Topological Spaces on Fuzzy Structures

MICHAEL GR. VOSKOGLOU
Mathematical Sciences, School of Technological Applications
University of Peloponnese (ex T.E.I. of Western Greece)
Meg. Alexandrou 1, 26334 Patras
GREECE

Abstract: - Apart from their applications to almost all sectors of the human activity, fuzzy mathematics is also importantly developed on a theoretical basis providing useful links even to classical branches of pure mathematics, like Algebra, Analysis, Geometry, Topology, etc. The present paper comes across the steps that enabled the extension of the concept of topological space, the most general category of mathematical spaces, to fuzzy structures. Fuzzy and soft topological spaces are introduced in particular, the fundamental concepts of limits, continuity, compactness and Hausdorff space are defined on them and examples are provided illustrating them.

Key-Words: - Fuzzy set, soft set, fuzzy topological space (FTS), soft topological space (STS).

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1 Introduction

Since Zadeh introduced the *fuzzy set theory* in 1965 [1], a lot of research was carried out for improving its effectiveness to deal with uncertain, ambiguous and vague situations. As a result a series of extensions and generalizations of the concept of fuzzy set have been reported (e.g. interval-valued fuzzy set, type-2 fuzzy set, intuitionistic fuzzy set, neutrosophic set, etc.) and several theories have been proposed (e.g. rough sets, soft sets, grey systems, etc.) as alternatives to the fuzzy set theory (e.g. see [2]).

Those new mathematical tools gave to experts the opportunity to model under conditions which are vague or not precisely defined, thus succeeding to solve mathematically problems whose statements are expressed in our natural language without exact numerical data. As a consequence, the spectre of applications of fuzzy sets and of the related to them extensions/theories has been rapidly extended covering nowadays almost all sectors of the human activity (Physical Sciences, Economics and Management, Expert Systems, Industry, Robotics, Decision Making, Programming, Medicine, Biology, Humanities, Human Reasoning, Education, etc.); e.g. see [3, Chapter 6], [4-7], etc.

Fuzzy mathematics, however, has been also importantly developed on a theoretical basis providing useful links even to classical branches of pure mathematics, like Algebra, Analysis, Geometry, Topology, etc.

Topological spaces [8], for instance, is the most general category of mathematical spaces, where fundamental concepts like limits, continuity, compactness, etc., are defined. The present paper comes across the steps that enabled the extension of topological spaces to fuzzy structures. The concepts of *fuzzy topological space (FTS)* and of *soft topological spaces (STS)* are introduced in particular, and examples are presented illustrating these concepts. More explicitly, FTSs are defined in Section 2, STSs are defined in Section 3 and the article closes with the final conclusions and some hints for future research, contained in its last Section 4.

2 Fuzzy Topological Spaces

2.1 Fuzzy Sets

Zadeh introduced the concept of fuzzy set as follows [1]:

Definition 1: A fuzzy set A in the universe U is defined with the help of its *membership function* m: $U \rightarrow [0,1]$ as the set of the ordered pairs

$$A = \{(x, m(x)): x \in U\} \quad (1)$$

The real number m(x) is called the *membership degree* of x in A**.** The greater m(x), the more x satisfies the characteristic property of A. Many authors, for reasons of simplicity, identify a fuzzy set with its membership function.

There is a difficulty, however, with the definition of the membership function, which is not unique, depending on the "signals" that each one receives from the real word. An individual of height 1.80 m, for example, may be considered as being "tall" by one observer and as having a regular height (not "tall") by another one. The methods used for defining the membership functions are usually statistical or empirical. The only restriction concerning the definition of a membership function is to be compatible with common logic; otherwise the fuzzy set does not give a reliable representation of the corresponding real situation. This could happen, for instance, if individuals of height $\leq$1.50 m possess membership degrees $\geq$ 0.5 in the fuzzy set of "tall people".

A crisp subset A of U is a fuzzy set in U with membership function taking the values m(x)=1, if x belongs to A, and 0 otherwise.

The basic definitions of crisp sets are generalized in a natural way to fuzzy sets as follows [3]:

Definition 2: The *universal fuzzy set* F_U and the *empty fuzzy set* $F_\emptyset$ in the universe U are defined as the fuzzy sets in U with membership functions m(x)=1 and m(x)=0 respectively, for all x in U.

It is straightforward to check that for each fuzzy set A in U is $A\cup F_U = F_U$, $A\cap F_U = A$, $A\cup F_\emptyset = A$ and $A\cap F_\emptyset = F_\emptyset$.

Definition 3: Let A and B be two fuzzy sets in the universe U with membership functions m_A and m_B respectively. Then A is said to be a *fuzzy subset* of B, if $m_A(x)\leq m_B(x)$, for all x in U. We write then $A \subseteq B$. If $m_A(x)< m_B(x)$, for all x in U, then A is called a *proper fuzzy subset* of B and we write $A \subset B$.

Definition 4: Let A and B be two fuzzy sets in the universe U with membership functions m_A and m_B respectively. Then:

- The *union* A∪B is defined to be the FS in U with membership function $m_{A\cup B}(x)=\max\{m_A(x), m_B(x)\}$, for each x in U.
- The *intersection* A∩B is defined to be the FS in U with membership function $m_{A\cap B}(x)=\min\{m_A(x), m_B(x)\}$, for each x in U.
- The *complement* of A is defined as the FS A^C in U with membership function $m^C(x) = 1- m(x)$, for all x in U.

Example 1: Let U be the set of the human ages U={5,10,20,30,40,50,60,70,80}. Table 1 gives the membership degrees of the elements of U with respect to the fuzzy sets A=young, B=adult and C=old in U.

1. Prove that $C \subseteq B$. Is $C \subset B$?
2. Calculate the fuzzy sets $A \cap C$ and $(A\cap C)\cup B$.

Table 1: Human ages

U	A	B	C
5	1	0	0
10	1	0	0
20	0.8	0.8	0
30	0.5	1	0.2
40	0.2	1	0.4
50	0.1	1	0.6
60	0	1	0.8
70	0	1	1
80	0	1	1

Solution: 1) From Table 1 turns out that $m_C(x) \leq \mu_B(x),\ \forall\ x \in U$. Thus $C \subseteq B$. But $m_C(70) = m_B(70) = 1$, therefore it is not true that $C \subset B$.

2) Calculating $\min\{m_A(x), m_C(x)\},\ \forall\ x \in U$ one finds that A∩C = {(5, 0), (10, 0), (20, 0), (30, 0.2), (40, 0.2), (50, 0.1), (60, 0), (70, 0), (80, 0)}.

Also, calculating $\max\{m_{A\cap C}(x), m_B(x)\}$, $\forall$ x $\in$ U, one finds that $(A\cap C)\cup B$ = {(5, 0), (10, 0), (20, 0), (30, 0.2), (40, 1), (50, 1), (60, 1), (70, 1), (80, 1)}.

2.2 Fuzzy Topologies

Definition 5 [9]: A *fuzzy topology (FT)* T on a non-empty set U is a collection of FSs in U such that:

- The universal and the empty FSs belong to T
- The intersection of any two elements of T and the union of an arbitrary (finite or infinite) number of elements of T belong also to T.

Examples 2: Trivial examples of FTs are the *discrete FT* $\{F_\emptyset, F_U\}$ and the *non-discrete FT* consisting of all FSs in U. Another example is the collection of all *constant FSs* in U, i.e. all FSs in U

with membership function of the form m(x)=c, for some c in [0, 1] and all x in U.
The elements of a FT T on U are called *open fuzzy sets* in U and their complements are called *closed fuzzy sets* in U. The pair (U, T) defines a *fuzzy topological space (FTS)* on U.
Next we describe how one can extend the concepts of *limit, continuity, compactness, and Hausdorff space* to FTSs [9].
Definition 6: Given two fuzzy sets A and B of the FTS (U, T), B is said to be a *neighborhood* of A, if there exists an open fuzzy set O such that $A \subseteq O \subset B$.
Definition 7: We say that a sequence $\{A_n\}$ of fuzzy sets of (U, T) converges to the fuzzy set A of (U, T), if there exists a positive integer m, such that for each integer n≥m and each neighborhood B of A we have that $A_n \subset B$. Then A is called the *limit* of $\{A_n\}$.
Lemma 1: (*Zadeh's extension principle*) Let X and Y be two non-empty crisp sets and let f: $X \to Y$ be a function. Then f can be extended to a function F mapping FSs in X to fuzzy sets in Y.
Proof: Let A be a fuzzy set in X with membership function m_A. Then its image F(A) is the fuzzy set B in Y with membership function m_B, which is defined as follows: Given y in Y, consider the set $f^{-1}(y)=\{x \in X: f(x)=y\}$. If $f^{-1}(y)=\emptyset$, then $m_B(y)=0$, and if $f^{-1}(y)\neq\emptyset$, then $m_B(y)$ is equal to the maximal value of all $m_A(x)$ such that $x \in f^{-1}(y)$. Conversely, the inverse image $F^{-1}(B)$ is the fuzzy set A in X with membership function $m_A(x)=m_B(f(x))$, for each $x \in X$.
Definition 8: Let (X, T) and (Y, S) be two FTSs and let f: $X \to Y$ be a function. By Lemma 1, f can be extended to a function F mapping fuzzy sets in X to fuzzy sets in Y. We say then that f is a *fuzzy continuous* function, if, and only if, the inverse image of each open fuzzy set in Y through F is an open fuzzy set in X.
Definition 9: A family $A=\{A_i, i\in I\}$ of fuzzy sets of a FTS (U, T) is called a *cover* of U, if $U=\bigcup_{i\in I} A_i$. If the elements of A are open fuzzy sets, then A is called an *open cover* of U. Also, each subset of A being also a cover of U is called a *sub-cover* of A. The FTS (U, T) is said to be *compact*, if every open cover of U contains a sub-cover with finitely many elements.
Definition 10: A FTS (U, T) is said to be:

1. A *T_1-FTS*, if, and only if, for each pair of elements u_1, u_2 of U, $u_1\neq u_2$, there exist at least two open fuzzy sets O_1 and O_2 such that $u_1\in O_1$, $u_2 \notin O_1$ and $u_2\in O_2$, $u_1 \notin O_2$.
2. A *T_2-FTS* (or a *separable* or a *Hausdorff FTS*), if, and only if, for each pair of elements u_1, u_2 of U, $u_1\neq u_2$, there exist at least two open fuzzy sets O_1 and O_2 such that $u_1\in O_1$, $u_2\in O_2$ and $O_1\cap O_2 = \emptyset_F$.

Obviously a T_2-FTS is always a T_1-FTS.

3 Soft Topological Spaces

3.1 Soft Sets

The need of passing through the existing difficulty to define properly the membership function of a fuzzy set gave the hint to D. Molodstov, Professor of Mathematics at the Russian Academy of Sciences, to introduce in 1999 the concept of *soft set* [10] as a tool to tackle the existing in real world uncertainty in a parametric manner. A soft set is defined as follows:
Definition 11: Let E be a set of parameters, let A be a subset of E, and let f be a map from A into the power set P(U) of all subsets of the universe U. Then the soft set (f, A) in U is defined to be the set of the ordered pairs

$$(f, A) = \{(e, f(e)): e \in A\} \quad (2)$$

In other words, a soft set in U is a parametrised family of subsets of U. The name "soft" was given because the form of (f, A) depends on the parameters of A. For each $e \in A$, its image f(e) is called the *value set* of e in (f, A), while f is called the *approximation function* of (f, A).
Example 3: Let $U= \{C_1, C_2, C_3\}$ be a set of cars and let $E = \{e_1, e_2, e_3\}$ be the set of the parameters e_1=cheap, e_2=hybrid (petrol and electric power) and e_3= expensive. Let us further assume that C_1, C_2 are cheap, C_3 is expensive and C_2, C_3 are the hybrid cars. Then, a map f: $E \to P(U)$ is defined by $f(e_1)=\{C_1, C_2\}$, $f(e_2)=\{C_2, C_3\}$ and $f(e_3)=\{C_3\}$. Therefore, the soft set (f, E) in U is the set of the ordered pairs $(f, E) = \{(e_1, \{C_1, C_2\}), (e_2, \{C_2, C_3\}, (e_3, \{C_3\}\}$.
A fuzzy set in U with membership function y = m(x) is a soft set in U of the form (f, [0, 1]), where $f(\alpha)=\{x\in U: m(x)\geq\alpha\}$ is the corresponding $\alpha - cut$ of the fuzzy set, for each α in [0, 1].
It is of worth noting that, apart from soft sets, which overpass the existing difficulty of defining properly membership functions through the use of the parameters of E, alternative theories for managing the uncertainty have been also developed, where the definition of a membership function is either not necessary (*grey systems/numbers* [11]), or it is overpassed by using a pair of crisp sets which give the lower and the upper approximation of the original crisp set (*rough sets* [12]).

The basic definitions on soft sets are introduced in a way analogous to fuzzy sets (see section 2.1)

Definition 12: The *absolute soft set* A_U is defined to be the soft set (f, A) such that f(e)=U, ∀ e ∈ A, and the *null soft set* $A_\emptyset$ is defined to be the soft set (f, A) such f(e)=∅, ∀ e ∈A.

It is straightforward to check that for each soft set A in U is $A \cup A_U = A_U$, $A \cap A_U = A$, $A \cup A_\emptyset = A$ and $A \cap A_\emptyset = A_\emptyset$.

Definition 13: If (f, A) and (g, B) are two soft sets in U, (f, A) is called a *soft subset* of (g, B), if $A \subseteq B$ and $f(e) \subseteq g(e)$, ∀ e ∈A. We write then $(f, A) \subseteq (g, B)$. If $A \subset B$, then (f, A) is called a *proper soft subset* of B and we write $(f, A) \subset (g, B)$.

Definition 14: Let (f, A) and (g, B) be two soft sets in U. Then:

- The *union* (f, A) ∪ (g, B) is the SS (h, A∪B) in U, with h(e)=f(e) if e ∈ A-B, h(e)=g(e) if e ∈ B-A and h(e)= f(e)∪g(e) if e ∈ A∩B.
- The *intersection* (f, A) ∩ (g, B) is the soft set (h, A∩B) in U, with h(e)=f(e)∩g(e), ∀ e ∈ A∩B.
- The *complement* $(f, A)^C$ of the soft SS (f, A) in U, is the SS (f^C, A) in U, for which the function f^C is defined by $f^C(e) = U\text{-}f(e)$, ∀ e ∈ A.

For general facts on soft sets we refer to [13].

Example 4: Let $U=\{H_1, H_2, H_3\}$, $E=\{e_1, e_2, e_3\}$ and $A= \{e_1, e_2\}$. Consider the soft set

$S = (f, A) = \{(e_1, \{H_1, H_2\}), (e_2, \{H_2, H_3\})\}$ in U. Then the soft subsets of S are the following:

$S_1=\{(e_1, \{H_1\})\}$, $S_2=\{(e_1, \{H_2\})\}$, $S_3=\{(e_1, \{H_1, H_2\})\}$, $S_4=\{(e_2, \{H_2\})\}$, $S_5=\{(e_2, \{H_3\})\}$, $S_6=\{(e_2, \{H_2, H_3\})\}$, $S_7=\{(e_1, \{H_1\}, (e_2, \{H_2\})\}$, $S_8=\{(e_1, \{H_1\}, (e_2, \{H_3\})\}$, $S_9=\{(e_1, \{H_2\}, (e_2, \{H_2\})\}$, $S_{10}=\{(e_1, \{H_2\}, (e_2, \{H_3\})\}$, $S_{11}=\{(e_1, \{H_1, H_2\}, (e_2, \{H_2\})\}$, $S_{12}=\{(e_1, \{H_1, H_2\}, (e_2, \{H_3\})\}$, $S_{13}=\{(e_1, \{H_1\}, (e_2, \{H_2, H_3\})\}$, $S_{14}=\{(e_1, \{H_2\}, (e_2, \{H_2, H_3\})\}$, S, $A_\emptyset=\{(e_1, \emptyset), (e_2, \emptyset)\}$

It is also easy to check that $(f, A)^C= \{(e_1, \{H_3\}), (e_2, \{H_1\})\}$.

3.3 Soft Topologies

Observe that the concept of FTS (Definition 5) is obtained from the classical definition of TS [8] by replacing in it the expression "a collection of subsets of U" by the expression "a collection of FSs in U". In an analogous way one can define the concepts of *intuitionistic FTS (IFTS)* [14], of *neutrosophic TS (NTS)* [15, 16], of *rough TS (RTS)* [17], of *soft TS (STS)* [18], etc. In particular, a STS is defined as follows:

Definition 15: A *soft topology* T on a non-empty set U is a collection of SSs in U with respect to a set of parameters E such that:

- The absolute and the null soft sets E_U and $E_\emptyset$ belong to T
- The intersection of any two elements of T and the union of an arbitrary (finite or infinite) number of elements of T belong also to T.

The elements of a ST T on U are called *open SS* and their complements are called *closed SS.* The triple (U, T, E) is called a STS on U.

Examples 5: Trivial examples of STs are the *discrete ST* $\{E_\emptyset, E_U\}$ and the *non-discrete ST* consisting of all SSs in U. Reconsider also Example 4. It is straightforward to check then that $T = \{E_U$, $E_\emptyset$, S, S_2, S_9, $S_{11}\}$ is a ST on U.

The concepts of limit, continuity, compactness, and Hausdorff TS are extended to STSs in a way analogous of FTSs [19, 20]. In fact, Definitions 7, 9 and 10 are easily turned to corresponding definitions of STSs by replacing the expression "fuzzy sets" with the expression "soft sets". For the concept of continuity we need the following Lemma ([19], definition 3.12) :

Lemma 2: Let (U, T. A), (V, S, B) be STSs and let u: U→V, p: A→B be given maps. Then a map f_{pu} is defined with respect to u and p mapping the soft sets of T to soft sets of S.

Proof: If (F, A) is a soft set of T, then its image $f_{pu}((F, A))$ is a soft set of S defined by

$f_{pu}((F, A))=(f_{pu}(F), p(A))$, where, ∀ y ∈ B is

$$f_{pu}(F)(y)= \bigcup_{x\in p^{-1}(y)\cap A} u(F(x)) \quad \text{if} \quad p^{-1}(y)\cap A\neq\emptyset \quad \text{and}$$

$f_{pu}(F)(y) = \emptyset$ otherwise.

Conversely, if (G, B) is a soft set of S, then its inverse image $f_{pu}^{-1}((G, B))$ is a soft set of T defined by $f_{pu}^{-1}((G, B))=(f_{pu}^{-1}(G), p^{-1}(B))$, where ∀ x ∈ A is $f_{pu}^{-1}(G)(x)=u^{-1}(G(p(x))$.

Definition 16: Let (U, T. A), (V, S, B) be STSs and let u: U→V, p: A→B be given maps. Then the map f_{pu}, defined by Lemma 2, is said to be *soft pu-continuous*, if, and only if, the inverse image of each open soft set in Y through f_{pu} is an open soft set in X.

4. Discussion and Conclusions

In this review paper we extended the classical concept of TS to FTS and STS and we generalized in them the fundamental notions of limit, continuity, compactness and Hausdorff space. Further extensions of the concept of TS to IFTS, NTS and to

other fuzzy structures are very important and could be studied by the author in a future review paper.
In general, apart from its many and important practical applications (e.g. [21,22], etc.), fuzzy mathematics has been also significantly developed on a theoretical basis, providing useful links even to classical branches of pure mathematics, like Algebra, Analysis, Geometry, Topology, etc. (e.g. [23,24], etc.) This is, therefore, a promising area for future research.

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