Some classes of quasi *-algebras

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Abstract: In this paper we will continue the analysis undertaken in [1] and in [2] [20] our investigation on the structure of Quasi-local quasi *-algebras possessing a sufficient family of bounded positive tracial sesquilinear forms. In this paper it is shown that any Quasi-local quasi *-algebras (A, A0), possessing a ”sufficient state” can be represented as non-commutative L2-spaces.

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1. Introduction

For reader convenience we collect below some preliminary definitions and propositions that will be used in what follows. A *quasi *-algebra* is a couple $(\mathcal{A}, \mathcal{A}_0)$, where $\mathcal{A}$ is a vector space with involution *, $\mathcal{A}_0$ is a *-algebra and a vector subspace of $\mathcal{A}$ and $\mathcal{A}$ is an $\mathcal{A}_0$-bimodule whose module operations and involution extend those of $\mathcal{A}_0$. The *unit* of $(\mathcal{A}, \mathcal{A}_0)$ is an element $e \in \mathcal{A}_0$ such that $xe = ex = x$, for every $x \in \mathcal{A}$. A quasi *-algebra $(\mathcal{A}, \mathcal{A}_0)$ is said to be *locally convex* if $\mathcal{A}$ is endowed with a topology τ which makes of $\mathcal{A}$ a locally convex space and such that the involution $a \mapsto a^*$ and the multiplications $a \mapsto ab$, $a \mapsto ba$, $b \in \mathcal{A}_0$, are continuous. If τ is a norm topology and the involution is isometric with respect to the norm, we say that $(\mathcal{A}, \mathcal{A}_0)$ is a *normed quasi *-algebra* and, if it is complete, we say it is a *Banach quasi*-algebra.*

Let $\mathcal{A}_\#$ be a C*-algebra, with involution $_\#$ and norm $\| \, \|_\#$, and $\mathcal{X}[\| \, \|]$ a left Banach $\mathcal{A}_\#$-module. This means, in particular, that there is a bounded bilinear map

$$(a, x) \to ax$$

from $\mathcal{A}_\# \times \mathcal{X}$ into $\mathcal{X}$ such that

$$(a_1a_2)x = a_1(a_2x), \quad \forall a_1, a_2 \in \mathcal{A}_\#, x \in \mathcal{X}.$$

We will always assume that the following inequality holds:

$$\|ax\| \leq \|a\|_\# \|x\|, \quad \forall x \in \mathcal{X}, a \in \mathcal{A}_\#.$$

This implies, as shown in [7], that

$$\|a\|_\# = \sup_{x \in \mathcal{X}; \|x\| \leq 1} \|ax\|, \quad a \in \mathcal{A}_\#.$$

A left Banach $\mathcal{A}_\#$-module $\mathcal{X}$ is called a CQ*-algebra if

(i) in $\mathcal{X}$ an involution * is defined and $\|x^*\| = \|x\|$ for every $x \in \mathcal{X}$;

(ii) $\mathcal{A}_{\#}$ is a $\| \|$-dense subspace of $\mathcal{X}$ and the module left-multiplication extends the multiplication of $\mathcal{A}_{\#}$;

(iii) $\mathcal{A}_{\#} \cap \mathcal{A}_{\#}^{*}$ is a *-algebra and it is dense in $\mathcal{A}_{\#}$ with respect to $\| \|_{\#}$;

A CQ*-algebra is denoted with $(\mathcal{X}, *, \mathcal{A}_{\#}, {}_{\#})$ to underline the different involutions.

If $* = {}_{\#}$ on $\mathcal{A}_{\#}$, the CQ*-algebra is called proper.

The following basic definitions and results on non-commutative measure theory and integration are also needed in what follows.

Let $\mathcal{M}$ be a von Neumann algebra and φ a normal faithful semifinite trace defined on $\mathcal{M}_{+}$.

Put

$$\mathcal{J} = \{X \in \mathcal{M} : \varphi(|X|) < \infty\}.$$

$\mathcal{J}$ is a *-ideal of $\mathcal{M}$.

Let $P \in \mathrm{Proj}(\mathcal{M})$, the lattice of projections of $\mathcal{M}$. We say that P is φ-finite if $P \in \mathcal{J}$. Any φ-finite projection is finite.

Definition 1 *A vector subspace $\mathcal{D}$ of $\mathcal{H}$ is said to be strongly dense (resp., strongly φ-dense) if*

- *$U'\mathcal{D} \subset \mathcal{D}$ for any unitary U' in $\mathcal{M}'$*
- *there exists a sequence $P_n \in \mathrm{Proj}(\mathcal{M})$: $P_n\mathcal{H} \subset \mathcal{D}$, $P_n^{\perp} \downarrow 0$ and $(P_n^{\perp})$ is a finite projection (resp., $\varphi(P_n^{\perp}) < \infty$).*

Clearly, every strongly φ-dense domain is strongly dense.

Throughout this paper, when we say that an operator T is affiliated with a von Neumann algebra, written $T \,\eta\, \mathcal{M}$, we always mean that T is closed, densely defined and $TU \supseteq UT$ for every unitary operator $U \in \mathcal{M}'$.

Definition 2 *An operator $T \,\eta\, \mathcal{M}$ is called*

- *measurable (with respect to $\mathcal{M}$) if its domain $D(T)$ is strongly dense;*
- *φ-measurable if its domain $D(T)$ is strongly φ-dense.*

From the definition itself it follows that, if T is φ-measurable, then there exists $P \in \mathrm{Proj}(\mathcal{M})$ such that TP is bounded and $\varphi(P^{\perp}) < \infty$.

We remind that any operator affiliated with a finite von Neumann algebra is measurable [12, Cor. 4.1] but it is not necessarily φ-measurable.

The following statements will be used later.

(i) Let $T \,\eta\, \mathcal{M}$ and $Q \in \mathcal{M}$. If $D(TQ) = \{\xi \in \mathcal{H} : Q\xi \in D(T)\}$ is dense in $\mathcal{H}$, then $TQ \,\eta\, \mathcal{M}$.

(ii) If $Q \in \mathrm{Proj}(\mathcal{M})$, then $Q\mathcal{M}Q = \{QXQ \restriction_{Q\mathcal{H}}; X \in \mathcal{M}\}$ is a von Neumann algebra on the Hilbert space $Q\mathcal{H}$; moreover $(Q\mathcal{M}Q)' = Q\mathcal{M}'Q$. If $T \,\eta\, \mathcal{M}$ and $Q \in \mathcal{M}$ and $D(TQ) = \{\xi \in \mathcal{H} : Q\xi \in D(T)\}$ is dense in $\mathcal{H}$, then $QTQ \,\eta\, Q\mathcal{M}Q$.

Let $\mathcal{M}$ be a von Neumann algebra and φ a normal faithful semifinite trace defined on $\mathcal{M}_{+}$. For each $p \geq 1$, let

$$\mathcal{J}_p = \{X \in \mathcal{M} : \varphi(|X|^p) < \infty\}.$$

Then $\mathcal{J}_p$ is a *-ideal of $\mathcal{M}$. Following [13], we denote with $L^p(\varphi)$ the Banach space completion of $\mathcal{J}_p$ with respect to the norm

$$\|X\|_p := \varphi(|X|^p)^{1/p}, \quad X \in \mathcal{J}_p.$$

One usually defines $L^{\infty}(\varphi) = \mathcal{M}$. Thus, if φ is a finite trace, then $L^{\infty}(\varphi) \subset L^p(\varphi)$ for every $p \geq 1$. As shown in [13], if $X \in L^p(\varphi)$, then X is a measurable operator.

2. Quasi Local Structure

We consider now the case where $\mathcal{A}$ has a *local structure.* Following [10] we construct the local net of C*-algebras as follows.

Let $\mathcal{F}$ be a set of indexes directed upward and with an orthonormality relation $\perp$ such that

(i.) $\forall \alpha \in \mathcal{F}$ there exists $\beta \in \mathcal{F}$ such that $\alpha \perp \beta$;

(ii.) if $\alpha \leq \beta$ and $\beta \perp \gamma$, $\alpha, \beta, \gamma \in \mathcal{F}$, then $\alpha \perp \gamma$;

(iii.) if, for $\alpha, \beta, \gamma \in \mathcal{F}$, $\alpha \perp \beta$ and $\alpha \perp \gamma$, there exists $\delta \in \mathcal{F}$ such that $\alpha \perp \delta$ and $\delta \geq \beta, \gamma$.

Definition 3 *Let $(\mathcal{A}, \mathcal{A}_0)$ be a quasi *-algebra with unit e. We say that $(\mathcal{A}, \mathcal{A}_0)$ has a* local structure *if there exists a net $\{\mathcal{A}_\alpha(\|.\|_\alpha), \alpha \in \mathcal{F}\}$ of subspaces of $\mathcal{A}_0$, indexed by $\mathcal{F}$, such that every $\mathcal{A}_\alpha$ is a C*-algebra (with norm $\|.\|_\alpha$ and unit e) with the properties*

(a.) $\mathcal{A}_0 = \bigcup_{\alpha \in \mathcal{F}} \mathcal{A}_\alpha$

(b.) if $\alpha \geq \beta$ then $\mathcal{A}_\alpha \supset \mathcal{A}_\beta$;

(c.) if $\alpha \perp \beta$, then $xy = yx$ for every $x \in \mathcal{A}_\alpha$, $y \in \mathcal{A}_\beta$.

(d.) if $x \in \mathcal{A}_\alpha \cap \mathcal{A}_\beta$, then $\|x\|_\alpha = \|x\|_\beta$.

*A quasi *-algebra $(\mathcal{A}, \mathcal{A}_0)$ with a local structure will be shortly called a* quasi-local quasi*-algebra.

If $(\mathcal{A}, \mathcal{A}_0)$ is a quasi-local quasi-*algebra, and $x \in \mathcal{A}_0$, there will be some $\beta \in \mathcal{F}$ such that $x \in \mathcal{A}_\beta$. We put $J_x = \{\alpha \in \mathcal{F} : x \in \mathcal{A}_\alpha\}$ and define

$$\|x\| = \inf_{\alpha \in J_x} \|x\|_\alpha.$$

Then $\mathcal{A}_0$ is a C*-normed algebra with norm $\| \cdot \|$.

For instance if we consider the quasi *-algebra $(L^p(I), L^\infty(I))$, where $L^p(I)$ $(p \geq 1)$ and $L^\infty(I)$ are the usual Lebesgue spaces on $I := [0,1]$. Put $\omega(f) = \int_0^1 f(x)dx$, $f \in L^p(I)$. If $1 \leq p < 2$, ω is not representable. Indeed, if $f \in L^p(I) \setminus L^2(I)$, there cannot exist any $\gamma_f > 0$ such that

$$|\omega(f\varphi)| = \left| \int_0^1 f(x)\varphi(x)dx \right| \leq \gamma_f \, \omega(\varphi^*\varphi)^{1/2} = \gamma_f \, \|\varphi\|_2, \quad \forall \varphi \in L^\infty(I),$$

since this would imply that $f \in L^2(I)$ (see [1]).

The following proposition extends the GNS construction to quasi *-algebras. The prof hers in [9].

Proposition 4 *Let ω be a linear functional on $\mathcal{A}$ $\tau-$ continuous satisfying the following requirements:*

(L1) $\omega(a^*a) = \omega(aa^*) \geq 0$ for all $a \in \mathcal{A}_0$;

(L2) $\omega(b^*x^*a) = \overline{\omega(a^*xb)}$, $\forall\, a, b \in \mathcal{A}_0$, $x \in \mathcal{A}$;

(L3) $\forall x \in \mathcal{A}$ there exists $\gamma_x > 0$ such that $|\omega(x^*a)| \leq \gamma_x \, \omega(a^*a)^{1/2}$.

Then there exists a triple $(\pi_\omega, \lambda_\omega, \mathcal{H}_\omega)$ such that

- *π_ω is a ultra-cyclic *-representation of $\mathcal{A}$ with ultra-cyclic vector ξ_ω;*
- *λ_ω is a linear map of $\mathcal{A}$ into $\mathcal{H}_\omega$ with $\lambda_\omega(\mathcal{A}_0) = \mathcal{D}_{\pi_\omega}$, $\xi_\omega = \lambda_\omega(e)$ and $\pi_\omega(x)\lambda_\omega(a) = \lambda_\omega(xa)$, for every $x \in \mathcal{A}$, $a \in \mathcal{A}_0$.*
- *$\omega(x) = \langle \pi_\omega(x)\xi_\omega | \xi_\omega \rangle$, for every $x \in \mathcal{A}$.*
- *$\pi^*_\omega(a)\lambda_\omega(x) = \lambda_\omega(ax)$, for every $x \in \mathcal{A}$, $a \in \mathcal{A}_o$.*

Definition 5 *A linear functional ω $\tau-$ continuous satisfying (L1),(L2),(L3) is called* representable. *We denote by $\mathcal{T}(\mathcal{A})$ the set of representable linear functionals on $\mathcal{A}$.*

3. A Representation Theorem

Once we have constructed in the previous section some Quasi-local quasi *-algebras of operators affiliated to a given von Neumann algebra, it is natural to pose the question under which conditions can an abstract Quasi-local quasi *-algebras be realized as a Quasi-local quasi *-algebras of this type.

We denotes by $[\mathcal{M}]$ the closed subspace of $\mathcal{H}$ spanned by $[\mathcal{M}]$ for any subset $\mathcal{M}$ of $\mathcal{H}$.

Thus for every $\omega \in \mathcal{T}(\mathcal{A})$ we put $\mathcal{H}_\alpha := [\pi_\omega(\mathcal{A}_\alpha)\xi_\omega]$ then $\{\pi_\omega \upharpoonright_{\mathcal{A}_\alpha}, \mathcal{H}_\alpha, \xi_\omega\}\}$ it is a representation of $\mathcal{A}_\alpha$.

Thus, let π_ω be the ultra-cyclic *-representation of $\mathcal{A}$ with ultra-cyclic vector ξ_ω and $\pi_\omega(\mathcal{A}_\alpha)''$ the von Neumann algebra generated by $\pi_\omega(\mathcal{A}_\alpha)$.

For every $\omega \in \mathcal{T}(\mathcal{A})$ and $a \in \mathcal{A}_\alpha$, we put

$$\varphi_{\omega,\alpha}(\pi_\omega(a)) = \omega(a) = \langle \pi_\omega(a)\xi_\omega | \xi_\omega \rangle.$$

Then, for each $\omega \in \mathcal{T}(\mathcal{A})$, $\varphi_{\omega,\alpha}$ is a positive bounded linear functional on the operator algebra $\pi_\omega(\mathcal{A}_\alpha)$. Clearly, for every $a \in \mathcal{A}_\alpha$

$$| \varphi_\omega(\pi_\omega(a)) | = | \langle \pi_\omega(a)\xi_\omega | \xi_\omega \rangle | \leq \|\pi_\omega(a)\| \|\xi_\omega\|^2$$

Thus $\varphi_{\omega,\alpha}$ is continuous on $\pi_\omega(\mathcal{A}_\alpha)$.

By [17, Theorem 10.1.2], $\varphi_{\omega,\alpha}$ is weakly continuous and so it extends uniquely to $\pi_\omega(\mathcal{A}_\alpha)''$. Moreover, since $\varphi_{\omega,\alpha}$ is a trace on $\pi_\omega(\mathcal{A}_\alpha)$:

$$\varphi_{\omega,\alpha}(a^*a) = \omega(a^*a) = \omega(aa^*) = \varphi_{\omega,\alpha}(aa^*)$$

the extension $\widetilde{\varphi_{\omega,\alpha}}$ is a trace on $\mathfrak{M}^\omega := \pi_\omega(\mathcal{A}_\alpha)''$ too.

Theorem 6 *If* $(\mathcal{A}, \mathcal{A}_0)$ *has a* local structure*, ω a state over $\mathcal{A}$ satisfying (L1, L2, L3) such that π_ω, the canonical cyclic representation of $\mathcal{A}$ associated with ω ([9]), is continuous, therefore* $(\pi_\omega(\mathcal{A}), \pi_\omega(\mathcal{A}_0))$ *has a* local structure

Proof: It is easy to verify the following property

$$\pi_\omega(A_0) = \cup_\alpha\{\pi_\omega \upharpoonright_{A_\alpha} (\mathcal{A}_\alpha)\} = \cup_\alpha\{\pi_\omega(\mathcal{A}_\alpha)\} \subseteq B(\mathcal{H}_\omega)$$

with $\pi_\omega(\mathcal{A}_\alpha) \subseteq B(\mathcal{H}_\alpha)$ is a C* algebra which norm $\|\cdot\|_{B(\mathcal{H}_\alpha)} \leq \|\cdot\|$.

The family of C*-algebras $\{\pi_\omega(\mathcal{A}_\alpha), \alpha \in \mathcal{F}\}$ with C*-norm $\|.\|_{B(\mathcal{H}_\alpha)}$, indexed by $\mathcal{F}$, satisfies the following property (a.) if $\alpha \geq \beta$ then $\pi_\omega(\mathcal{A}_\alpha) \supset \pi_\omega(\mathcal{A}_\beta)$; (b.) there exists a unique identity $\pi_\omega(e)$ for all $\pi_\omega(\mathcal{A}_\alpha)$ and the C*-norm $\|.\|_{B(\mathcal{H}_\alpha)}$ are equals a $\|.\|_{B(\mathcal{H})}$; (c.) if $\alpha \perp \beta$ then for all $X \in \pi_\omega(\mathcal{A}_\alpha)$, $Y \in \pi_\omega(\mathcal{A}_\beta)$ there exist $x \in \mathcal{A}_\alpha$, $y \in \mathcal{A}_\beta$ such that $\pi_\omega(x) = X$ and $\pi_\omega(y) = Y$ but $xy = yx$ for all $x \in \mathcal{A}_\alpha$, $y \in \mathcal{A}_\beta$ therefore $XY = \pi_\omega(x)\pi_\omega(y) = \pi_\omega(xy) = \pi_\omega(yx) = \pi_\omega(y)\pi_\omega(x) = YX$.

Thus $\pi_\omega(\mathcal{A}_0)$ is, a quasi-local C*-algebra of operator.

□

Let P_α the operator of projection of $\mathcal{H}$ in $\mathcal{H}_\alpha$ Put $\mathcal{M}_\alpha^\omega := \mathfrak{M}^\omega P_\alpha$, where, as before, P_α denotes the support of $\widetilde{\varphi_{\omega,\alpha}}$.

Each $\mathcal{M}_\alpha^\omega$ is a von Neumann algebra and $\widetilde{\varphi_{\omega,\alpha}}$ is faithful in $\mathcal{M}P_\alpha$ [6, Proposition V. 2.10].

More precisely,

$$\mathcal{M}_\alpha^\omega := \mathcal{M}^\omega P_\alpha = \{Z = XP_\alpha, \text{ for some } X \in \mathcal{M}^\omega\}.$$

In this case, putting $\mathcal{H}_\alpha = P_\alpha\mathcal{H}$, we have

$$\mathcal{H} = \bigoplus_{\alpha\in\mathcal{I}} \mathcal{H}_\alpha = \{(f_\alpha) : f_\alpha \in \mathcal{H}_\alpha, \sum_{\alpha\in I} \|f_\alpha\|^2 < \infty\}.$$

Each vector $X = \{f_\alpha\}_{\alpha\in\mathcal{I}} \in \mathcal{H}$ is denoted by $X = \sum_{\alpha\in I}^{\bigoplus} f_\alpha$ (Definition 3.4, [6]). For each bounded sequencese $\{A_\alpha\}_{\alpha\in\mathcal{I}} \in \prod_{\alpha\in\mathcal{I}} \mathcal{M}_\alpha^\omega$, we define an operator A (following [6]) on $\mathcal{H}$ by

$$AX := A\sum_{\alpha\in I}^{\bigoplus} f_\alpha = \sum_{\alpha\in I}^{\bigoplus} A_\alpha f_\alpha.$$

Clearly A is a bounded operator on $\mathcal{H}$ we denote it by $A = \sum_{\alpha\in I}^{\bigoplus} A_\alpha$.

Let $\sum_{\alpha\in I}^{\bigoplus} \mathcal{M}_\alpha^\omega$ the set of all such A, by Proposition 3.3 [6], $\sum_{\alpha\in I}^{\bigoplus} \mathcal{M}_\alpha^\omega$ is a von Neumann algebra on $\mathcal{H}$. The algebra $\sum_{\alpha\in I}^{\bigoplus} \mathcal{M}_\alpha^\omega$ is called the diret sum of $\{\mathcal{M}_\alpha\}$. Of course for every ω we have $\pi_\omega(\mathcal{A}_0) = \sum_{\alpha\in I}^{\bigoplus} \mathcal{M}_\alpha^\omega$ and

$\widetilde{\varphi_\omega} := \sum_{\alpha\in I}\widetilde{\varphi_{\omega,\alpha}}$ is a faithful semifinite normal trace on $\mathcal{M}^\omega$.

The previous discussion can be summerized in the following

Theorem 7 *If* $(\mathcal{A},\mathcal{A}_0)$ *has a* local structure*, ω a state over $\mathcal{A}$ satisfying (L1, L2, L3) such that π_ω, the canonical cyclic representation of $\mathcal{A}$ associated with ω ([9]), is continuous, therefore* $(L^2(\widetilde{\varphi_\omega}),\pi_\omega(\mathcal{A}_0)'')$ *has a* local structure.

Theorem 8 *If* $(\mathcal{A},\mathcal{A}_0)$ *has a* local structure *the CQ*-Algebra* $(L^2(\widetilde{\varphi_\omega}),\pi_\omega(A_o)''))$ *consists of operators affiliated with* $\pi_\omega(A_o)''$.

Proposition 9 *If* $(\mathcal{A},\mathcal{A}_0)$ *is a quasi-local quasi-*algebra,* $\omega\in\mathcal{T}(\mathcal{A})$ *and* $(\mathcal{H}_\omega,\pi_\omega,\xi_\omega)$ *is a the canonical cyclic representation of* $\mathcal{A}$ *associated with* ω*, let* τ *the weakest locally topology on* $\mathcal{A}$ *such that* π_ω *is continuous from* $\mathcal{A}_o(\tau)$ *into* $\pi_\omega(\mathcal{A}_o)$.

*then there exist a quasi-local quasi *-algebra* $(L^2(\widetilde{\varphi_\omega}),\pi_\omega(\mathcal{A}_0))$ *and a onomorphism*

$$\Phi : x\in\mathcal{A}\to\Phi(x) := \widetilde{X}\in L^2(\widetilde{\varphi_\omega})$$

with the following properties:

(i) Φ *extends the representation* π_ω *of* $\mathcal{A}_0$*;*

(ii) $\Phi(x^*) = \Phi(x)^*, \quad \forall x\in\mathcal{A}$*;*

(iii) $\Phi(xy) = \Phi(x)\Phi(y)$ *for every* $x,y\in\mathcal{A}$ *such that* $x\in\mathcal{A}_0$ *or* $y\in\mathcal{A}_0$.

Proof:

For every element $x\in\mathcal{A}$, there exists a sequence $\{a_n\}$ of elements of $\mathcal{A}_0$ converging to x with respect τ. Put $X_n = \pi_\omega(a_n)$. Then, $\varphi_\omega(|X_n - X_m|^2)\to 0$. Let $\widetilde{X}$ be the $\|\cdot\|_2$-limit of the sequence (X_n). We define $\Phi(x) := \widetilde{X}$. □

Concluding remark – We have discussed the possibility of constructing an rappresentation of Quasi-local quasi *-algebras $(\mathcal{A},\mathcal{A}_0)$, possessing a "sufficient state" on a non-commutative L^2-spaces. A more restricted choice could only be obtained by requiring that the Radon-Nikodym theorem in quasi * -algebras is satisfied [11]. We hope to discuss this aspect in a further paper.

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References:

[1] F. Bagarello, C.Trapani and S.Triolo, *Representable states on quasilocal quasi *-algebras* Journal of Mathematical Physics Volume 52, Issue 15 January 2011 Article number 013510

[2] F. Bagarello, A. Inoue and C. Trapani, *Representations and derivations of quasi *-algebras induced by* local modifications *of states*, J.Math.Anal.Appl. **356** (2009), 615-623

[3] F. Bagarello, C. Trapani and S. Triolo *Quasi *-algebras of measurable operators*, Studia Mathematica. **172** (3) (2006).

[4] G. L. Sewell, *Quantum Mechanics and its Emergent Macrophysics*, Princeton University Press, Princeton and Oxford, 2002.

[5] J.-P. Antoine, A. Inoue, C. Trapani, *Partial *-algebras and their operator realizations*, Kluwer, Dordrecht, 2002.

[6] M. Takesaki, *Theory of Operator Algebras.I*, Springer-Verlag, New York, 1979.

[7] C.Trapani and S.Triolo, *Representations of certain Banach C*-modules* Mediterr. J. Math. 1 (2004), no. 4, 441-461

[8] C. Trapani, *Quasi *-algebras of operators and their applications*, Reviews Math. Phys. 7, (1995), 1303-1332.

[9] C. Trapani, **-Representations, seminorms and structure properties of normed quasi *-algebras*, Studia Mathematica, Vol. 186, 47-75 (2008).

[10] O. Bratteli and D.W. Robinson, *Operator algebras and Quantum statistical mechanics 1*, Springer-Verlag, New York, 1987.

[11] C. La Russa, S. Triolo; *Radon-Nikodym theorem in quasi *-algebras*, Journal of Operator Theory, 2013, 69(2), pp. 423–433

[12] I. E. Segal, A noncommutative extension of abstract integration, *Ann. Math.* **57** (1953), 401–457.

[13] E. Nelson, *Note on non-commujtative integration*, J. Funct. Anal., **15** (1974) 103-116

[14] Burderi, F., Trapani, C., Triolo, S, *Extensions of hermitian linear functionals*, Banach Journal of Mathematical Analysis, 16(3), 45, (2022).

[15] F. Bagarello, M. Fragoulopoulou, A. Inoue and C. Trapani, *Locally convex quasi C^*-normed algebras*, preprint 2009

[16] S.Triolo, *WQ*-algebras of measurable operators Indian Journal of Pure and Applied Mathematics, 2012, 43(6), pp. 601–617.*

[17] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Vol.II*, Academic Press, New York, 1986.

[18] S. Triolo *Extensions of the Noncommutative Integration* Complex Analysis and Operator Theory, 2016, 10(7), pp. 1551–1564

[19] S. Triolo; *Possible extensions of the noncommutative integral*, Rend. Circ. Mat. Palermo, (2) 60 (2011), no 3, 409–416. DOI 10.1007/s12215-011-0063-1

[20] S. Triolo *CQ *-algebras of measurable operators Moroccan Journal of Pure and Applied Analysis* vol.8, no.2, 2022, pp.279-285.