Finding an Optimal Proximity Bound in a Very Special Scenario

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Abstract—Given A ∈ $Z^{m\times n}$ and b ∈ Z^m, we provide a sharp upper bound for the ℓ∞-distance from any vertex of the polyhedron P(A, b) = {x ∈ R^n ≥0 : Ax = b} to a nearby feasible integral point under a strong assumption regarding the dimensions of the matrix A. It is hoped that this result provides motivation for conducting further research into providing more such upper bounds under certain additional assumptions.

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1. Introduction

In order to introduce the main result and also provide some background material, we require some notation. Let $A = (a_{ij}) \in \mathbb{Z}^{m\times n}$ with $m < n$ and let $\tau = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ with $i_1 < \dots < i_k$ be an index set. We will use the notation A_τ for the $m \times k$ submatrix of A with columns indexed by τ. In a similar manner, given $\boldsymbol{x} \in \mathbb{R}^n$, we will denote by $\boldsymbol{x}_\tau$ the vector $(\boldsymbol{x}_{i_1}, \dots, \boldsymbol{x}_{i_k})^T$. The complement of τ in $\{1, \dots, n\}$ will be denoted by $\bar{\tau} = \{1, \dots, n\}\backslash\tau$. We will say that τ is a *basis* if $|\tau| = m$ and the submatrix A_τ is nonsingular. In the scenario that τ is a basis, we will replace τ by γ. Further, denote by $\Delta(A)$ the maximum absolute valued m-dimensional subdeterminant of A, namely

$$\Delta(A) = \max\{|\det(A_\tau)| : \tau \subset \{1, \dots, n\} \text{ with } |\tau| = m\}.$$

If $\Delta(A)$ is positive, the notation $\gcd(A)$ will denote the greatest common divisor of all m-dimensional subdeterminants of the matrix A.

We assume without loss of generality that $A \in \mathbb{Z}^{m\times n}$ with $m < n$ has full rank m and let $\boldsymbol{b} \in \mathbb{Z}^m$. Consider the polyhedron

$$P = P(A, \boldsymbol{b}) = \left\{\boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : A\boldsymbol{x} = \boldsymbol{b}\right\}.$$

Upon further assuming that P is nonempty, we take any vertex $\boldsymbol{x}^*$ of P. Since A has rank m by assumption, it follows that there exits a basis γ of A such that

$$\boldsymbol{x}^*_\gamma = A_\gamma^{-1}\boldsymbol{b} \text{ and } \boldsymbol{x}^*_{\bar{\gamma}} = \boldsymbol{0}. \tag{1}$$

It is worth noting that in general, given a vertex $\boldsymbol{x}^*$ of P, the basis γ need not be unique, however, if one assumes that $\boldsymbol{x}^*$ is nondegenerate, i.e. exactly m of the x_i^*'s are nonzero, then there is indeed a unique choice for the basis γ.

We will estimate the ℓ_∞-distance from a vertex $\boldsymbol{x}^*$ of the polyhedron P, which is given by a basis γ as in (1), to the set of its (feasible) lattice points $\boldsymbol{z} \in P \cap \mathbb{Z}^n$. It is worth noting that bounds of this form provide information regarding the level of accuracy when one "relaxes" an integer program (IP) and instead solves the related linear program (LP). This type of relaxation is often (see e.g. [6, Section 11.5]) implemented since solving the decision version of an IP is in general $\mathcal{NP}$-complete (see e.g. [12, Chapter 18]), however, it is well-known that one can solve an LP in polynomial time via the ellipsoid [9] or the interior-point method [8].

Before stating our result, we recall some of the upper bounds on the distance from a vertex $\boldsymbol{x}^*$ of P to the set of its lattice points $\boldsymbol{z} \in P \cap \mathbb{Z}^n$. For this purpose, let us assume that P is integer feasible, i.e. that $P \cap \mathbb{Z}^n \neq \emptyset$. The classical sensitivity theorem of Cook et al. [4] implies that

$$\|\boldsymbol{x}^* - \boldsymbol{z}\|_\infty \leq n \cdot \Delta(A) \tag{2}$$

holds. Eisenbrand and Weismantel [5] improved on this classical bound through a novel use of Steinitz Lemma [14] in order to demonstrate that

$$\|\boldsymbol{x}^* - \boldsymbol{z}\|_1 \leq m\,(2m\|A\|_\infty + 1)^m$$

holds, where $\|A\|_\infty = \max_{i,j}|a_{ij}|$ is the maximum absolute entry of A. Lee et al. [10] utilise bounds on the sparsity of feasible integer points to show that

$$\|\boldsymbol{x}^* - \boldsymbol{z}\|_1 < 3m^2 \log\left(2\sqrt{m} \cdot \Delta(A)^{1/m}\right) \cdot \Delta(A)$$

holds. Lee et al. [11] demonstrate that

$$\|\boldsymbol{x}^* - \boldsymbol{z}\|_1 \leq m(m+1)^2 \cdot \Delta^3(A) + (m+1) \cdot \Delta(A)$$

holds in a slightly more general setting. A very recent strong improvement over the classical bound of Cook et al. [4] obtained by Celaya et al. [3] demonstrates that we can replace n by $n/2$ in (2) provided $n \geq 2$. It remains an open question how tight these bounds actually are. Despite this, in the knapsack scenario, i.e. when $m = 1$, upon following traditional vector notation, Aliev et al. [2] show that

$$\|\boldsymbol{x}^* - \boldsymbol{z}\|_\infty \leq \|\boldsymbol{a}\|_\infty - 1$$

holds and that this bound is optimal.

Since the main result in this article provides an upper bound on the ℓ_∞-distance from any vertex $\boldsymbol{x}^*$ of P to the set of its lattice points, it is useful to define (as in [2]) the *(maximum) vertex distance* $d(A, \boldsymbol{b})$ as

$$d(A,\boldsymbol{b}) = \begin{cases} \max\limits_{\boldsymbol{x}^*} \min\limits_{\boldsymbol{z}\in P\cap\mathbb{Z}^n} \|\boldsymbol{x}^* - \boldsymbol{z}\|_\infty, & \text{if } P\cap\mathbb{Z}^n \neq \emptyset, \\ & -\infty, \ \text{otherwise}, \end{cases}$$

where the maximum is taken over all vertices $\boldsymbol{x}^*$ of the polyhedron P.

The following theorem provides a sharp upper bound for the (maximum) vertex distance when $n = m+1$. As stated in the abstract, it is hoped that this result provides motivation for conducting further research into finding more such upper bounds under additional assumptions. It should be noted that by sharp we mean that we can construct an example where the equality appearing in (3) is attained. We provide this example directly after the statement of the theorem.

Theorem 1. *Let $A \in \mathbb{Z}^{m\times(m+1)}$ with full rank m and $\boldsymbol{b}\in\mathbb{Z}^m$. If $P\cap\mathbb{Z}^{m+1}\neq\emptyset$, then*

$$d(A,\boldsymbol{b}) \leq \frac{\Delta(A)}{\gcd(A)} - 1. \tag{3}$$

The following example demonstrates the sharpness of Theorem 1 in the case when $m = 2$. Consider

$$A = \begin{pmatrix} 3 & 11 & 7 \\ -5 & 7 & 3 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 154 \\ 58 \end{pmatrix}.$$

The absolute values of the 2×2 subdeterminants of A are here 76, 44 and 16, respectively. In particular, $\Delta(A) = 76$ and $\gcd(A) = 4$. The polyhedron P is in this case a line segment connecting $\boldsymbol{x}_1^* = (110/19, 236/19, 0)^T$ and $\boldsymbol{x}_2^* = (14/11, 0, 236/11)^T$ and, in addition, the only feasible integer point is $\boldsymbol{z} = (2,2,18)^T$. The ℓ_∞-norm distances from the two vertices to $\boldsymbol{z}$ are

$$\|\boldsymbol{x}_1^* - \boldsymbol{z}\|_\infty = \max\left(\left|\frac{110}{19} - 2\right|, \left|\frac{236}{19} - 2\right|, 18\right) = 18$$

and

$$\|\boldsymbol{x}_2^* - \boldsymbol{z}\|_\infty = \max\left(\left|\frac{14}{11} - 2\right|, 2, \left|\frac{236}{11} - 18\right|\right) = \frac{38}{11},$$

respectively. In particular, we have $d(A,\boldsymbol{b}) = 18$ and simply noting that

$$\frac{\Delta(A)}{\gcd(A)} - 1 = \frac{76}{4} - 1 = 19 - 1 = 18$$

holds demonstrates that the equality in (3) is attained.

Theorem 1 can be easily applied to estimate the (additive) integrality gap for an IP with the assumed dimension. Given $A\in\mathbb{Z}^{m\times(m+1)}$ with full rank m, $\boldsymbol{b}\in\mathbb{Z}^m$ and a cost vector $\boldsymbol{c}\in\mathbb{Q}^n$, we now consider the IP

$$\min\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x}\in P\cap\mathbb{Z}^{m+1}\} \tag{4}$$

and assume that (4) is feasible and bounded.

Denote by $IP(\boldsymbol{c}, A, \boldsymbol{b})$ and $LP(\boldsymbol{c}, A, \boldsymbol{b})$ the optimal values of the IP (4) and its linear programming relaxation

$$\min\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x}\in P\}, \tag{5}$$

respectively. The *(additive) integrality gap* $IG(\boldsymbol{c}, A, \boldsymbol{b})$ associated with the IP (4) is

$$IG(\boldsymbol{c}, A, \boldsymbol{b}) = IP(\boldsymbol{c}, A, \boldsymbol{b}) - LP(\boldsymbol{c}, A, \boldsymbol{b}).$$

Clearly, in this case, we have

$$IG(\boldsymbol{c}, A, \boldsymbol{b}) \leq d(A,\boldsymbol{b})\cdot\|\boldsymbol{c}\|_1. \tag{6}$$

Notice that the upper bound (3) is independent of the right-hand side $\boldsymbol{b}$ and, as such, it is possible to bound the *integer programming gap* [7]. Given a pair $(A,\boldsymbol{b})$, the integer programming gap is the maximum $IG(\boldsymbol{c}, A, \boldsymbol{b})$ over all suitable integral $\boldsymbol{b}$, namely

$$\mathrm{Gap}(\boldsymbol{c}, A) = \max_{\boldsymbol{b}} IG(\boldsymbol{c}, A, \boldsymbol{b}),$$

where $\boldsymbol{b}$ ranges over all integer vectors such that the IP (4) is feasible and bounded.

As a corollary of Theorem 1, we obtain the following upper bound on the integer programming gap under the current assumptions.

Corollary 1. *Let $A\in\mathbb{Z}^{m\times(m+1)}$ with full rank m, $\boldsymbol{b}\in\mathbb{Z}^m$ and $\boldsymbol{c}\in\mathbb{Q}^n$. If $P\cap\mathbb{Z}^{m+1}\neq\emptyset$, then*

$$\mathrm{Gap}(\boldsymbol{c}, A) \leq \left(\frac{\Delta(A)}{\gcd(A)} - 1\right)\cdot\|\boldsymbol{c}\|_1.$$

The proof of this corollary follows immediately from (6) and the statement of Theorem 1.

2. Proof of Theorem 1

In order to simply the notation slightly during the proof of the main result, we let $\Lambda = \Lambda(A,\boldsymbol{b})$ denote the affine lattice in $\mathbb{R}^{m+1}$ formed by taking integer points in the (affine) flat that is described by the linear system $A\boldsymbol{x} = \boldsymbol{b}$, namely

$$\Lambda = \Lambda(A,\boldsymbol{b}) = \left\{\boldsymbol{x}\in\mathbb{R}^{m+1} : A\boldsymbol{x} = \boldsymbol{b}\right\}\cap\mathbb{Z}^{m+1}.$$

Further, let $\pi(\cdot) : \mathbb{R}^{m+1}\to\mathbb{R}$ denote the projection onto the final coordinate, i.e. the projection which forgets about the first m coordinates.

Proof. Reordering the columns of the matrix A if necessary, we may assume without loss of generality that

$$\gamma = \{1, 2, \ldots, m\},$$

i.e. that $A = (A_\gamma, A_{\bar\gamma})$, where $\det(A_\gamma)\neq 0$. It should be noted that $A_{\bar\gamma}$ is here an m-dimensional column vector. The polyhedron P can be written as

$$P = \left\{\boldsymbol{x}\in\mathbb{R}^{m+1}_{\geq 0} : A_\gamma\boldsymbol{x}_\gamma + A_{\bar\gamma}x_{\bar\gamma} = \boldsymbol{b}\right\},$$

where $\boldsymbol{x} = (\boldsymbol{x}_\gamma, x_{\bar\gamma})^T$, i.e. $\boldsymbol{x}_\gamma\in\mathbb{R}^m$ and $x_{\bar\gamma} = x_{m+1}\in\mathbb{R}$ contain the entries of the vector $\boldsymbol{x}$ corresponding to A_γ and $A_{\bar\gamma}$, respectively. In this case, the conditions (1) on $\boldsymbol{x}^*$ become

$$\boldsymbol{x}^*_\gamma = A_\gamma^{-1}\boldsymbol{b} \text{ and } x^*_{\bar\gamma} = x^*_{m+1} = 0.$$

Further, using Cramer's Rule (see e.g. [13, Theorem 2.20]), the vertex $\boldsymbol{x}^*$ has the form

$$\boldsymbol{x}^* = \left(\frac{\det\left(A_\gamma^1(\boldsymbol{b})\right)}{\det\left(A_\gamma\right)}, \ldots, \frac{\det\left(A_\gamma^m(\boldsymbol{b})\right)}{\det\left(A_\gamma\right)}, 0 \right)^T, \tag{7}$$

where $A_\gamma^i(\boldsymbol{b})$ denotes the submatrix A_γ whose i-th column has been replaced by $\boldsymbol{b}$. Observe that if the polyhedron P is bounded, then P is a line segment in $\mathbb{R}^{m+1}$ connecting its two vertices. If instead P is unbounded, then the polyhedron P is a ray in $\mathbb{R}^{m+1}$, where $\boldsymbol{x}^*$ is the only vertex of P.

Recall that we are upper bounding the distance with respect to the ℓ_∞-norm from the vertex $\boldsymbol{x}^*$ to some (feasible) integral point. We denote such an integral point by

$$\boldsymbol{z} = (z_1, \ldots, z_{m+1})^T \in P \cap \mathbb{Z}^{m+1}.$$

Further, we assume for technical reasons that $\boldsymbol{z}$ is the feasible lattice point with minimal final entry. In other words, we assume the lattice point $\boldsymbol{z}$ is the closest (feasible) integral point to the vertex $\boldsymbol{x}^*$ in the x_{m+1}-th coordinate direction with respect to the ℓ_∞-norm.

Note that by the form (7) of the vertex $\boldsymbol{x}^*$, its projection is $\pi(\boldsymbol{x}^*) = 0$. Firstly, we upper bound

$$\|\pi(\boldsymbol{x}^*) - \pi(\boldsymbol{z})\|_\infty = z_{m+1}$$

before "lifting" to a (feasible) lattice point in $\mathbb{R}^{m+1}$. Recall that Λ denotes the affine lattice in $\mathbb{R}^{m+1}$ containing all integer points which satisfy $A\boldsymbol{x} = \boldsymbol{b}$. In particular, its projection $\pi(\Lambda)$ is a one-dimensional affine lattice which, in light of [1, Lemma 10], has determinant

$$\det(\Lambda) = \frac{|\det(A_\gamma)|}{\gcd(A)}.$$

It follows that all projected affine lattice points from $\pi(\Lambda)$ belong to the same congruence class. Further, upon noting that the least residue in this congruence class is one of the integers $\{0, 1, \ldots, \det(\Lambda) - 1\}$, it follows since $\boldsymbol{z}$ is by assumption the closest to the vertex $\boldsymbol{x}^*$ in the final coordinate direction that $\pi(\boldsymbol{z})$ satisfies

$$\pi(\boldsymbol{z}) = z_{m+1} \leq \det(\Lambda) - 1 = \frac{|\det(A_\gamma)|}{\gcd(A)} - 1. \tag{8}$$

Observe that if one fixes the value of z_{m+1}, then the corresponding m-dimensional integer solution $\boldsymbol{z}_\gamma$ is uniquely determined by

$$\boldsymbol{z}_\gamma = A_\gamma^{-1}\Big(\boldsymbol{b} - A_{\bar{\gamma}} z_{m+1}\Big)$$

since the submatrix A_γ is nonsingular by assumption. In order to simplify subsequent notation, we let

$$\tilde{\boldsymbol{b}} = \boldsymbol{b} - A_{\bar{\gamma}} z_{m+1}.$$

In particular, using Cramer's Rule, for fixed z_{m+1}, the corresponding m-dimensional integral solution $\boldsymbol{z}_\gamma$ is given by

$$\begin{aligned}\boldsymbol{z}_\gamma &= (z_1, \ldots, z_m)^T \\ &= \left(\frac{\det\left(A_\gamma^1(\tilde{\boldsymbol{b}})\right)}{\det\left(A_\gamma\right)}, \ldots, \frac{\det\left(A_\gamma^m(\tilde{\boldsymbol{b}})\right)}{\det\left(A_\gamma\right)} \right)^T.\end{aligned} \tag{9}$$

Further, the m-dimensional solution (9) to the linear system $A_\gamma \boldsymbol{x}_\gamma = \tilde{\boldsymbol{b}}$ can be "lifted" to yield a solution to the original linear system by simply appending the fixed value z_{m+1} to the $(m+1)$-th entry. In other words, we can write

$$\begin{aligned}\boldsymbol{z} &= (z_1, \ldots, z_m, z_{m+1})^T \\ &= \left(\frac{\det\left(A_\gamma^1(\tilde{\boldsymbol{b}})\right)}{\det\left(A_\gamma\right)}, \ldots, \frac{\det\left(A_\gamma^m(\tilde{\boldsymbol{b}})\right)}{\det\left(A_\gamma\right)}, z_{m+1} \right)^T.\end{aligned}$$

Using (7), the vertex distance is

$$\begin{aligned}&\|\boldsymbol{x}^* - \boldsymbol{z}\|_\infty \\ &\quad = \max\left(\left| \frac{\det\left(A_\gamma^j(\boldsymbol{b})\right) - \det\left(A_\gamma^j(\tilde{\boldsymbol{b}})\right)}{\det\left(A_\gamma\right)} \right|, z_{m+1} \right),\end{aligned} \tag{10}$$

where $1 \leq j \leq m$. Recall (8) and, in particular, clearly

$$|x_{m+1}^* - z_{m+1}| = z_{m+1} \leq \frac{\Delta(A)}{\gcd(A)} - 1$$

holds as required by (3).

It remains to finally consider the other differences appearing in (10). In particular, we consider

$$|x_j^* - z_j| = \left| \frac{\det\left(A_\gamma^j(\boldsymbol{b})\right) - \det\left(A_\gamma^j(\tilde{\boldsymbol{b}})\right)}{\det\left(A_\gamma\right)} \right| \tag{11}$$

for $j \in \{1, \ldots, m\}$. Upon noting that $A_\gamma^j(\boldsymbol{b})$ and $A_\gamma^j(\tilde{\boldsymbol{b}})$ differ only by their j-th column and recalling that

$$\tilde{\boldsymbol{b}} = \boldsymbol{b} - A_{\bar{\gamma}} z_{m+1}$$

for fixed z_{m+1}, it follows using several fundamental properties of determinants that (11) can be equivalently expressed as

$$|x_j^* - z_j| = \left| \frac{z_{m+1} \cdot \left|\det\left(A_\gamma^j(A_{\bar{\gamma}})\right)\right|}{|\det\left(A_\gamma\right)|} \right|. \tag{12}$$

It is worth noting that we could alternatively deduce (12) from (11) by applying Laplace's expansion formula (see e.g. [13, Theorem 10.33]) and then performing some algebraic manipulation.

Upon recalling (8), we can bound (12) by

$$\begin{aligned}&\left| \frac{z_{m+1} \cdot \left|\det\left(A_\gamma^j(A_{\bar{\gamma}})\right)\right|}{|\det\left(A_\gamma\right)|} \right| \\ &\quad \leq \left| \left|\det\left(A_\gamma^j(A_{\bar{\gamma}})\right)\right| \cdot \left(\frac{1}{\gcd(A)} - \frac{1}{|\det(A_\gamma)|} \right) \right|.\end{aligned} \tag{13}$$

Notice that the matrix $A_\gamma^j(A_{\bar{\gamma}})$ contains only columns from A and, in consequence,

$$\left|\det\left(A_\gamma^j(A_{\bar{\gamma}})\right)\right| \leq \Delta(A)$$

holds. Finally, since

$$\gcd(A) \leq \left|\det\left(A_\gamma\right)\right| \leq \Delta(A)$$

holds, it follows that (13) is bounded by

$$\left| \left| \det\left(A_{\gamma}^{j}(A_{\bar{\gamma}})\right) \right| \cdot \left(\frac{1}{\gcd(A)} - \frac{1}{|\det(A_{\gamma})|} \right) \right|$$
$$\leq \left| \Delta(A) \cdot \left(\frac{1}{\gcd(A)} - \frac{1}{\Delta(A)} \right) \right| = \frac{\Delta(A)}{\gcd(A)} - 1,$$

which implies that (3) holds as required and concludes the proof of Theorem 1. □

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References

[1] Iskander Aliev, Marcel Celaya, Martin Henk, and Aled Williams. Distance-sparsity transference for vertices of corner polyhedra. *SIAM Journal on Optimization*, 31(1):200–216, 2021.
[2] Iskander Aliev, Martin Henk, and Timm Oertel. Distances to lattice points in knapsack polyhedra. *Mathematical Programming*, 2019.
[3] Marcel Celaya, Stefan Kuhlmann, Joseph Paat, and Robert Weismantel. Improving the cook et al. proximity bound given integral valued constraints. In Karen Aardal and Laura Sanità, editors, *Integer Programming and Combinatorial Optimization*, pages 84–97, Cham, 2022. Springer International Publishing.
[4] William Cook, Albertus M H Gerards, Alexander Schrijver, and Éva Tardos. Sensitivity theorems in integer linear programming. *Mathematical Programming*, 34(3):251–264, 1986.
[5] Friedrich Eisenbrand and Robert Weismantel. Proximity results and faster algorithms for integer programming using the steinitz lemma. *ACM Transactions on Algorithms (TALG)*, 16(1):1–14, 2019.
[6] Frederick S Hillier and Gerald J Lieberman. *Introduction to Operations Research*. McGraw-Hill, 9th edition, 2009.
[7] Serkan Hoşten and Bernd Sturmfels. Computing the integer programming gap. *Combinatorica*, 27(3):367–382, 2007.
[8] Narendra Karmarkar. A new polynomial-time algorithm for linear programming. In *Proceedings of the sixteenth annual ACM symposium on Theory of computing*, pages 302–311, 1984.
[9] Leonid G Khachiyan. A polynomial algorithm in linear programming. In *Doklady Akademii Nauk*, volume 244.5, pages 1093–1096. Russian Academy of Sciences, 1979.
[10] Jon Lee, Joseph Paat, Ingo Stallknecht, and Luze Xu. Improving proximity bounds using sparsity. In Mourad Baïou, Bernard Gendron, Oktay Günlük, and A. Ridha Mahjoub, editors, *Combinatorial Optimization*, pages 115–127, Cham, 2020. Springer International Publishing.
[11] Jon Lee, Joseph Paat, Ingo Stallknecht, and Luze Xu. Polynomial upper bounds on the number of differing columns of δ-modular integer programs. *arXiv preprint arXiv:2105.08160*, 2021.
[12] Alexander Schrijver. *Theory of Linear and Integer Programming*. John Wiley & Sons, 1998.
[13] Igor R Shafarevich and Alexey O Remizov. *Linear algebra and geometry*. Springer Science & Business Media, 2012.
[14] Ernst Steinitz. Bedingt konvergente reihen und konvexe systeme. *Journal für die reine und angewandte Mathematik*, 143:128–176, 1913.