Approximations of Fixed Point of Nonexpansive Mappings in Banach Spaces

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Abstract: -In this paper, we used condition $(\mathcal{E})$ to study approximation on the Banach space, demonstrating the convergence theorem as well as an example that supports the main theorem. Iterative fixed point approximation for nonlinear operators is a novel area of investigation. As a result, the literature contains a number of iterative techniques for overcoming such impediments and improving the rate of convergence.

Key-Words: Approximations of Fixed Point, Nonexpansive Mappings, Banach Spaces

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1 Introduction

Iterative fixed point approximation for nonlinear operators is a novel area of investigation (see, for example, [1, 2, 3, 4, 5] and others). Using a Picard iterative technique, the Banach contraction principle determines the unique fixed point of a contraction mapping. On the other hand, the Picard iterative technique does not necessarily converge to the fixed point of a nonexpansive mapping.

Let $\mathcal{X}$ be a Banach space, whereas $\emptyset \neq \mathcal{C} \subseteq \mathcal{X}$, and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$. If $v = \mathcal{T}v$, an element $v \in \mathcal{C}$ is regarded to as a fixed point for $\mathcal{T}$. The set of all fixed points of the map $\mathcal{T}$ is denoted by $\mathcal{F}(\mathcal{T})$. The set of all natural numbers shall be denoted by $\mathbb{N}$ throughout the work. When $\mathcal{T}$ is nonexpansive, that is, for all $\eta, \delta \in \mathcal{C}$,

$$\|\mathcal{T}\eta - \mathcal{T}\delta\| \leq \|\eta - \delta\|.$$

If $\mathcal{X}$ is uniformly convex and $\mathcal{C}$ is convex closed bounded, then $\mathcal{F}(\mathcal{T})$ is nonempty. In 2008, Suzuki [6] presented a new class of nonlinear mappings that is just a generalization of the nonexpansive mappings class. A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is said to obey the condition $(\mathcal{C})$ (or Suzuki mapping) if for all $\eta, \delta \in \mathcal{C}$,

$$\frac{1}{2}\|\eta - \mathcal{T}\eta\| \leq \|\eta - \delta\| \Rightarrow \|\mathcal{T}\eta - \mathcal{T}\delta\| \leq \|\eta - \delta\|.$$

García-Falset et al. [7] extended condition $(\mathcal{C})$ to the following general formulations in 2011. A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is said to satisfy condition $(\mathcal{E}_\mu)$ if there exists some $\mu \geq 1$ such that

$$\|\mathcal{T}\eta - \delta\| \leq \mu\|\mathcal{T}\delta - \delta\| + \|\eta - \delta\| \quad \text{for all } \eta, \delta \in \mathcal{C}.$$

A mapping $\mathcal{T}$ is said to satisfy condition $(\mathcal{E}_\mu)$ (or García-Falset mapping) when it does so for some $\mu \geq 1$. García-Falset et al. demonstrated that every Suzuki mapping meets condition $(\mathcal{E})$ with $\mu = 3$. It is also worth noting that the class of Garcia-Falset mappings includes many other classes of generalized nonexpansive mappings (see [8] for details).

Agarwal iteration process introduced in [9], also called S-iteration process, is defined as:

$$\begin{cases} \eta_0 \in \mathcal{C}, \\ \zeta_n = (1 - \iota_n)\eta_n + \iota_n \mathcal{T}\eta_n, \\ \eta_{n+1} = (1 - \tau_n)\mathcal{T}\eta_n + \tau_n \mathcal{T}\zeta_n, \end{cases} \tag{1}$$

where $\{\iota_n\}$, $\{\tau_n\}$ are sequences in $[0, 1]$.

Suantai and Phuengrattana [10] also introduced one another three-step iteration process known as SP- iteration process, defined as:

$$\begin{cases} \eta_0 \in \mathcal{C}, \\ \zeta_n = (1 - \iota_n)\eta_n + \iota_n \mathcal{T}\eta_n, \\ \vartheta_n = (1 - \tau_n)\zeta_n + \tau_n \mathcal{T}\zeta_n, \\ \eta_{n+1} = (1 - \sigma_n)\vartheta_n + \sigma_n \mathcal{T}\vartheta_n, \end{cases} \tag{2}$$

where $\{\iota_n\}$, $\{\tau_n\}$, $\{\sigma_n\}$ are sequences in $[0, 1]$. Thakur et. al. [11] used a new iteration process, defined as:

$$\begin{cases} \eta_0 \in \mathcal{C}, \\ \zeta_n = (1 - \iota_n)\eta_n + \iota_n \mathcal{T}\eta_n, \\ \vartheta_n = \mathcal{T}((1 - \tau_n)\eta_n + \tau_n \zeta_n) \\ \eta_{n+1} = \mathcal{T}\vartheta_n, \end{cases} \tag{3}$$

where $\{\iota_n\}$, $\{\tau_n\}$ are sequences in $[0, 1]$.

Hussain et al. [12] also introduced the K-iteration process, a three-step iteration procedure defined as:

$$\begin{cases} \eta_0 \in \mathcal{C}, \\ \zeta_n = (1 - \iota_n)\eta_n + \iota_n \mathcal{T}\eta_n, \\ \vartheta_n = \mathcal{T}((1 - \tau_n)\mathcal{T}\eta_n + \tau_n \mathcal{T}\zeta_n), \\ \eta_{n+1} = \mathcal{T}\vartheta_n, \end{cases} \tag{4}$$

Ullah et al. [13] also created the K^*-iteration process, a three-step iteration procedure defined as:

$$\begin{cases} \eta_0 \in \mathcal{C}, \\ \zeta_n = (1-\iota_n)\eta_n + \iota_n \mathcal{T}\eta_n, \\ \vartheta_n = \mathcal{T}((1-\tau_n)\zeta_n + \tau_n \mathcal{T}\zeta_n), \\ \eta_{n+1} = \mathcal{T}\vartheta_n, \end{cases} \tag{5}$$

where $\{\iota_n\}$, $\{\tau_n\}$ are sequences in $[0, 1]$.

Ullah et al. [14] also described the AK-iteration process, which consists of three steps:

$$\begin{cases} \eta_0 \in \mathcal{C}, \\ \zeta_n = \mathcal{T}((1-\iota_n)\eta_n + \iota_n \mathcal{T}\eta_n), \\ \vartheta_n = \mathcal{T}((1-\tau_n)\zeta_n + \tau_n \mathcal{T}\zeta_n), \\ \eta_{n+1} = \mathcal{T}\vartheta_n, \end{cases} \tag{6}$$

where $\{\iota_n\}$, $\{\tau_n\}$ are sequences in $[0, 1]$.

In this paper, we present a novel iteration process designated as the SP^*-iteration process, which is formally defined:

$$\begin{cases} \eta_0 \in \mathcal{C}, \\ \zeta_n = \mathcal{T}((1-\iota_n)\eta_n + \iota_n \mathcal{T}\eta_n), \\ \vartheta_n = \mathcal{T}((1-\tau_n)\zeta_n + \tau_n \mathcal{T}\zeta_n), \\ \eta_{n+1} = \mathcal{T}((1-\sigma_n)\vartheta_n + \sigma_n \mathcal{T}\vartheta_n), \end{cases} \tag{7}$$

where $\{\iota_n\}$, $\{\tau_n\}$, $\{\sigma_n\}$ are sequences in $[0, 1]$ such that $0 < a \le \iota_n, \tau_n, \sigma_n \le b < 1$ for all $n \ge 1$.

2 Preliminaries

Let $\mathcal{C}$ be a nonempty closed convex subset of a Banach space $\mathcal{X}$, and let $\{\eta_n\}$ be a bounded sequence in $\mathcal{X}$. For $\rho \in \mathcal{X}$, we set $r(\rho, \{\eta_n\}) = \limsup_{n\to\infty} \|\eta_n - \rho\|$. The asymptotic radius of $\{\eta_n\}$ relative to $\mathcal{C}$ is given by $r(\mathcal{C}, \{\eta_n\}) = \inf\{r(\rho, \{\eta_n\}) : \rho \in \mathcal{C}\}$ and the asymptotic center of $\{\eta_n\}$ relative to $\mathcal{C}$ is the set

$$\mathcal{A}(\mathcal{C}, \{\eta_n\}) = \{\rho \in \mathcal{C} : r(\rho, \{\eta_n\}) = r(\mathcal{C}, \{\eta_n\})\}.$$

It is known that, in a uniformly convex Banach space, $\mathcal{A}(\mathcal{C}, \{\eta_n\})$ consists of exactly one point.

We say that a Banach space $\mathcal{X}$ has Opial property [15] if and only if for all $\{\eta_n\}$ in $\mathcal{C}$ which weakly converges to $\rho \in \mathcal{X}$ and

Lemma 2.1. [7] *Let $\mathcal{T}$ be a mapping on a subset $\mathcal{C}$ of a Banach space $\mathcal{X}$ having the Opial property. Assume that $\mathcal{T}$ satisfies the condition $(\mathcal{E})$. If $\{\eta_n\}$ converges weakly to ω and $\lim_{n\to\infty} \|\mathcal{T}\eta_n - \eta_n\| = 0$, then $\omega \in \mathcal{F}(\mathcal{T})$.*

Lemma 2.2. [7] *Let $\mathcal{T}$ be a mapping on a subset $\mathcal{C}$ of a Banach space $\mathcal{X}$. If $\mathcal{T}$ satisfies condition $(\mathcal{E})$, then for all $v \in \mathcal{F}(\mathcal{T})$ and $\rho \in \mathcal{C}$, we have*

$$\|\mathcal{T}\rho - v\| \le \|\rho - v\|.$$

Lemma 2.3. [7] *Let $\mathcal{T}$ be a mapping on a subset $\mathcal{C}$ of a Banach space $\mathcal{X}$. If $\mathcal{T}$ satisfies condition $(\mathcal{C})$, then $\mathcal{T}$ also satisfies condition $(\mathcal{E}_\mu)$ with $\mu = 3$.*

Lemma 2.4. [16] *Let $\mathcal{X}$ be a uniformly convex Banach space and $0 < a \le \sigma_n \le b < 1$ for all $n \ge 1$. If $\{\eta_n\}$ and $\{\delta_n\}$ are two sequences in $\mathcal{X}$ such that $\limsup_{n\to\infty} \|\eta_n\| \le \lambda$, $\limsup n \to \infty \|\delta_n\| \le \lambda$, and $\lim_{n\to\infty} \|\sigma_n\eta_n + (1-\sigma_n)\delta_n\| = \lambda$ for some $\lambda \ge 0$, then $\lim_{n\to\infty} \|\eta_n - \delta_n\| = 0$.*

3 Main results

Lemma 3.1. *Let $\mathcal{C}$ be a nonempty closed convex subset of $\mathcal{X}$, which is a uniformly convex Banach space, and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a mapping obeying condition $(\mathcal{E})$ with $\mathcal{F}(\mathcal{T}) \neq \emptyset$. For all $n \ge 1$, $\eta_0 \in \mathcal{C}$. Authorize the sequence $\{\eta_n\}$ to be produced by (7), then the $\lim_{n\to\infty} \|\eta_n - v\|$ exists for any $v \in \mathcal{F}(\mathcal{T})$.*

Proof. Let $v \in \mathcal{F}(\mathcal{T})$ and $\eta_n, \zeta_n, \vartheta_n \in \mathcal{C}$. Because $\mathcal{T}$ is a mapping satisfying condition $(\mathcal{E})$, we obtain

$$\begin{aligned} \|\mathcal{T}\eta_n - v\| &\le \mu\|\mathcal{T}v - v\| + \|\eta_n - v\|, \\ \|\mathcal{T}\zeta_n - v\| &\le \mu\|\mathcal{T}v - v\| + \|\zeta_n - v\|, \\ \|\mathcal{T}\vartheta_n - v\| &\le \mu\|\mathcal{T}v - v\| + \|\vartheta_n - v\|. \end{aligned} \tag{8}$$

So,

$$\begin{aligned} &\|\zeta_n - v\| \\ &= \|\mathcal{T}((1-\iota_n)\eta_n + \iota_n\mathcal{T}\eta_n) - v\| \\ &\le \|(1-\iota_n)\eta_n + \iota_n\mathcal{T}\eta_n - v\| \\ &\le (1-\iota_n)\|\eta_n - v\| + \iota_n\|\mathcal{T}\eta_n - v\| \\ &\le (1-\iota_n)\|\eta_n - v\| + \iota_n[\mu\|\mathcal{T}v - v\| + \|\eta_n - v\|] \\ &= (1-\iota_n)\|\eta_n - v\| + \iota_n\|\eta_n - v\| \\ &= \|\eta_n - v\|. \end{aligned} \tag{9}$$

Using (9), we obtain

$$\begin{aligned} &\|\vartheta_n - v\| \\ &= \|\mathcal{T}((1-\tau_n)\zeta_n + \tau_n\mathcal{T}\zeta_n) - v\| \\ &\le \|(1-\tau_n)\zeta_n + \tau_n\mathcal{T}\zeta_n - v\| \\ &\le (1-\tau_n)\|\zeta_n - v\| + \tau_n\|\mathcal{T}\zeta_n - v\| \\ &\le (1-\tau_n)\|\zeta_n - v\| + \tau_n[\mu\|\mathcal{T}v - v\| + \|\zeta_n - v\|] \\ &= \|\zeta_n - v\| \\ &\le \|\eta_n - v\|. \end{aligned} \tag{10}$$

Similarly, using (10), we obtain

$$\begin{aligned} &\|\eta_{n+1} - v\| \\ &= \|\mathcal{T}((1-\sigma_n)\vartheta_n + \sigma_n\mathcal{T}\vartheta_n) - v\| \\ &\le \|(1-\sigma_n)\vartheta_n + \sigma_n\mathcal{T}\vartheta_n - v\| \\ &\le (1-\sigma_n)\|\vartheta_n - v\| + \sigma_n\|\mathcal{T}\vartheta_n - v\| \\ &\le (1-\sigma_n)\|\vartheta_n - v\| + \sigma_n[\mu\|\mathcal{T}v - v\| + \|\vartheta_n - v\|] \\ &= \|\vartheta_n - v\| \\ &\le \|\eta_n - v\|. \end{aligned} \tag{11}$$

This means that for all $v \in \mathcal{F}(\mathcal{T})$, $\{\|\eta_n - v\|\}$ is bounded and non-increasing. As a result,$\lim_{n\to\infty} \|\eta_n - v\|$ exists, as necessary. □

Theorem 3.2. *Let $\mathcal{C}$ be a nonempty closed convex subset of $\mathcal{X}$, which is a uniformly convex Banach space, and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a mapping obeying condition $(\mathcal{E})$.For all $n \geq 1, \eta_0 \in \mathcal{C}$. Authorize the sequence $\{\eta_n\}$ to be produced by (7), where $\{\iota_n\}$, $\{\tau_n\}$, $\{\sigma_n\}$ are sequences in $[0,1]$ such that $0 < a \leq \iota_n, \tau_n, \sigma_n \leq b < 1$. Then $\mathcal{F}(\mathcal{T}) \neq \emptyset$ if and only if $\lim_{n\to\infty} \|\mathcal{T}\eta_n - \eta_n\| = 0$.*

Proof. Suppoes $\mathcal{F}(\mathcal{T}) \neq \emptyset$ and let $v \in \mathcal{F}(\mathcal{T})$. Then, by Lemma 3.1, $\lim_{n\to\infty} \|\eta_n - v\|$ exists and $\{\eta_n\}$ is bounded. Put

$$\lim_{n\to\infty} \|\eta_n - v\| = \lambda \geq 0. \tag{12}$$

From (9) and (12), we obtain

$$\limsup_{n\to\infty} \|\zeta_n - v\| \leq \limsup_{n\to\infty} \|\eta_n - v\| = \lambda. \tag{13}$$

From (8), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \|\mathcal{T}\eta_n - v\| \\
&\leq \limsup_{n\to\infty} \mu\|\mathcal{T}v - v\| + \limsup_{n\to\infty} \|\eta_n - v\| \\
&= \limsup_{n\to\infty} \|\eta_n - v\| = \lambda.
\end{aligned} \tag{14}$$

From (10) and (12), we obtain

$$\limsup_{n\to\infty} \|\vartheta_n - v\| \leq \limsup_{n\to\infty} \|\eta_n - v\| = \lambda. \tag{15}$$

From (8), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \|\mathcal{T}\vartheta_n - v\| \\
&\leq \limsup_{n\to\infty} \mu\|\mathcal{T}v - v\| + \limsup_{n\to\infty} \|\vartheta_n - v\| \\
&= \limsup_{n\to\infty} \|\eta_n - v\| = \lambda.
\end{aligned} \tag{16}$$

From (11), we obtain

$$\|\eta_{n+1} - v\| \leq \|\vartheta_n - v\|$$

Therefore,

$$\lambda \leq \liminf_{n\to\infty} \|\vartheta_n - v\|. \tag{17}$$

Using (15) and (17), we have

$$\lambda = \lim_{n\to\infty} \|\vartheta_n - v\|. \tag{18}$$

From (10), we obtain

$$\|\vartheta_n - v\| \leq \|\zeta_n - v\|.$$

So,

$$\lambda \leq \liminf_{n\to\infty} \|\zeta_n - v\|. \tag{19}$$

From (13) and (19), we have

$$\lambda = \lim_{n\to\infty} \|\zeta_n - v\|. \tag{20}$$

Using (20), (12) and (14), we obtain

$$\begin{aligned}
\lambda &= \lim_{n\to\infty} \|\zeta_n - v\| \\
&= \lim_{n\to\infty} \|\mathcal{T}((1-\iota_n)\eta_n + \iota_n\mathcal{T}\eta_n) - v\| \\
&\leq \lim_{n\to\infty} \|(1-\iota_n)\eta_n + \iota_n\mathcal{T}\eta_n - v\| \\
&\leq (1-\iota_n)\lim_{n\to\infty} \|\eta_n - v\| + \iota_n \lim_{n\to\infty} \|\mathcal{T}\eta_n - v\| \\
&= \lambda.
\end{aligned}$$

Therefore,

$$\lambda = \lim_{n\to\infty} \|\mathcal{T}((1-\iota_n)\eta_n + \iota_n\mathcal{T}\eta_n) - v\|. \tag{21}$$

Applying Lemma 2.4, we have

$$\lim_{n\to\infty} \|\mathcal{T}\eta_n - \eta_n\| = 0. \tag{22}$$

Conversely, suppose that $\{\eta_n\}$ is bounded and $\lim_{n\to\infty} \|\mathcal{T}\eta_n - \eta_n\| = 0$. Let $\mathcal{A}(\mathcal{C}, \{\eta_n\})$. By (8), we have

$$\begin{aligned}
r(\mathcal{T}v, \{\eta_n\}) &= \limsup_{n\to\infty} \|\eta_n - \mathcal{T}v\| \\
&\leq \limsup_{n\to\infty} (\mu\|\mathcal{T}\eta_n - \eta_n\| + \|\eta_n - v\|) \\
&\leq \limsup_{n\to\infty} \|\eta_n - v\| \\
&= r(v, \{\eta_n\}).
\end{aligned}$$

This means that $\mathcal{T}v \in \mathcal{A}(\mathcal{C}, \{\eta_n\})$. Because $\mathcal{X}$ is uniformly convex, is a singleton set and hence we have $\mathcal{T}v = v$. Thus, $\mathcal{F}(\mathcal{T}) \neq \emptyset$. □

Theorem 3.3. *Let $\mathcal{C}$ be a nonempty closed convex subset of $\mathcal{X}$, which is a uniformly convex Banach space with the Opial property, and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a mapping obeying condition $(\mathcal{E})$. For all $n \geq 1$, $\eta_0 \in \mathcal{C}$. Authorize the sequence $\{\eta_n\}$ to be produced by (7), where $\{\iota_n\}$, $\{\tau_n\}$, $\{\sigma_n\}$ are sequences in $[0,1]$ such that $0 < a \leq \iota_n, \tau_n, \sigma_n \leq b < 1$. Then $\{\eta_n\}$ converges weakly to a fixed point of $\mathcal{T}$.*

Proof. Let $\mathcal{F}(\mathcal{T}) \neq \emptyset$ implies that $\{\eta_n\}$ is bounded and $\lim_{n\to\infty} \|\mathcal{T}\eta_n - \eta_n\| = 0$. Because $\mathcal{X}$ is uniformly convex, it is reflexive. According to Eberlin's theorem, there exists a subsequence of $\{\eta_{n_j}\}$ of $\{\eta_n\}$ which thus converges weakly to some $\omega \in \mathcal{X}$. Because $\mathcal{C}$ is closed and convex, Mazur's theorem states that $\omega \in \mathcal{C}$. And $\omega \in \mathcal{F}(\mathcal{T})$, that according Lemma

2.1. We also show that $\{\eta_n\}$ weakly converges to ω. If this is not the circumstance, then there must be a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ such that $\{\eta_{n_k}\}$ converges weakly to $\varpi \in \mathcal{C}$ and $\omega \neq \varpi$. By Lemma 2.1, $\varpi \in \mathcal{F}(\mathcal{T})$. Because $\lim_{n\to\infty} \|\eta_n - v\|$ exists for every $v \in \mathcal{F}(\mathcal{T})$. By Lemma 3.1 and Opial property, we have

$$\begin{aligned}\lim_{n\to\infty} \|\eta_n - \omega\| &= \lim_{j\to\infty} \|x_{n_j} - \omega\| \leq \lim_{j\to\infty} \|\eta_{n_j} - \varpi\| \\ &= \lim_{n\to\infty} \|\eta_n - \varpi\| = \lim_{k\to\infty} \|\eta_{n_k} - \varpi\| \\ &< \lim_{k\to\infty} \|\eta_{n_k} - \omega\| = \lim_{n\to\infty} \|\eta_n - \omega\|,\end{aligned}$$

which is a contradiction. So $\omega = \varpi$. This means that $\{\eta_n\}$ converges weakly to a fixed point of $\mathcal{T}$. □

Next we prove the strong convergence theorem.

Theorem 3.4. *Let $\mathcal{C}$ be a nonempty closed convex subset of $\mathcal{X}$, which is a uniformly convex Banach space, and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a mapping obeying condition $(\mathcal{E})$. For all $n \geq 1$, $\eta_0 \in \mathcal{C}$. Authorize the sequence $\{\eta_n\}$ to be produced by* (7)*, where $\{\iota_n\}$, $\{\tau_n\}$, $\{\sigma_n\}$ are sequences in $[0,1]$ such that $0 < a \leq \iota_n, \tau_n, \sigma_n \leq b < 1$. Then $\{\eta_n\}$ converges strongly to a fixed point of $\mathcal{T}$.*

Proof. We know that $\mathcal{F}(\mathcal{T}) \neq \emptyset$ according to Lemma 2.1, and that $\lim_{n\to\infty} \|\mathcal{T}\eta_n - \eta_n\| = 0$ owing to Theorem 3.2. Because $\mathcal{C}$ is compact, there exists a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ such that $\{\eta_{n_k}\}$ converges strongly to v for some $v \in \mathcal{C}$. Because $\mathcal{T}$ satisfies condition $(\mathcal{E})$, we have

$$\|\eta_{n_k} - \mathcal{T}v\| \leq \mu\|\mathcal{T}\eta_{n_k} - v\| + \|\eta_{n_k} - v\|, \quad \text{for all} \quad n \geq 1.$$

Taking $k \to \infty$, we obtain $\mathcal{T}v = v$ *i.e.*, $v \in \mathcal{F}(\mathcal{T})$. By Lemma 3.1, $\lim_{n\to\infty} \|\eta_n - v\|$ exists for every $v \in \mathcal{F}(\mathcal{T})$ and so $\{\eta_n\}$ converge strongly to v. □

Senter and Dotson [19] suggested the concept of a mapping obeying the condition $(\mathcal{I})$, which will be defined as: If there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(u) > 0$ for all $u > 0$ such that $\|\rho - \mathcal{T}\rho\| \geq f(\text{dist}(\mathcal{F}(\mathcal{T}))$ for all $\rho \in \mathcal{C}$, where $\text{dist}(\rho, \mathcal{F}(\mathcal{T})) = \inf_{v\in\mathcal{F}(\mathcal{T})} \|\rho - v\|$, and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$.

We now use condition $(\mathcal{I})$ to show the strong convergence theorem.

Theorem 3.5. *Let $\mathcal{C}$ be a nonempty closed convex subset of $\mathcal{X}$, which is a uniformly convex Banach space, and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a mapping obeying condition $(\mathcal{E})$. For all $n \geq 1$, $\eta_0 \in \mathcal{C}$. Authorize the sequence $\{\eta_n\}$ to be produced by* (7)*, where $\{\iota_n\}$, $\{\tau_n\}$, $\{\sigma_n\}$ are sequences in $[0,1]$ such that $0 < a \leq \iota_n, \tau_n, \sigma_n \leq b < 1$ with $\mathcal{F}(\mathcal{T}) \neq \emptyset$. If $\mathcal{T}$ satisfies condition $(\mathcal{I})$, then $\{\eta_n\}$ converges strongly to a fixed point of $\mathcal{T}$.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|\eta_n - v\|$ exists for every $v \in \mathcal{F}(\mathcal{T})$ and $\lim_{n\to\infty} d(\eta_n, \mathcal{F}(\mathcal{T}))$ exists. Assume that $\lim_{n\to\infty} \|\eta_n - v\|$ for some $u \geq 0$. If $u = 0$ then the result follows. Suppose $u > 0$. From the hypothesis and condition $(\mathcal{I})$,

$$f(\text{dist}(\eta_n, \mathcal{F}(\mathcal{T}))) \leq |\mathcal{T}\eta_n - \eta_n\|. \tag{23}$$

Because $\mathcal{F}(\mathcal{T}) \neq \emptyset$, by Theorem 3.2, we have $\lim_{n\to\infty} \|\mathcal{T}\eta_n - \eta_n\| = 0$. So (23) implies that

$$\lim_{n\to\infty} f(\text{dist}(\eta_n, \mathcal{F}(\mathcal{T}))) = 0. \tag{24}$$

Because f is nondecreasing function, as a result of (24), we have $\lim_{n\to\infty} \text{dist}(\eta_n, \mathcal{F}(\mathcal{T})) = 0$. Thus, we have a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ and a sequence $\{\vartheta_n\}$ such that

$$\|\eta_{n_k} - \vartheta_k\| \leq \frac{1}{2^k} \quad \text{for all} \quad k \in \mathbb{N}.$$

So, using (11), we get

$$\|\eta_{n_{k+1}} - \vartheta_k\| \leq \|\vartheta_{n_k} - \vartheta_k\| \leq \frac{1}{2^k}.$$

Hence

$$\begin{aligned}\|\vartheta_{n_{k+1}} - \vartheta_k\| &\leq \|\vartheta_{n_{k+1}} - \eta_{n_{k+1}}\| + \|\eta_{n_{k+1}} - \vartheta_k\| \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \to 0 \quad \text{as} \quad k \to \infty.\end{aligned}$$

This shows that $\{\vartheta_n\}$ is Cauchy sequence in $\mathcal{F}(\mathcal{T})$ and so it converges to a point v. Because $\mathcal{F}(\mathcal{T})$ is closed, $v \in \mathcal{F}(\mathcal{T})$ and then $\{\eta_{n_k}\}$ converges strongly to v. Because $\lim_{n\to\infty} \|\eta_n - v\|$ exists, we have that $\eta_n \to v \in \mathcal{F}(\mathcal{T})$. □

4 Numerical Example

Let $\mathcal{E} = (-\infty, \infty)$ with usual norm and $\mathcal{C} = [1, 10]$. Define $\mathcal{T}$ on $\mathcal{C}$ satisfying condition $(\mathcal{E})$ with $\mu = 3$ as follow:

$$\mathcal{T}\eta = \frac{2\eta + 5}{3}.$$

We will show that

$$|\mathcal{T}\eta - \delta| \leq 3|\mathcal{T}\delta - \delta| + |\eta - \delta| \quad \text{for all} \quad \eta, \delta \in \mathcal{C}.$$

In fact,

$$\begin{aligned}|\mathcal{T}\eta-\delta| &\leq |\delta-\mathcal{T}\delta|+|\mathcal{T}\delta-\mathcal{T}\eta| \\ &\leq |\delta-\mathcal{T}\delta|+\left|\frac{2\delta+5}{3}-\frac{2\eta+5}{3}\right| \\ &= |\delta-\mathcal{T}\delta|+\frac{2}{3}|\eta-\delta| \\ &\leq 3|\mathcal{T}\delta-\delta|+|\eta-\delta|.\end{aligned}$$

Now, we conclude that $\mathcal{T}$ satisfies condition $(\mathcal{E})$. Using the initial value $\eta_1 = 8.5$ and letting the stopping criteria $|\eta_n - 5| < 10^{-6}$, reckoning the iterative values of K^*-iteration process, AK-iteration process and SP^*-iteration process for choose $\iota_n = \frac{8n}{9n+1}$, $\tau_n = \frac{9n}{10n+1}$, and $\sigma_n = \frac{7n}{8n+1}$ as show in Table 1 and Figure 1.

Table 1: Comparative sequence

Iter.	K^*	AK	SP^*
1	8.500000	8.500000	8.500000
2	5.829630	5.553086	5.446939
3	5.189445	5.084198	5.053214
4	5.042681	5.012646	5.006173
5	5.009549	5.001886	5.000706
6	5.002127	5.000280	5.000080
7	5.000473	5.000041	5.000009
8	5.000105	5.000006	5.000001
9	5.000023	5.000006	5.000000
10	5.000005	5.000000	5.000000
11	5.000001	5.000000	5.000000
12	5.000000	5.000000	5.000000

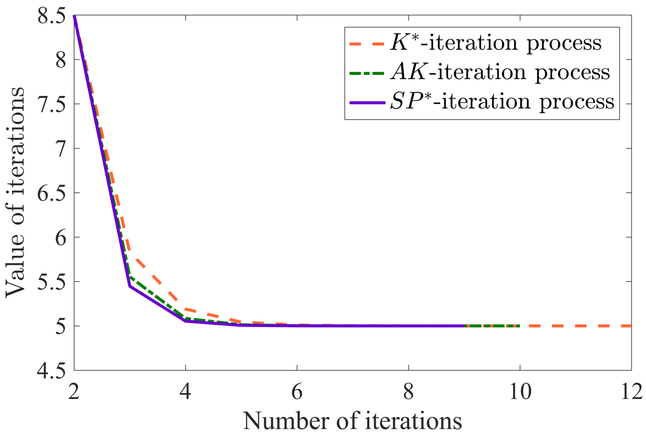

Figure 1: The plotting of comparative sequence in Table 1

5 Conclusion

In this study, we proposed a new modified fixed point algorithms to approximate the solution of fixed points problem of a nonexpansive mapping in the framework of Banach space. We performed convergence analysis of the proposed algorithm and hence proved some convergence theorems. Also, we provided some illustrative numerical examples to show the efficiency of the proposed algorithm.

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