New results for degenerated generalized Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials

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Abstract- The main objective of this work is to deduce some interesting algebraic relationships that connect the degenerated generalized Apostol–Bernoulli, Apostol–Euler and Apostol– Genocchi polynomials and other families of polynomials such as the generalized Bernoulli polynomials of level m and the Genocchi polynomials. Futher, find new recurrence formulas for these three families of polynomials to study.

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1. Introduction

Let $m \in \mathbb{N}$. For parameters $a, b \in \mathbb{R}$ and $\lambda, \alpha \in \mathbb{C}$, Ramírez et al. in [4] introduces three new classes of the Apostol-Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x; a, b; \lambda)$, the degenerated generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x; a, b; \lambda)$ and the degenerated generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{[m-1,\alpha]}(x; a, b; \lambda)$ of level m by means of the following generating functions, defined in a suitable neighborhood of $t = 0$:

$$t^{m\alpha}[\sigma(\lambda; a, b; t)]^{\alpha}(1 + at)^{\frac{x}{a}} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{[m-1,\alpha]}(x; a, b; \lambda)\frac{t^n}{n!},$$

$$2^{m\alpha}[\psi(\lambda; a, b; t)]^{\alpha}(1 + at)^{\frac{x}{a}} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{[m-1,\alpha]}(x; a, b; \lambda)\frac{t^n}{n!},$$

$$(2t)^{m\alpha}[\psi(\lambda; a, b; t)]^{\alpha}(1+at)^{\frac{x}{a}} = \sum_{n=0}^{\infty} \mathfrak{G}_n^{[m-1,\alpha]}(x; a, b; \lambda)\frac{t^n}{n!},$$

where

$$\sigma(\lambda; a, b; t) = \left(\lambda(1 + at)^{\frac{1}{a}} - \sum_{l=0}^{m-1} \frac{(t \log b)^l}{l!}\right)^{-1}$$

and,

$$\psi(\lambda; a, b; t) = \left(\lambda(1 + at)^{\frac{1}{a}} + \sum_{l=0}^{m-1} \frac{(t \log b)^l}{l!}\right)^{-1}.$$

The following proposition summarizes some elementary properties of the degenerated generalized the Apostol-Bernoulli polynomials, the degenerated generalized Apostol-Euler polynomials and the degenerated generalized Apostol-Genocchi polynomials, in the variable x, (cf. [4]).

Proposition I..1 *For a $m \in \mathbb{N}$ fixed, let $\{\mathfrak{B}_n^{[m-1,\alpha]}(x; a, b; \lambda)\}_{n\geq 0}$, $\{\mathfrak{E}_n^{[m-1,\alpha]}(x; a, b; \lambda)\}_{n\geq 0}$ and $\{\mathfrak{G}_n^{[m-1,\alpha]}(x; a, b; \lambda)\}_{n\geq 0}$ be the sequence of degenerated generalized Apostol-type polynomials in the variable x, $a, b \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level m. Then the followings identities (Addition theorem of the argument) hold.*

$$\mathfrak{B}_n^{[m-1,\alpha+\beta]}(x + y; a, b; \lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_k^{[m-1,\alpha]}(x; a, b; \lambda)\mathfrak{B}_{n-k}^{[m-1,\beta]}(y; a, b; \lambda),$$

$$\mathfrak{B}_n^{[m-1,\alpha]}(x + y; a, b; \lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{B}_k^{[m-1,\alpha]}(y; a, b; \lambda)(x|a)_{n-k}, \quad (1)$$

$$\mathfrak{E}_n^{[m-1,\alpha+\beta]}(x + y; a, b; \lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(x; a, b; \lambda)\mathfrak{E}_{n-k}^{[m-1,\beta]}(y; a, b; \lambda),$$

$$\mathfrak{E}_n^{[m-1,\alpha]}(x + y; a, b; \lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y; a, b; \lambda)(x|a)_{n-k},$$

$$\mathfrak{G}_n^{[m-1,\alpha+\beta]}(x + y; a, b; \lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{G}_k^{[m-1,\alpha]}(x; a, b; \lambda)\mathfrak{G}_{n-k}^{[m-1,\beta]}(y; a, b; \lambda),$$

$$\mathfrak{G}_n^{[m-1,\alpha]}(x + y; a, b; \lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{G}_k^{[m-1,\alpha]}(y; a, b; \lambda)(x|a)_{n-k}.$$

On the subject of the Appell–type polynomials and their various extensions, a remarkably large number of investigations have appeared in the literature, see for example (see, [1, 3, 7, 10]).

On the other hand, the first–kind Stirling number $s(n,k)$ is the number of ways in which n objects can be divided among k non–empty cycles and the second-kind Stirling numbers $S(n,k)$ count the number of ways to partition a set of n elements into exactly k nonempty subsets. The generating functions are given, respectively, by (see [8]):

$$\frac{1}{k!}\left[\ln(1+t)\right]^k = \sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!}$$

and,

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!}.$$

The generalized falling factorial $(x|a)_n$ with increment a is defined by (see [9, Definition 2.3]):

$$(x|a)_n = \prod_{k=0}^{n-1}(x-ak),$$

for positive integer n, with the convention $(x|a)_0 = 1$, it follows that

$$(x|a)_n = \sum_{k=0}^{n} s(n,k)a^{n-k}x^k. \tag{2}$$

Proposition I..2 *For $m \in \mathbb{N}$. Let $\{B_n^{[m-1]}(x)\}_{n\geq 0}$ and $\{G_n(x)\}_{n\geq 0}$ be the sequences of generalized Bernoulli polynomials of level m and Genocchi polynomials, respectively. Then, the following identities are satisfied.*

1) [6, Equation (2.6)].

$$x^n = \sum_{k=0}^{n}\binom{n}{k}\frac{k!}{(k+m)!}B_{n-k}^{[m-1]}(x), \tag{3}$$

2) [5, Remark 7].

$$x^n = \frac{1}{2(n+1)}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}G_k(x) + G_{n+1}(x)\right]. \tag{4}$$

2. Some connection formulas for degenerated generalized Apostol–Bernoulli, Apostolï Euler and Apostolï Genocchi polynomials

From the Proposition I..2 it is possible to deduce some interesting algebraic relations connecting the degenerated generalized Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials and other families of polynomials such as generalized Bernoulli polynomials of level m, Genocchi polynomials and Apostol-Euler polynomials.

Theorem II..1 *For $m \in \mathbb{N}$, degenerated generalized Apostol–Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, are related with the generalized Bernoulli polynomials $B_n^{[m-1]}(x)$ of level m, by means of the following identity.*

$$\mathfrak{B}_n^{[m-1,\alpha]}(x+y;a,b;\lambda) = \sum_{k=0}^{n}\sum_{j=0}^{n-k}\sum_{r=0}^{\nu}\binom{n}{k}\binom{\nu}{r}\frac{r!a^{n-k-j}}{(r+m)!} \times \mathfrak{B}_k^{[m-1,\alpha]}(y;a,b;\lambda)s(n-k,j)B_{\nu-r}^{[m-1]}(x).$$

Proof 1 *By substituting* (3) *and* (2) *into the right-hand side of* (1), *we have*

$$\begin{aligned}
&\mathfrak{B}_n^{[m-1,\alpha]}(x+y;a,b;\lambda)\\
&= \sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_k^{[m-1,\alpha]}(y;a,b;\lambda)(x|a)_{n-k}\\
&= \sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_k^{[m-1,\alpha]}(y;a,b;\lambda)\sum_{j=0}^{n-k}s(n-k,j)a^{n-k-j}x^j.\\
&= \sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_k^{[m-1,\alpha]}(y;a,b;\lambda)\sum_{j=0}^{n-k}s(n-k,j)a^{n-k-j}\\
&\times \sum_{r=0}^{\nu}\binom{\nu}{r}\frac{r!}{(r+m)!}B_{\nu-r}^{[m-1]}(x)\\
&= \sum_{k=0}^{n}\sum_{j=0}^{n-k}\sum_{r=0}^{\nu}\binom{n}{k}\binom{\nu}{r}\frac{r!a^{n-k-j}}{(r+m)!}\\
&\times \mathfrak{B}_k^{[m-1,\alpha]}(y;a,b;\lambda)s(n-k,j)B_{\nu-r}^{[m-1]}(x).
\end{aligned}$$

Therefore, Theorem II..1 holds.

The proofs of Theorem II..2 and Theorem II..3, it is analogously to Theorem II..1.

Theorem II..2 *For $m \in \mathbb{N}$, degenerated generalized Apostol–Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, are related with the generalized Bernoulli polynomials $B_n^{[m-1]}(x)$ of level m, by means of the following identity.*

$$\begin{aligned}
&\mathfrak{E}_n^{[m-1,\alpha]}(x+y;a,b;\lambda)\\
&= \sum_{k=0}^{n}\sum_{j=0}^{n-k}\sum_{r=0}^{\nu}\binom{n}{k}\binom{\nu}{r}\frac{r!a^{n-k-j}}{(r+m)!}\\
&\times \mathfrak{E}_k^{[m-1,\alpha]}(y;a,b;\lambda)s(n-k,j)B_{\nu-r}^{[m-1]}(x).
\end{aligned}$$

Theorem II..3 *For $m \in \mathbb{N}$, degenerated generalized Apostol–Genocchi polynomials $\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, are related with the generalized Bernoulli polynomials $B_n^{[m-1]}(x)$ of level m , by means of the following identity.*

$$\begin{aligned}
&\mathfrak{G}_n^{[m-1,\alpha]}(x+y;a,b;\lambda)\\
&= \sum_{k=0}^{n}\sum_{j=0}^{n-k}\sum_{r=0}^{\nu}\binom{n}{k}\binom{\nu}{r}\frac{r!a^{n-k-j}}{(r+m)!}\\
&\times \mathfrak{G}_k^{[m-1,\alpha]}(y;a,b;\lambda)s(n-k,j)B_{\nu-r}^{[m-1]}(x).
\end{aligned}$$

Theorem II..4 *For $m \in \mathbb{N}$, degenerated generalized Apostol–Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, are* *related with the Genocchi polynomials $G_n(x)$, by means of the following identity.*

$$\begin{aligned}&\mathfrak{B}_n^{[m-1,\alpha]}(x+y;a,b;\lambda)\\&=\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\frac{a^{n-k-j}}{2(\nu+1)}\sum_{r=0}^{\nu+1}G_r(x)\\&\times\mathfrak{B}_k^{[m-1,\alpha]}(y;a,b;\lambda)s(n-k,j)\binom{\nu+1}{r}\\&+\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}s(n-k,j)G_{\nu+1}(x).\end{aligned}$$

Proof 2 *By substituting* (4) *and* (2) *into the right-hand side of* (1), *we obtain*

$$\begin{aligned}&\mathfrak{B}_n^{[m-1,\alpha]}(x+y;a,b;\lambda)\\&=\sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_k^{[m-1,\alpha]}(y;a,b;\lambda)(x|a)_{n-k}\\&=\sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_k^{[m-1,\alpha]}(y;a,b;\lambda)\sum_{j=0}^{n-k}s(n-k,j)a^{n-k-j}x^j\\&=\sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_k^{[m-1,\alpha]}(y;a,b;\lambda)\sum_{j=0}^{n-k}s(n-k,j)a^{n-k-j}\\&\times\left[\frac{1}{2(\nu+1)}\sum_{r=0}^{\nu+1}\binom{\nu+1}{k}G_r(x)+\frac{1}{2(\nu+1)}G_{\nu+1}(x)\right]\\&=\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\frac{a^{n-k-j}}{2(\nu+1)}\sum_{r=0}^{\nu+1}\binom{\nu+1}{k}s(n-k,j)\\&\times\mathfrak{B}_k^{[m-1,\alpha]}(y;a,b;\lambda)G_r(x)\\&+\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\frac{a^{n-k-j}}{2(\nu+1)}G_{\nu+1}(x)\\&\times\mathfrak{B}_k^{[m-1,\alpha]}(y;a,b;\lambda)s(n-k,j).\end{aligned}$$

Therefore, Theorem II..4 holds.

The proofs of Theorem II..5 and Theorem II..6, it is analogously to Theorem II..4.

Theorem II..5 *For $m\in\mathbb{N}$, degenerated generalized Apostol–Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, are related with the Genocchi polynomials $G_n(x)$, by means of the following identity.*

$$\begin{aligned}&\mathfrak{E}_n^{[m-1,\alpha]}(x+y;a,b;\lambda)=\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\frac{a^{n-k-j}}{2(\nu+1)}\\&\sum_{r=0}^{\nu+1}G_r(x)\times\mathfrak{E}_k^{[m-1,\alpha]}(y;a,b;\lambda)s(n-k,j)\binom{\nu+1}{r}\\&+\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\frac{a^{n-k-j}}{2(\nu+1)}\mathfrak{E}_k^{[m-1,\alpha]}(y;a,b;\lambda)\\&\times s(n-k,j)G_{\nu+1}(x).\end{aligned}$$

Theorem II..6 *For $m\in\mathbb{N}$, degenerated generalized Apostol–Genocchi polynomials $\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, are related with the Genocchi polynomials $G_n(x)$, by means of the following identity.*

$$\begin{aligned}&\mathfrak{G}_n^{[m-1,\alpha]}(x+y;a,b;\lambda)\\&=\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\frac{a^{n-k-j}}{2(\nu+1)}\sum_{r=0}^{\nu+1}G_r(x)\\&\times\mathfrak{G}_k^{[m-1,\alpha]}(y;a,b;\lambda)s(n-k,j)\binom{\nu+1}{r}\\&+\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}\frac{a^{n-k-j}}{2(\nu+1)}\mathfrak{G}_k^{[m-1,\alpha]}(y;a,b;\lambda)\\&\times s(n-k,j)G_{\nu+1}(x).\end{aligned}$$

Theorem II..7 *For $m\in\mathbb{N}$, degenerated generalized Apostol–Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, they satisfy the following relation.*

$$\begin{aligned}\mathfrak{B}_n^{[m-1,\alpha]}(ax+x;a,b;\lambda)&=\sum_{k=0}^{n-1}\binom{x}{k+1}\binom{n-1}{k}a^{k+1}nk!\\&\times\mathfrak{B}_{n-1-k}^{[m-1,\alpha]}(x;a,b;\lambda)+\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda).\end{aligned}$$

Proof 3 *By the generating function of degenerated generalized Apostol–Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)$ and considering $\varphi_n=\mathfrak{B}_n^{[m-1,\alpha]}(ax+x;a,b;\lambda)$ and $\psi_n=\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}[\varphi_n-\psi_n]\frac{t^n}{n!}=t^{m\alpha}[\sigma(\lambda;a,b;t)]^{\alpha}(1+at)^{\frac{ax+x}{a}}\\&\quad-t^{m\alpha}[\sigma(\lambda;a,b;t)]^{\alpha}(1+at)^{\frac{x}{a}}\\&=t^{m\alpha}[\sigma(\lambda;a,b;t)]^{\alpha}(1+at)^{\frac{x}{a}}\left[(1+at)^x-1\right]\\&=\sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\binom{x}{n+1}a^{k+1}z^{n+1}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{x}{k+1}\binom{n}{k}k!a^{k+1}\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^{n+1}}{n!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\binom{x}{k+1}\binom{n-1}{k}k!na^{k+1}\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of the equation, the result is

$$\begin{aligned}\mathfrak{B}_n^{[m-1,\alpha]}(ax+x;a,b;\lambda)&=\sum_{k=0}^{n-1}\binom{x}{k+1}\binom{n-1}{k}a^{k+1}nk!\\&\times\mathfrak{B}_{n-1-k}^{[m-1,\alpha]}(x;a,b;\lambda)+\mathfrak{B}_n^{[m-1,\alpha]}(x;a,b;\lambda).\end{aligned}$$

Theorem II..8 *For $m\in\mathbb{N}$, degenerated generalized Apostol–Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)$, they sat-*

isfy the following relation.

$$\begin{aligned}
&\mathfrak{E}_n^{[m-1,\alpha]}(ax+x;a,b;\lambda)\\
&=\sum_{k=0}^{n-1}\binom{x}{k+1}\binom{n-1}{k}a^{k+1}nk!\\
&\times\mathfrak{E}_{n-1-k}^{[m-1,\alpha]}(x;a,b;\lambda)+\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda).
\end{aligned}$$

Proof 4 *By the generating function of degenerated generalized Apostol–Euler polynomials* $\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)$ *and considering* $\varphi_n=\mathfrak{E}_n^{[m-1,\alpha]}(ax+x;a,b;\lambda)$ *and* $\psi_n=\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)$*, we have*

$$\begin{aligned}
&\sum_{n=0}^{\infty}[\varphi_n-\psi_n]\frac{t^n}{n!}=2^{m\alpha}[\psi(\lambda;a,b;t)]^{\alpha}(1+at)^{\frac{ax+x}{a}}\\
&-2^{m\alpha}[\psi(\lambda;a,b;t)]^{\alpha}(1+at)^{\frac{x}{a}}\\
&=2^{m\alpha}[\psi(\lambda;a,b;t)]^{\alpha}(1+at)^{\frac{x}{a}}\left[(1+at)^x-1\right]\\
&=\sum_{n=0}^{\infty}\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\binom{x}{n+1}a^{k+1}z^{n+1}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{x}{k+1}\binom{n}{k}k!a^{k+1}\\
&\times\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^{n+1}}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\binom{x}{k+1}\binom{n-1}{k}k!na^{k+1}\\
&\times\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of the equation, the result is

$$\begin{aligned}
\mathfrak{E}_n^{[m-1,\alpha]}(ax+x;a,b;\lambda)&=\sum_{k=0}^{n-1}\binom{x}{k+1}\binom{n-1}{k}a^{k+1}nk!\\
&\times\mathfrak{E}_{n-1-k}^{[m-1,\alpha]}(x;a,b;\lambda)+\mathfrak{E}_n^{[m-1,\alpha]}(x;a,b;\lambda).
\end{aligned}$$

Theorem II..9 *For* $m\in\mathbb{N}$*, degenerated generalized Apostol–Genocchi polynomials* $\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)$*, they satisfy the following relation.*

$$\begin{aligned}
\mathfrak{G}_n^{[m-1,\alpha]}(ax+x;a,b;\lambda)&=\sum_{k=0}^{n-1}\binom{x}{k+1}\binom{n-1}{k}a^{k+1}nk!\\
&\times\mathfrak{G}_{n-1-k}^{[m-1,\alpha]}(x;a,b;\lambda)+\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda).
\end{aligned}$$

Proof 5 *By the generating function of degenerated generalized Apostol–Genocchi polynomials* $\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)$ *and considering* $\varphi_n=\mathfrak{G}_n^{[m-1,\alpha]}(ax+x;a,b;\lambda)$ *and* $\psi_n=\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)$*, we have*

$$\begin{aligned}
&\sum_{n=0}^{\infty}[\varphi_n-\psi_n]\frac{t^n}{n!}=(2t)^{m\alpha}[\sigma(\lambda;a,b;t)]^{\alpha}(1+at)^{\frac{ax+x}{a}}\\
&-(2t)^{m\alpha}[\psi(\lambda;a,b;t)]^{\alpha}(1+at)^{\frac{x}{a}}\\
&=(2t)^{m\alpha}[\psi(\lambda;a,b;t)]^{\alpha}(1+at)^{\frac{x}{a}}\left[(1+at)^x-1\right]\\
&=\sum_{n=0}^{\infty}\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\binom{x}{n+1}a^{k+1}z^{n+1}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{x}{k+1}\binom{n}{k}k!a^{k+1}\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^{n+1}}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\binom{x}{k+1}\binom{n-1}{k}k!na^{k+1}\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides of the equation, the result is

$$\begin{aligned}
\mathfrak{G}_n^{[m-1,\alpha]}(ax+x;a,b;\lambda)&=\sum_{k=0}^{n-1}\binom{x}{k+1}\binom{n-1}{k}a^{k+1}nk!\\
&\times\mathfrak{G}_{n-1-k}^{[m-1,\alpha]}(x;a,b;\lambda)+\mathfrak{G}_n^{[m-1,\alpha]}(x;a,b;\lambda).
\end{aligned}$$

3. Conclusion

In this work, new properties of the degenerated generalized Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials are studied, using various generating function methods. The generalization of these results can lead to other interesting results, which can be useful for fractional calculus theory.

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