{e}$ *of order* n. *Indeed, we have:*

$$\mathbf{e} \star x \;=\; \frac{\mathbf{e}x + x\mathbf{e}}{2} = \frac{x+x}{2} = \frac{2x}{2} = x = x \star \mathbf{e}.$$

Let $\mathcal{A}$ be a n dimensional real Euclidean Jordan algebra with the vector product $\star$, the inner product $\bullet|\bullet$ and with the unit $\mathbf{e}$. Then $\mathcal{A}$ is a power associative algebra, this is for any of it's element $\mathbf{x}$ the algebra spanned by x and $\mathbf{e}$ is associative.

The rank of an element a in $\mathcal{A}$ is the least natural number k such that $\{e, a^{1\star}, \ldots, a^{k\star}\}$ is a linearly dependent set and we write $\text{rank}(a) = k$. Since for any $a \in \mathcal{A}$ we have $\text{rank}(x) \leq n$, then we define the rank of $\mathcal{A}$ as being the natural number $r = \text{rank}(\mathcal{A}) = \max\{\text{rank}(a) : a \in \mathcal{A}\}$. An element a of $\mathcal{A}$ is regular if $\text{rank}(a) = r$, Let x be a regular element of $\mathcal{A}$ and $r = \text{rank}(x)$. Then, there exist real scalars $\beta_1(x), \beta_2(x), \ldots, \beta_{r-1}(x)$ and $\beta_r(x)$ such that

$$x^{r\star} - \beta_1(x)x^{r-1\star} + \cdots + (-1)^r\beta_r(x)x^{0\star} = 0, \quad (2)$$

where 0 is the null vector of $\mathcal{A}$. Taking into account (2) we conclude that the polynomial

$$p(x, \lambda) = \lambda^r - \beta_1(x)\lambda^{r-1} + \cdots + (-1)^r\beta_r(x) \quad (3)$$

is the minimal polynomial of x. When x is not regular the minimal polynomial of x has a degree less than r. The roots of the minimal polynomial of x are the eigenvalues of x.

An element $x \in \mathcal{A}$ is an idempotent if $x^{2\star} = x$. Two idempotent a and b are orthogonal if $a \star b = 0$. The set $\{g_1, g_2, \ldots, g_l\}$ is a complete system of orthogonal idempotent if $g_i^{2\star} = g_i$, for $i = 1, \ldots, l$, $g_i \star g_j = 0$, if $i \neq j$ and $1 \leq i, j \leq l$, and $\sum_{i=1}^{l} g_i = \mathbf{e}$. An idempotent is primitive if is a nonzero idempotent of $\mathcal{A}$ and cannot be written as a sum of two nonzero orthogonal idempotent. We say that $\{g_1, g_2, \ldots, g_k\}$ is a Jordan frame if $\{g_1, g_2, \ldots, g_k\}$ is a complete system of orthogonal idempotent such that each idempotent is primitive.

Example 3 *Let's consider the Euclidean Jordan algebra* $\mathcal{A} = \textit{Sym}(n, \mathbb{R})$ *with the Jordan product* $\star$ *such that* $x \star y = \frac{xy+yx}{2}, \forall x, y \in \mathcal{A}$ *and the inner product* $\bullet|\bullet$ *such that for any* x *and* y *elements of* $\mathcal{A}$ *we have* $x|y = \textit{trace}(x \star y)$. *Let's consider* i *and* j *be natural numbers such that* $1 \leq i, j \leq n$, *the matrices* E_{ij} *of* $\mathcal{A}$ *such that the only non null entry of* E_{ij} *is the entry* ij *and it's value is* 1. *Then, the set of matrices* $\mathcal{B}_1 = \{E_{11}, E_{22}, \cdots, E_{nn}\}$ *is a Jordan frame of* $\mathcal{A}$ *and the set of matrices* $\mathcal{B}_2 = \{E_{11} + E_{22}, \sum_{i=3}^{n} E_{ii}\}$ *is a complete system of orthogonal idempotent of* $\mathcal{A}$.

Theorem 1 *([1], p. 43). Let* $\mathcal{V}$ *be a real Euclidean Jordan algebra. Then for* x *in* $\mathcal{V}$ *there exist unique real numbers* $\lambda_1, \lambda_2, \ldots, \lambda_k$, *all distinct, and a unique complete system of orthogonal idempotent* $\{g_1, g_2, \ldots, g_k\}$ *such that*

$$x = \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_k g_k. \quad (4)$$

The numbers λ_j's of (4) are the eigenvalues of x and the decomposition (4) is the first spectral decomposition of x.

Theorem 2 *([1], p. 44). Let* $\mathcal{V}$ *be a real Euclidean Jordan algebra with* $\textit{rank}(\mathcal{V}) = r$. *Then for each* $\mathbf{x}$ *in* $\mathcal{V}$ *there exists a Jordan frame* $\{g_1, g_2, \cdots, g_r\}$ *and real numbers* $\lambda_1, \cdots, \lambda_{r-1}$ *and* λ_r *such that*

$$x \;=\; \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_r g_r. \quad (5)$$

The decomposition (5) is called the second spectral decomposition of x.

3 Some results about strongly regular graphs

Along this paper we consider only non empty, simple and non complete graphs. By simple graphs we mean graphs without loops and parallel edges. Strongly regular graphs were firstly introduced by R. C. Bose in the paper [21].

One says that $\overline{G}$ is the complement of the graph G if it has the same set o vertices as G and if any of its two distinct vertices are adjacent vertices in $\overline{G}$ if and only if are non adjacent vertices in G.

A non null and non complete graph G, whose order is greater or equal than 3 is called a strongly regular graph with parameters $(n, k; \lambda, \mu)$ if G is $k-$regular graph such that any pair of adjacent vertices have λ common neighbor vertices and any pair of non adjacent vertices have μ common neighbor vertices.

If G is a $(n, k; \lambda, \mu)$ strongly regular graph then the complement graph of $G, \overline{G}$ is a $(n, n-k-1; n-2k+\mu-2, n-2k+\lambda)$ strongly regular graph.

Let's consider a graph G. We call a set of edges and vertices a walk of vertices in G to every sequence $v_0e_1v_1e_2\ldots e_{l-1}v_{l-1}e_lv_l$ such that $v_1, v_2, \ldots, v_{l-1}$ and v_l are vertices and $e_1, e_2\ldots, e_{l-1}, e_l$ are edges of

G and each edge e_i has extreme vertices v_{i-1} and vertice v_i for $i = 1, \cdots, l$. The walk is closed if $v_0 = v_l$ and is open otherwise. One says that a walk in G is a path if all the vertices v_is are distinct with the exception of the initial vertex v_0 and the final vertex v_l.

One says that a path is a closed path or a cycle if the initial vertex and final vertex of the path are the same.

A graph G is connected if for any pair of distinct vertices exists a path that joins them. A $(n, k; \lambda, \mu)$ strongly regular graph G is primitive if and only if G and $\overline{G}$ are connected. Otherwise one says that G is disconnected.

A primitive strongly regular graph $(n, k; \lambda, \mu)$ is a non primitive strongly regular graph if and only if $\mu = k$ or $\mu = 0$. In the following text we only consider primitive strongly regular graphs.

From now we only consider primitive strongly regular graphs.

Let G be a $(n, k; \lambda, \mu)$ strongly regular graph. The adjacency matrix of G, $A = [a_{ij}]$, is a binary matrix of order n such that $a_{ij} = 1$, if the vertex i is adjacent to j and 0 otherwise. The adjacency matrix of G satisfies the equation $A^2 = kI_n + \lambda A + \mu(J_n - A - I_n)$, where J_n is the all ones matrix of order n. It is well known (see, for instance, [22]) that the eigenvalues of A are k, θ and τ, where θ and τ are given by $\theta = (\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)})/2$ and $\tau = (\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)})/2$,(see [22]). One defines the eigenvalues of G as being the eigenvalues of A. And, we also know that the multiplicities f_θ and f_τ of the eigenvalues θ of τ are given respectively by the relations (6) and (7).

$$f_\theta = \frac{1}{2}\left(n - 1 + \frac{2k + (n-1)(\lambda - \mu)}{\tau - \theta}\right), \quad (6)$$

$$f_\tau = \frac{1}{2}\left(n - 1 - \frac{2k + (n-1)(\lambda - \mu)}{\tau - \theta}\right). \quad (7)$$

Since f_θ and f_τ are integer positive numbers, then the conditions present on (8) and on (9) are known as integrability conditions

$$f_\theta \in \mathbb{N}, \quad (8)$$

$$f_\tau \in \mathbb{N}. \quad (9)$$

.

In the context of strongly regular graphs one of the problems to analyse is to know if given the real numbers n, k, λ and μ if there exists a $(n, k; \lambda, \mu)$ strongly regular graph. The more referenced admissibility conditions for the existence of a $(n, k; \lambda, \mu)$ strongly regular graph are the inequalities (10),(11),(12),(12),(13), and (14).

$$k(k - 1 - a) = (n - k - 1)\mu, \quad (10)$$

$$(\tau + 1)(k + \tau + 2\theta\tau) \leq (k + \tau)(\theta + 1)^2, (11)$$

$$(\theta + 1)(k + \theta + 2\theta\tau) \leq (k + \theta)(\tau + 1)^2, (12)$$

$$n \leq \frac{1}{2} f_\theta(f_\theta + 3), \quad (13)$$

$$n \leq \frac{1}{2} f_\tau(f_\tau + 3), \quad (14)$$

The inequalities (11) and (12) are known as the Krein conditions of the strongly regular graph G, and the inequalities (13) and (14) are known as the absolute bounds. In the next section we establish some new inequalities over the spectrum of a strongly regular graph and it's parameters, over certain conditions, but relating only the parameters of a strongly regular graph or only one eigenvalue of the strongly regular graph and it's parameters.

4 Some new inequalities over the parameters of a strongly regular graph

Let's G be a primitive $(n, k; \lambda, \mu)$ strongly regular such that $0 < \mu < k - 1, k < \frac{n}{2}, \lambda > \mu$, and ϵ a positive real number such that $\lambda k + |\tau|^3 + \epsilon > (k - \mu) + (\lambda - \mu)\lambda + \mu k$ and such that $\lambda K + |\tau|^3 + \epsilon > (\lambda - \mu)\mu + \mu k$, A it's adjacency matrix and finally let's consider the 3-dimension Euclidean subalgebra $\mathcal{A}$ of rank three of the Euclidean Jordan algebra $\mathrm{Sym}(n, \mathbb{R})$ spanned by I_n and the natural powers of A. Next, let's consider the unique Jordan frame $\mathcal{B} = \{G_1, G_2, G_3\}$ where we have: $G_1 = \frac{1}{n}I_n + \frac{1}{n}A + \frac{1}{n}(J_n - A - I_n) = \frac{J_n}{n}$, $G_2 = \frac{|\tau|n + \tau - k}{n(\theta - \tau)} I_n + \frac{n + \tau - k}{n(\theta - \tau)} A + \frac{\tau - k}{n(\theta - \tau)}(J_n - A - I_n$, $G_3 = \frac{\theta n + k - \theta}{n(\theta - \tau)} I_n + \frac{-n + k - \theta}{n(\theta - \tau)} A + \frac{k - \theta}{n(\theta - \tau)}(J_n - A - I_n)$. Now, we know that $A^2 = kI_n + \lambda A + \mu(J_n - A - I_n)$ where J_n is the matrix where each of it's entries is the real number 1. And so after some algebraic manipulation we conclude that (15) is verified.

$$A^2 = (k - \mu)I_n + (\lambda - \mu)A + \mu J_n \quad (15)$$

and therefore we conclude (16)

$$A^3 = (k - \mu)A + (\lambda - \mu)A^2 + \mu k J_n). \quad (16)$$

And, noting that

$$A^2 = kI_n + \lambda A + \mu(J_n - A - I_n$$

we deduce the equality $A^3 = (k - \mu)A + (\lambda - \mu)(kI_n + \lambda A + \mu(J_n - A - I_n)) + \mu k J_n$

Hence, we can write the inequality (17)

$$\begin{aligned} A^3 + |\tau|^3 I_n &= (\lambda k + |\tau|^3) I_n + \\ &+ ((k - \mu) + (\lambda - \mu)\lambda + \mu k)A + \\ &+ ((\lambda - \mu)\mu + \mu k)(J_n - A - I_n). \quad (17) \end{aligned}$$

Now, since $\lambda k+|\tau|^3+\epsilon > (k-\mu)+(\lambda-\mu)\lambda+\mu k$ and $\lambda K+|\tau|^3+\epsilon > (\lambda-\mu)\mu+\mu k$ then, let's consider the Hadammard series $S = \sum_{k=0}^{+\infty}\left(\frac{A^3+|\tau|^3 I_n}{\lambda k+|\tau|^3+\epsilon}\right)^{k\circ}$. Considering the notation

$$\begin{aligned}\alpha_1 &= \lambda k+|\tau|^3,\\ \alpha_2 &= (k-\mu)+(\lambda-\mu)\lambda+\mu k,\\ \alpha_3 &= (\lambda-\mu)\mu+\mu k,\\ \alpha_0 &= \lambda k+|\tau|^3+\epsilon\end{aligned}$$

we can write $S = \frac{1}{1-\frac{\alpha_1}{\alpha_0}}I_n + \frac{1}{1-\frac{\alpha_2}{\alpha_0}}A + +\frac{1}{1-\frac{\alpha_3}{\alpha_0}}(J_n-A-I_n)$. Next, let's consider the element $G_3\circ S$ of $\mathcal{A}$. So we conclude that (18) is verified.

$$\begin{aligned}G_3\circ S = \frac{\theta n+k-\theta}{n(\theta-\tau)}\frac{1}{1-\frac{\alpha_1}{\alpha_0}}I_n + \\ + \quad \frac{-n+k-\theta}{n(\theta-\tau)}\frac{1}{1-\frac{\alpha_2}{\alpha_0}}A + \\ + \quad \frac{k-\theta}{n(\theta-\tau)}\frac{1}{1-\frac{\alpha_3}{\alpha_0}}(J_n-A-I_n) \qquad (18)\end{aligned}$$

Now, we consider the spectral decomposition $q_3\circ S = q_{31}G_1+q_{32}G_2+q_{33}G_3$. We deduce that

$$\begin{aligned}q_{31} &= \frac{\theta n+k-\theta}{n(\theta-\tau)}\frac{1}{1-\frac{\alpha_1}{\alpha_0}}\\ &+ \frac{-n+k-\theta}{n(\theta-\tau)}\frac{1}{1-\frac{\alpha_2}{\alpha_0}}k+\\ &+ \frac{k-\theta}{n(\theta-\tau)}\frac{1}{1-\frac{\alpha_3}{\alpha_0}}(n-k-1). \qquad (19)\end{aligned}$$

Since $\frac{\theta n+k-\theta}{n(\theta-\tau)}+\frac{-n+k-\theta}{n(\theta-\tau)}k+\frac{k-\theta}{n(\theta-\tau)}(n-k-1)=0$, from (19) we conclude that:

$$\begin{aligned}q_{31} &= \frac{\theta n+k-\theta}{n(\theta-\tau)}\left(\frac{1}{1-\frac{\alpha_1}{\alpha_0}}-\frac{1}{1-\frac{\alpha_3}{\alpha_0}}\right)+\\ &+ \frac{-n+k-\theta}{n(\theta-\tau)}\left(\frac{1}{1-\frac{\alpha_2}{\alpha_0}}-\frac{1}{1-\frac{\alpha_3}{\alpha_0}}\right)k\end{aligned}$$

Now, from a spectral analysis of $S_3\circ S$ we conclude that $q_{3i}\geq 0$, for $i=1,\cdots,3$, and therefore since $q_{31}\geq 0$ so we can write inequality (20).

$$\begin{aligned}\frac{\theta n+k-\theta}{n-k+\theta}\left(\frac{1}{1-\frac{\alpha_1}{\alpha_0}}-\frac{1}{1-\frac{\alpha_3}{\alpha_0}}\right)\geq\\ \geq \left(\frac{1}{1-\frac{\alpha_2}{\alpha_0}}-\frac{1}{1-\frac{\alpha_3}{\alpha_0}}\right)k. \qquad (20)\end{aligned}$$

From an algebraic manipulation of (20) we deduce the inequality (21).

$$\begin{aligned}\frac{\theta n+k-\theta}{n-k+\theta}\left(\frac{1}{\alpha_0-\alpha_1}-\frac{1}{\alpha_0-\alpha_3}\right)\geq\\ \geq \left(\frac{1}{\alpha_0-\alpha_2}-\frac{1}{\alpha_0-\alpha_3}\right)k. \qquad (21)\end{aligned}$$

So, from (21) we deduce the inequality (22).

$$\begin{aligned}\frac{\theta n+k-\theta}{n-k+\theta}\left(\frac{\alpha_1-\alpha_3}{(\alpha_0-\alpha_1)(\alpha_0-\alpha_3)}\right)\\ \geq \left(\frac{\alpha_2-\alpha_3}{(\alpha_0-\alpha_2)(\alpha_0-\alpha_3)}\right)k. \qquad (22)\end{aligned}$$

By, rewriting the inequality (22) we obtain the inequality (23).

$$\begin{aligned}\frac{\theta n+k-\theta}{n-k+\theta}\left(\frac{\alpha_1-\alpha_3}{\alpha_0-\alpha_1}\right)\geq\\ \geq \left(\frac{\alpha_2-\alpha_3}{\alpha_0-\alpha_2}\right)k. \qquad (23)\end{aligned}$$

After, some calculations from (23), and noting that $\alpha_0-\alpha_1=\epsilon$ and $\alpha_0-\alpha_2=(\lambda-\mu)(k-\lambda)-(k-\mu)+|\tau|^3+\epsilon$, considering $\alpha_4=(\lambda-\mu)(k-\lambda)-(k-\mu)+|\tau|^3$, we deduce (24).

$$\frac{\theta n+k-\theta}{n-k+\theta}(\alpha_1-\alpha_3)\geq\frac{\epsilon}{\alpha_4+\epsilon}(\alpha_2-\alpha_3)k. \qquad (24)$$

Hence, we conclude that

$$\frac{\theta n+k-\theta}{n-k+\theta}(\alpha_1-\alpha_3) \quad\geq\quad (\alpha_2-\alpha_3)k. \qquad (25)$$

But, since $\alpha_2-\alpha_3=(k-\mu)+(\lambda-\mu)^2$ and $\alpha_1-\alpha_3=(\lambda-\mu)(k-\mu)+|\tau|^3$ then from (25) we deduce (26).

$$\begin{aligned}\frac{\theta n+k-\theta}{n-k+\theta}((\lambda-\mu)(k-\mu)+|\tau|^3)\\ \geq ((k-\mu)+(\lambda-\mu)^2)k. \qquad (26)\end{aligned}$$

Next, we suppose that $k<\frac{n}{2}$, then in this case we conclude that $\theta n+k-\theta\leq\frac{2\theta+1}{2}$ and $\frac{1}{n-k+\theta}\leq\frac{2}{n}$ and therefore from (26) we conclude that the inequality (27) is verified.

$$\begin{aligned}(2\theta+1)((\lambda-\mu)(k-\mu)+|\tau|^3)\\ \geq ((k-\mu)+(\lambda-\mu)^2)k. \qquad (27)\end{aligned}$$

Next, since $|\tau|<\frac{k-\mu}{\lambda-\mu}$ we obtain (28).

$$\begin{aligned}(2\theta+1)((\lambda-\mu)^4(k-\mu)+(k-\mu)^3)\\ \geq ((k-\mu)(\lambda-\mu)^3+(\lambda-\mu)^5)k. \qquad (28)\end{aligned}$$

Then, we have establish the Theorem 3.

Theorem 3 *Let G be a primitive $(n, k; \lambda, \mu)$ strongly regular graph such that $0 < \mu < k-1, \lambda > \mu, k < \frac{n}{2}$ then we have the inequality (29).*

$$\begin{aligned} & (2\theta+1)((\lambda-\mu)^4(k-\mu)+(k-\mu)^3) \\ \geq\ & (k-\mu)(\lambda-\mu)^3+(\lambda-\mu)^5. \end{aligned} \tag{29}$$

Making a similar spectral analysis of the element $G_3 \circ S$ and analyzing the eigenvalue q_{33} of $G_3 \circ S$ we deduce the inequality (30) presented on Theorem 4.

Theorem 4 *Let G be a primitive $(n, k; \lambda, \mu)$ strongly regular graph such that $0 < \mu < k-1, \lambda > \mu, k < \frac{n}{2}$ then we have the inequality (30).*

$$\begin{aligned} & (((\lambda-\mu)^4(k-\mu)+(k-\mu)^3)) \\ \geq\ & \frac{1}{3}((k-\mu)(\lambda-\mu)^3+(\lambda-\mu)^5). \end{aligned} \tag{30}$$

5 Conclusion

The research of this paper allow us to establish some new inequalities over the parameters of a primitive strongly regular graph and it's spectrum, but establishing relations over only the parameters of a primitive strongly regular graph or over the parameters of a primitive strongly regular and one of it's eigenvalues. In future research we will establish relations but relaxing the conditions over the parameters of a primitive strongly regular graph. To achieve that we will use spectral analysis of the Hadamard power series of the power of order n of the adjacency matrix of a primitive strongly regular graph with an asymptotic algebraic approach or with others spectral analysis methods.

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