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A Proof of the Twin Prime Conjecture in the *P x P* Space

YINGXU WANG
Dept. of Electrical and Software Engineering
Schulich School of Engineering and Hotchkiss Brain Institute, University of Calgary
2500 University Dr NW, Calgary, Alberta, CANADA T2N 1N4

Abstract: This work presents a formal proof of the twin prime conjecture based on a novel mathematical structure of mirror primes $\mathbb{P}_\mu \subset \mathbb{P} \times \mathbb{P}$ in the 2-dimensional space of primes. $\mathbb{P}_\mu$ is an infinite recursive set of pairs of symmetric primes adjacent to any pivotal even number $n_e/2 \geq 4 \in \mathbb{N}_e \subset \mathbb{N}$ in finite distances $1 \leq k \leq (n_e/2) - 2$. In the framework of $\mathbb{P}_\mu$, the set of twin primes $\mathbb{P}_\tau$ is deduced to a subset of the mirror primes $\mathbb{P}_\tau \subset \mathbb{P}_\mu$ where the half interval $k \equiv 1$. Therefore, the equivalence among sets of $\mathbb{P}_\tau$, $\mathbb{P}_\mu$, $\mathbb{P}$ and $\mathbb{N}$ is established, i.e., $\lim_{n\to\infty} |\mathbb{P}_\tau| = \lim_{n\to\infty} |\mathbb{P}_\mu| = \lim_{n\to\infty} |\mathbb{P}| = \lim_{n\to\infty} |\mathbb{N}| = \infty$ based on Cantor's principle of infinite countability, such that the twin prime conjecture holds. Experiments based on an Algorithm of Twin-Prime Sieve (ATPS) are designed to demonstrate and visualize the proven twin prime theorem.

Keywords: Number theory, twin prime conjecture, mirror primes, recursive sequence, algebraic number theory, and algorithm

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1. Introduction

Number theory in general and analytic primality in particular are a fundamental field of pure mathematics [1], [2], [3], [4], [5], [6], which are widely applied in computer science, AI, IT, algorithm design, coding theories, cryptography, Internet protocols, biology, and economics [4], [7], [8], [9], [10]. One of the most challenging questions in number theory is the *Twin Primes Conjecture* initiated by *de Polignac* in 1849 [11] who queried whether there exist infinitely many twin primes among nature numbers. There is no formal proof yet in general, because the nature and complexity of the problem in the infinite universe of discourse of number theory.

A typical form of the twin prime conjecture may be informally expressed according to de Polignac's suggestion [4], [11] as follows.

Definition 1. The *Twin Primes Conjecture* queries whether there exist infinitely many primes p such that $p + 2$ or $p - 2$ may also be prime.

Many key milestones towards proving the twin prime conjecture in the past 173 years have been represented by the following hypotheses, findings or theorems: a) V. Brun proved that the sum of the reciprocals of twin primes convergences to $\sum_{p,p+2\in\mathbb{P}} (\frac{1}{p} + \frac{1}{p+2}) < \infty$ in 1915 [12]; b) T. Tao explored obstructions to uniformity of primes and their arithmetic patterns in 2006 [13]; c) T. Goldston, J. Pintz and I. Yildirim derived that the relative gap of twin primes approaches $\lim_{n\to\infty} (p_{n+1} - p_n)/\log p_n = 0$ in 2009 [14]; c) Y. Zhang found that there are infinitely many pairs of twin primes within a bounded distance $0 < |p_{n+1} - p_n| \leq 70 \bullet 10^6$ in 2014 [15]; d) T. Tao initiated the International Polymath Project where optimizations of Zhang's work are conducted since 2014 [16]; and e) J. Maynard reduced Zhang's bound of prime gaps to $\lim_{n\to\infty}(p_{n+1} - p_n) \leq 600$, $\lim_{n\to\infty}(p_{n+1} - p_n) \leq 12$, and $\lim_{n\to\infty}(p_{n+1} - p_n) \leq 6$ successively based on the *Maynard–Tao theorem* in 2015 [17].

This work intends to present a formal proof of the twin prime conjecture based on a discovery of the mirror primes $\mathbb{P}_\mu$ and their universal distributions in the infinite set $\mathbb{P}_\mu \subset \mathbb{P} \times \mathbb{P}$ [18]. Then, the set of twin primes $\mathbb{P}_\tau \subset \mathbb{P}_\mu \subset \mathbb{P} \times \mathbb{P}$ is recognized as a subset of $\mathbb{P}_\mu$ where $p_{n+1} - p_n \equiv 2$ in Section II using the *big-R* calculus [19]. The twin prime conjecture is then deduced to a problem of Cantor's equivalent countability among sets of $\mathbb{P}_\tau$, $\mathbb{P}_\mu$. $\mathbb{P}$, and $\mathbb{N}$, which leads to a formal proof of the twin prime conjecture in Section III based on the recursive properties of the prime sequence and the infinite distribution of $\mathbb{P}_\mu$ as discovered for proving *Goldbach conjecture* [18]. Analytic experiments based on an Algorithm of Twin-Prime Sieve (ATPS) are designed in Section IV to demonstrate the proven twin prime theorem and its applications.

2. Formal Models and Properties of Twin Primes Underpinned by Mirror Primes

A novel structure and a set of interesting properties of mirror primes are introduced in recent basic research breakthroughs [18] as a set of symmetrically adjacent pair of primes with respect to any even number in $\mathbb{N}_e$. Based on the concept of mirror primes, the set of twin primes may be deduced to a special category of them where the distance (or difference of values) is always 2. This approach may significantly reduce the complex for recursive determination of twin primes and enables a rigorous inference on whether there are infinitely many pairs of twin primes.

2.1 Analytic Properties of Primes

It is used to be perceived that the sets of primes $\mathbb{P}$ and twin primes $\mathbb{P}_\tau$ seem to possess almost irregular members. However, a new perspective on the underpinned *recursive properties* of $\mathbb{P}$ and $\mathbb{P}_\tau$ is introduced as follows.

Definition 2. A *prime number p*, except 2, is a positive odd integer $p > 2 \in \mathbb{N}_o \subset \mathbb{N}$ that is not a product of two smaller integers:

$$p \triangleq (n \mid \mathop{R}_{m=2}^{\lfloor \sqrt{n} \rfloor} n \not\equiv 0 \ (\mathrm{mod}\, m)),\ \forall n \in \mathbb{N}_o \cup \{2\} \setminus \{1\} \tag{1}$$

where the big-*R* calculus [19] denotes a recursive structure or manipulates a recursive function.

Based on Definition 2, a general primality verification method may be derived [18]. Though, alternative sieve methodologies and algorithms exist [12], [20], [21], [22], [23], [24].

Definition 3. The *primality verification function* $\rho(n)$ determines, $\forall n \in \mathbb{N}_o \cup \{2\} \setminus \{1\}$, whether *n* is prime:

$$\rho(n) = \begin{cases} 0, & \bigvee_{m=2,\, m \in \mathbb{P}}^{\lfloor \sqrt{n} \rfloor} n \equiv 0 \ (\mathrm{mod}\, m) \quad // \, n \notin \mathbb{P} \\ 1, & \text{otherwise} \quad // \, n \in \mathbb{P} \end{cases} \tag{2}$$

where $\rho(n)$ results in a positive verification *iff* $n \not\equiv 0 \ (\mathrm{mod}\ m)$ for all $2 \le m \in \mathbb{P} \le \lfloor \sqrt{n} \rfloor$. Otherwise, as a shortcut, any negative result $n \equiv 0 \ (\mathrm{mod}\ m)$ will terminate the testing by returning false.

The primality checker $\rho(n)$ plays an important role in recursive prime generation, which knocks down any successive odd integer for being prime if it is divisible by any preceding prime up to $\lfloor \sqrt{n} \rfloor \in \mathbb{P}$.

Definition 4. The *generic pattern of the set of primes $\mathbb{P}$* is a recursive and infinite sequence of monotonously increasing odd integers (except 2) validated by the primality checker $\rho(n)$:

$$\begin{aligned} \mathbb{P} &\triangleq \{\mathop{R}_{i=1}^{\infty} p_i\} \\ &= \{p_1 = 2, p_2 = 3, \mathop{R}_{i=3}^{\infty} [\mathop{R}_{k=1}^{p_{i-1}} (p_i = p_{i-1} + 2k) \mid \rho(p_i) = 1]\}, k \in \mathbb{N} \end{aligned} \tag{3}$$

where $p_3 = (p_2 + 2) = 5$, $p_4 = (p_3 + 2) = 7$, and $p_5 = (p_4 + 4 \mid k = 2) = 11$ because $\rho(p_4 + 2 = 9 \mid k = 1) \ne 1$, and etc.

The generic mathematical model of the set of primes $\mathbb{P}$ reveals an important recursive property of primes that leads to the theorem of recursiveness of the prime sequence as proven in [19]. The recursive pattern of $\mathbb{P}$ does not only explains the nature of primality, but also indicates that any $p_n \in \mathbb{P}$ would remain indeterminable until the preceding primes $[p_1, \lfloor \sqrt{p_n} \rfloor] \in \mathbb{P}_n$ have been obtained. This mechanism enables a new perspective on the nature of primes in $\mathbb{P}$ and their manipulations as elaborated in the following subsection.

2.2 Analytic Properties of Mirror Primes

A novel mathematical concept of the set of mirror primes $\mathbb{P}_\mu$ is introduced in this work to model the pairs of mirror primes as a 2-dimensional structure $\mathbb{P}_\mu \subset \mathbb{P} \times \mathbb{P}$ [18]. $\mathbb{P}_\mu$ establishes a relation between the sequence of pairwise primes and each pivotal even numbers n_e as the center of them. $\mathbb{P}_\mu$ will be used to formally model the prime distribution pattern where at least a pair of mirror primes is symmetrically adjacent to each n_e on both sides within finite distance.

Definition 5. The *mirror primes* $p_\mu^{n_e/2}$ with respect to a pivotal even number $n_e = \frac{p_{\mu^-} + p_{\mu^+}}{2} \in \mathbb{N}_e \subset \mathbb{N}$ are pairwise primes symmetrically adjacent to the central n_e within finite $\pm k|\mathbb{N}$ distances:

$$\begin{aligned} p_\mu^{n_e/2} \triangleq \{ \mathop{R}_{k=1}^{\frac{n_e}{2}-2} (p_{\mu^-}^{n_e/2} = \frac{n_e}{2} - k,\ p_{\mu^+}^{n_e/2} = \frac{n_e}{2} + k) \\ \mid \rho(p_{\mu^-}^{n_e/2}) \wedge \rho(p_{\mu^+}^{n_e/2})) = 1)\} \end{aligned} \tag{4}$$

where *k* is called the *half interval* of a pair of mirror primes.

For instances, according to Definition 5, the following pairs of primes are mirror primes symmetrically adjacent to certain pivot $n_e/2$:

$$p_\mu^4 = \{\underset{k=1}{\overset{2}{R}} (4 \mp k) \mid \rho(4 \mp k) = 1)\}$$
$$= \{(3,5) \mid \rho(4 \mp 1) = 1,\ (2,6) \mid \rho(4 \mp 2) \neq 1\} = \{(3,5)\}$$
$$p_\mu^5 = \{\underset{k=1}{\overset{3}{R}} (5 \mp k) \mid \rho(5 \mp k) = 1)\} = \{(3,7)\}$$
$$p_\mu^{50} = \{\underset{k=1}{\overset{48}{R}} (50 \mp k) \mid \rho(50 \mp k) = 1\}$$
$$= \{(47,53), (41,59), (29,71), (11,89), (3,97)\}$$

Based on Definitions 5, the entire set of mirror primes $\mathbb{P}_\mu$ may be rigorously determined as follows.

Definition 6. The *set of mirror primes* $\mathbb{P}_\mu$ is all valid pairs of adjacent primes with respect to each of the pivotal sequence $4 \leq n_e / 2 \in \mathbb{N}_e$ bounded by the finite half interval $1 \leq k \leq \frac{n_e}{2} - 2$:

$$\mathbb{P}_\mu \triangleq \{\underset{n_e/2=4}{\overset{\infty}{R}} p_\mu^{n_e/2}\} \subset \mathbb{P} \times \mathbb{P}$$
$$= \{\underset{n_e/2=4}{\overset{\infty}{R}} [\underset{k=1}{\overset{\frac{n_e}{2}-2}{R}} \left(\begin{array}{c} (p_{\mu^-}^{n_e/2} = \frac{n_e}{2} - k,\ p_{\mu^+}^{n_e/2} = \frac{n_e}{2} + k) \\ \mid \rho(p_{\mu^-}^{n_e/2}) \wedge \rho(p_{\mu^+}^{n_e/2}) = 1 \end{array} \right)]\} \quad (5)$$

where all pairs of mirror primes $p_\mu^{n_e/2} \in \mathbb{P}_\mu$ in the scope $8 \leq n_e < \infty$ are determined by Definition 5.

On the basis of the properties of mirror primes $\mathbb{P}_\mu$, a key theorem of mirror-prime decomposition for all even numbers may be formally derived in the following theorem.

Theorem 1 (Mirror Prime Decomposition, MPD). Any even integer $n_e / 2 \geq 4 \in \mathbb{N}_e$ may be decomposed to the sum of at least a pair of mirror primes $p_\mu^{n_e/2} = (p_{\mu^-}^{n_e/2}, p_{\mu^+}^{n_e/2}) = (\frac{n_e}{2} - k,\ \frac{n_e}{2} + k) \in \mathbb{P}_\mu$ adjacent to $n_e/2$ as the pivot within $1 \leq k \leq \frac{n_e}{2} - 2$ steps:

$$\forall n_e / 2 \geq 4 \in \mathbb{N}_e \subset \mathbb{N},\ k \in \mathbb{N}:$$
$$n_e \equiv p_{\mu^-}^{n_e/2} + p_{\mu^+}^{n_e/2} = (\frac{n_e}{2} - k) + (\frac{n_e}{2} + k) \text{ bounded}$$
$$\text{by } 1 \leq k \leq \frac{n_e}{2} - 2 \quad (6)$$
$$\text{where } p_\mu^{n_e/2} = (p_{\mu^-}^{n_e/2}, p_{\mu^+}^{n_e/2}) \in \mathbb{P}_\mu,$$
$$\mathbb{P}_\mu = \underset{n_e/2=4}{\overset{\infty}{R}} \{\underset{k=1}{\overset{\frac{n_e}{2}-2}{R}} \left(\begin{array}{c} (p_{\mu^-}^{n_e/2} = \frac{n_e}{2} - k,\ p_{\mu^+}^{n_e/2} = \frac{n_e}{2} + k) \\ \mid \rho(p_{\mu^-}^{n_e/2}) \wedge \rho(p_{\mu^+}^{n_e/2}) = 1 \end{array} \right) \}.$$

The proof of Theorem 1 has been presented in [18]. Theorem 1 for *prime decomposition* of arbitrary even integers is a necessary counterpart of Euclid's *Fundamental Theorem of Arithmetic* [1] for *prime factorization*. The MPD theorem provides a general theory and methodology for finding all pairs of mirror primes, including twin primes, on both sides of any arbitrary even number n_e, $4 \leq n_e / 2 < \infty$, except the special case $n_e / 2 = 2$ where the mirror primes regress to a pair of *reflexive* primes $p_\gamma^4 = (2, 2),\ k = 0$. Theorem 1 will be adopted to explain the nature of twin primes $\mathbb{P}_\tau$ in the following subsection.

2.3 Analytic Properties of Twin Primes

Twin primes are used to be perceived as random pairs of primes with a constant half-interval $k \equiv 1$ in the spectrum of $\mathbb{P}$. However, according to the mathematical model of mirror primes $\mathbb{P}_\mu$, as introduced in Section 2.2, the set of twin primes $\mathbb{P}_\tau$ may be formally derived as a special subset of $\mathbb{P}_\mu$. Therefore, if the size of $\mathbb{P}_\mu$ may be determined as proven in [18], so do $\mathbb{P}_\tau$ towards solving the twin prime conjecture.

Definition 7. The *twin primes* $p_\tau^{n_e/2}$ with respect to a pivot n_e, $n_e \in \mathbb{N}_e \subset \mathbb{N}$, are a special pair of mirror primes with a constant *half interval* $k \equiv 1$:

$$p_\tau^{n_e/2} \triangleq p_\mu^{n_e/2}\big|_{k=1} = (p_{\tau^-}^{n_e/2} = \frac{n_e}{2} - 1,\ p_{\tau^+}^{n_e/2} = \frac{n_e}{2} + 1) \in \mathbb{P}_\mu,$$
$$(p_{\tau^+}^{n_e/2} - p_{\tau^-}^{n_e/2}) \equiv 2,\ \rho(\frac{n_e}{2} \mp 1) = 1 \quad (7)$$

where each pair of potential twin primes $p_\tau^{n_e/2} = (p_{\tau^-}^{n_e/2}, p_{\tau^+}^{n_e/2})$ must be validated by $\rho(p_{\tau^-}^{n_e/2}) \wedge \rho(p_{\tau^+}^{n_e/2}) = 1$ for sufficiently determining both of their primality.

For instances, according to Definition 7:

$$p_\tau^4 = p_\mu^4(8/2 \mid k=1) = \{(4 \mp 1) \mid \rho(4 \mp 1) = 1\} = \{(3,5)\}$$
$$p_\tau^6 = p_\mu^6(12/2 \mid k=1) = \{(6 \mp 1) \mid \rho(6 \mp 1) = 1\} = \{(5,7)\}$$
$$p_\tau^8 = p_\mu^8(16/2 \mid k=1) = \{(8 \mp 1) \mid \rho(8-1) = 1 \wedge \rho(8+1) = 0\} = \{(7, \not{9})\} = \{\varnothing\}$$
$$p_\tau^{10} = p_\mu^{10}(20/2 \mid k=1) = \{(10 \mp 1) \mid \rho(9) = 0 \wedge \rho(11) = 1\} = \{(\not{9}, 11)\} = \{\varnothing\}$$
$$p_\tau^{12} = p_\mu^{12}(24/2 \mid k=1) = \{(12 \mp 1) \mid \rho(12 \mp 1) = 1\} = \{(11,13)\}$$
$$\ldots$$
$$p_\tau^{2996863034895 \bullet 2^{1290000}}$$
$$= p_\mu^{2996863034895 \bullet 2^{1290000}}(2996863034895 \bullet 2^{1290000} \mid k=1)$$
$$= \{(2996863034895 \bullet 2^{1290000} \mp 1)\}$$

where the largest pair of twin primes ever known has been found by PrimeBios in 2016 [26].

Definition 8. The *set of twin primes* $\mathbb{P}_\tau$ with respect to the entire spectrum of pivotal even numbers $4 \le n_e / 2 \in \mathbb{N}_e < \infty$ are determined in the constant half interval $k \equiv 1$ dependent on a valid primality verification for each pair:

$$\begin{aligned}\mathbb{P}_\tau &\triangleq \{\mathop{R}_{n_e/2=4}^{\infty} p_\tau^{n_e/2}\} \subset \mathbb{P}_\mu \subset \mathbb{P}\times\mathbb{P} \\ &= \{\mathop{R}_{n_e/2=4}^{\infty} (p_{\tau^-}^{n_e/2} = \frac{n_e}{2} - 1, p_{\tau^+}^{n_e/2} = \frac{n_e}{2} + 1) \\ &\quad | \rho(p_{\tau^-}^{n_e/2}) \wedge \rho(p_{\tau^+}^{n_e/2})) = 1\}\end{aligned} \tag{8}$$

The mathematical models and the recursive properties of $\mathbb{P}_\tau \subset \mathbb{P}_\mu \subset \mathbb{P}\times\mathbb{P}$ pave a way to the formal proof of the twin prime conjecture in Section III.

3. Formal Proof of the Twin Prime Conjecture

As a long-term challenging problem in number theory, it is curious to find whether there are infinitely many twin primes $p_\tau^{n_e} = (p_{\tau^-}^{n_e}, p_{\tau^+}^{n_e}) \in \mathbb{P}_\tau$ as the twin prime conjecture queried [11]. A formal proof for twin prime conjecture is expected to be based on the fundamental properties of twin primes $\mathbb{P}_\tau$ as described in preceding sections including: a) The universe of discourse of $\mathbb{P}_\tau$ is constrained by the Cartesian product of the sets of primes $\mathbb{P}\times\mathbb{P}$; b) $\mathbb{P}_\tau$ is necessarily a subset of mirror primes, i.e., $\mathbb{P}_\tau \subset \mathbb{P}_\mu$; and c) $\mathbb{P}_\tau$ is sufficiently restricted by the uniform half-interval $k \equiv 1$ with respect to any pivotal $n_e / 2 \in \mathbb{N}_e \setminus \{2\}$.

Hypothesis 1. The *Twin Prime Conjecture* (TPC) queries whether there are infinitely many pairs of twin primes $\mathbb{P}_\tau \subset \mathbb{P}_\mu \subset \mathbb{P}\times\mathbb{P}$:

$$\begin{aligned}|\mathbb{P}_\tau| &= \left|\mathop{R}_{n_e/2=4}^{\infty} p_\tau^{n_e}\right| \\ &= \left|\mathop{R}_{n_e/2=4}^{\infty} (p_{\tau^-}^{n_e/2}, p_{\tau^+}^{n_e/2}) \,|\, \rho(p_{\tau^-}^{n_e/2}) \wedge \rho(p_{\tau^+}^{n_e/2}) = 1\right| \overset{?}{=} \infty\end{aligned} \tag{9}$$

The hypothesis of TPC may be formally proven based on the preparations in Section 2, particularly the recursiveness of the prime sequence and the mirror prime decomposition theorem for arbitrary even numbers. According to Theorem 1, any pair of potential twin primes symmetrically adjacent to a pivotal even number n_e may be efficiently elicited from the set of mirror primes $\mathbb{P}_\tau \subset \mathbb{P}_\mu = \mathop{R}_{n_e/2=4}^{\infty} (p_{\mu^+}^{n_e/2}, p_{\mu^-}^{n_e/2}) \,|\, p_{\mu^+}^{n_e/2} - p_{\mu^-}^{n_e/2} \equiv 2)$ when the distance of the pair is 2 or $k \equiv 1$.

Lemma 1. The *size of the set of mirror primes* $\mathbb{P}_\mu \subset \{\mathbb{P}\times\mathbb{P} \,|\, \mathop{R}_{n_e/2=4}^{\infty} \rho(p_{\mu^-}^{n_e/2}) \wedge \rho(p_{\mu^+}^{n_e/2}) = 1\}$ is infinite.

Proof. $\forall n_e \in \mathbb{N}$ and $\mathbb{P} \subset \mathbb{N}$, because $\mathbb{P}$ is infinite, so is $\mathbb{P}_\mu \subset \mathbb{P}\times\mathbb{P}$ according to Definition 6. That is, given:

$$\mathbb{P}_\mu = \{\mathop{R}_{n_e/2=4}^{\infty} (p_{\mu^-}^{n_e/2}, p_{\mu^+}^{n_e/2}) \,|\, \rho(p_{\mu^-}^{n_e/2}) \wedge \rho(p_{\mu^+}^{n_e/2}) = 1\} \subset \mathbb{P}\times\mathbb{P},$$

there exist at least a pair of mirror primes with respect to any pivotal $n_e \in \mathbb{N}$ according to Theorem 1,

such that:

$$|\mathbb{P}_\mu| = \left|\{\mathop{R}_{n_e/2=4}^{\infty} (p_{\mu^-}^{n_e/2}, p_{\mu^+}^{n_e/2}) \,|\, \rho(p_{\mu^-}^{n_e/2}) \wedge \rho(p_{\mu^+}^{n_e/2}) = 1\}\right| = \infty$$

or according to the *prime number theorem* [4] by one-to -one mapping,

$$|\mathbb{P}_\mu| = |\mathbb{P}|^2 = (\lim_{n_e\to\infty} \frac{n_e}{\log n_e})^2 = \lim_{n_e\to\infty} (|\mathbb{P}_{\mu^-}^{n_e}| \times |\mathbb{P}_{\mu^+}^{n_e}|) = \infty$$

Thus, the infinity of $|\mathbb{P}_\mu|$ holds. ■

Based on Lemma 1 and Definition 8, the relationship between $\mathbb{P}_\tau$ and $\mathbb{P}_\mu$ may be formally described in Lemma 2.

Lemma 2. The *set of twin primes* $\mathbb{P}_\tau$ is a special subset of mirror primes $\mathbb{P}_\mu$ determined at the sequential positions $4 \le n_e / 2 \in N_\tau < \infty$:

$$\begin{aligned}\mathbb{P}_\tau = \mathop{R}_{\frac{n_e}{2}=4 \wedge \frac{n_e}{2} \in N_\tau}^{\infty} \{(3,5), \mathop{R}_{k=1}^{\infty} [(p_\tau^{n_e/2} = 6k \mp 1) \\ | \rho(p_{\tau^-}^{n_e/2}) \wedge \rho(p_{\tau^+}^{n_e/2}) = 1]\}\end{aligned} \tag{10}$$

where the distribution pattern of the pivotal $n_e/2$ is constrained by $N_\tau = \{4, \mathop{R}_{k=1}^{\infty} 6k\} = \{4, 6, 12, 18, ..., 6k, ..., \infty\}$ confirmed by $\rho(p_{\tau^-}^{n_e/2}) \wedge \rho(p_{\tau^+}^{n_e/2}) = 1$.

Proof. Lemma 2 holds based on the following necessary and sufficient conditions:

a) $\forall p_\tau^{n_e/2} \in \mathbb{P}_\tau \subset \mathbb{P}_\mu$, the *necessary* condition for twin primes elicitation requires that $p_\tau^{n_e/2}$ must be symmetrically adjacent to the pivot $\frac{n_e}{2} \in N_\tau = \{4, \mathop{R}_{k=1}^{\infty} 6k\} = \{4, 6, 12, ..., 6k, ..., \infty\}$.

Otherwise, if $\frac{n_e}{2} \notin N_t$, the potential pair of twin primes p_τ^{6k} is disqualified because:

$$\mathop{R}_{k=1}^{\infty} \mathop{R}_{n_e/2 \ne 6k \wedge n_e/2 \notin N_\tau}^{\infty} p_{\bar{\tau}}^{n_e/2 \ne 6k} = (p_{\bar{\tau}}^{n_e/2} - 1, p_{\bar{\tau}}^{n_e/2} + 1) \notin \mathbb{P}_\tau,$$

e.g., $p_{\bar{\tau}}^{8 \ne 6k} = (7, \not{9}) \notin \mathbb{P}_\tau$ and $p_{\bar{\tau}}^{10 \ne 6k} = (\not{9}, 11) \notin \mathbb{P}_\tau$.

b) The *sufficient* condition for twin primes validation ensures any false exception such as:

$$p_\tau^{6k} = \mathop{R}_{k=1}^{\infty} \mathop{R}_{n_e/2=6k \wedge n_e/2 \in N_\tau}^{\infty} \{p_\tau^{n_e/2} = p_\tau^{6k} \,|\, \rho(\frac{n_e}{2} - 1 = 6k - 1) \vee \rho(\frac{n_e}{2} + 1 = 6k + 1) \ne 1)\}$$

be eliminated, *e.g.*, $p_{\bar{\tau}}^{24} = (23, \not{25})$ or $p_{\bar{\tau}}^{36} = (\not{35}, 37)$. ■

The necessary condition of twin primes in Lemma 2 is expressed in an informal way in the literature [27]. However, without the restriction of the sufficient condition, many false predictions would be resulted in $\mathbb{P}_\tau$.

Example 1. Given a sequence *s* of arbitrarily pivotal numbers $n_e \in \mathbb{N}$, certain pairs of twin primes can be elicited from the qualified pairs of mirror primes in $\mathbb{P}_\tau$ according to Lemma 2:

$$\begin{array}{lllllllll} s & = \{0, & 1, & 2, & ..., 4, & 5, & ..., 100, & ..., 2996863034895 \bullet 2^{1290000} / 6, & ...\} \\ n_e & = \{4, & 6, & 12, & ..., 24, & 30, & ..., 600, & ..., 2996863034895 \bullet 2^{1290000}, & ...\} \\ \mathbb{P}_\tau & = \{(3,5), & (5,7), & (11,13), & ..., (23, \not{25}), & (29,31), & ..., (599, 601), & ..., (2996863034895 \bullet 2^{1290000} \mp 1), & ...\} \\ d & = \{2, & 2, & 2, & ..., \varnothing, & 2, & ..., 2, & ..., 2, & ...\} \end{array}$$

where (s, n_e, $\mathbb{P}_\tau$, d) represent the serial number, pivotal center, derived twin primes, and their distances, respectively. It is noteworthy that the candidate pair of (23, 25) centered by n_e = 24 is not twin primes because the sufficient condition of Lemma 2 requires that both $p_\tau^{n_e} = (p_{\tau^-}^{n_e},\ p_{\tau^+}^{n_e})$ must be prime.

Both Lemmas 1 and 2 will enable the proof of the twin prime hypothesis to be true in order to establish the twin prime theorem based on Cantor's *principle of equivalent countability* between infinite sets [29].

Theorem 2 (Twin Prime Theorem, TPT). There are infinitively many pairs of twin primes in $\mathbb{P}_\tau \subset \mathbb{P}_\mu \subset \mathbb{P} \times \mathbb{P}$:

$$\begin{aligned} |\mathbb{P}_\tau| &= \left| \underset{n_e/2=4}{\overset{\infty}{R}} p_\tau^{n_e/2} \right| = \left| \underset{n_e/2=4}{\overset{\infty}{R}} (p_{\tau^-}^{n_e/2}, p_{\tau^+}^{n_e/2}) \right| \\ &= \left| \underset{n_e/2=4}{\overset{\infty}{R}} (\frac{n_e}{2}-1, \frac{n_e}{2}+1) \mid \rho(p_{\tau^-}^{n_e/2} = \frac{n_e}{2}-1) \wedge \rho(p_{\tau^+}^{n_e/2} = \frac{n_e}{2}+1) = 1 \right| = \infty \end{aligned} \quad (11)$$

Theorem 2 holds by a reductive proof based on the properties and relationships of $\mathbb{P}_\tau$ and $\mathbb{P}_\mu$ as given in Lemmas 1 and 2 as follows:

Proof. $\forall n_e / 2 \geq 4, \mathbb{P}_\tau \subset \mathbb{P}_\mu \subset \mathbb{P} \times \mathbb{P}$,

$$\mathbb{P}_\tau = \underset{n_e/2=4}{\overset{\infty}{R}} (p_{\tau^-}^{n_e/2}, p_{\tau^+}^{n_e/2}) = \underset{n_e/2=4}{\overset{\infty}{R}} (p_{\tau^-}^{n_e/2} = \frac{n_e}{2}-1, p_{\tau^+}^{n_e/2} = \frac{n_e}{2}+1 \mid \rho(p_{\tau^-}^{n_e/2}) \wedge \rho(p_{\tau^+}^{n_e/2}) = 1).$$

In the recursive process of twin prime varification for $\mathbb{P}_\tau = \underset{n_e/2=4}{\overset{\infty}{R}} (p_{\tau^-}^{n_e/2}, p_{\tau^+}^{n_e/2})$, although some ineligible pairs of primes $\rho(p_{\tau^-}^{n_e/2}) \vee \rho(p_{\tau^+}^{n_e/2}) \neq 1$ among $\mathbb{P}_\tau = \underset{n_e/2=4}{\overset{\infty}{R}} \underset{k=1}{\overset{\infty}{R}} (p_\tau^{n_e/2} = 6k \mp 1)$ must be eliminated according to Lemma 2, the remaining set of valid twin primes $\mathbb{P}_\tau = \mathbb{P}_\tau^{n_e} \setminus \mathbb{P}_{\mp}^{n_e} \subset \mathbb{P}_\mu$ maintain its infinity based on Lemma 1, because only finite potential primes in $|\mathbb{P}_{\mp}^{n_e}|$ may be eliminated by particular n_e. This mechanism ensures:

$$|\mathbb{P}_\tau| = \lim_{n_e \to \infty} |(\mathbb{P}_{\tau^-}^{n_e} \times \mathbb{P}_{\tau^+}^{n_e}) \setminus (\mathbb{P}_{\mp}^{n_e} \times \mathbb{P}_{\mp}^{n_e})| = \lim_{n_e \to \infty} (|\mathbb{P}_\tau^{n_e} \times \mathbb{P}_\tau^{n_e}| - |\mathbb{P}_{\mp}^{n_e} \times \mathbb{P}_{\mp}^{n_e}|) = \infty$$

Thus there are infinitely many twin primes, i.e.:

$$|\mathbb{P}_\tau| = \left| \{ \underset{n_\tau=4}{\overset{\infty}{R}} \left(p_{\tau^+}^{n_\tau/2} - p_{\tau^-}^{n_\tau/2} \equiv 2 \right) \} \right| = \infty. \quad \blacksquare$$

Theorem 2 indicates that, although some pairs of mirror primes in $\mathbb{P}_\mu$ would be ineligible because $k \neq 1$ during twin prime verification, the entire $\mathbb{P}_\tau$ still maintains infinity as what Cantor has proven [29] for the equivalent classes $|\mathbb{N}_e| \sim |\mathbb{N}_o| \sim |\mathbb{N}| = \infty$.

4. Numerical Experiments Based on the Twin Prime Theorem

The proven twin-prime conjecture in Theorem 2 will be experimentally elaborated in this section by an infinitively recursive sequence of twin primes $(p_{\tau^-}^{n_e/2}, p_{\tau^+}^{n_e/2}) \in \mathbb{P}_\tau \subset \mathbb{P}_\mu \subset \mathbb{P} \times \mathbb{P}$. A numerical algorithm is introduced to demonstrate and visualize Theorem 2, as well as the infinite distribution pattern of twin primes in $\mathbb{P}_\tau \subset \mathbb{P}_\mu$.

Algorithm 1. The *algorithm of Twin-Prime Sieve* (ATPS) is designed based on Theorem 2 as shown in Figure 1. It provides a twin-prime determination methodology for selecting $\mathbb{P}_\tau^{n_e^{\max}}$ from validated mirror primes $\mathbb{P}_\mu$ against each pivotal even integer $\underset{n_e/2=4}{\overset{\infty-}{R}} \frac{n_e|\mathbb{N}_e}{2}$ in type $|\mathbb{N}_e$, where $p_{\tau^+}^{n_e/2}|\mathbb{P} - p_{\tau^-}^{n_e/2}|\mathbb{P} \equiv 2$ and $\rho(p_{\tau^-}^{n_e/2}|\mathbb{P}) \wedge \rho(p_{\tau^+}^{n_e/2}|\mathbb{P}) = 1$ in $\mathbb{P}_\tau \subset \mathbb{P}_\mu \subset \mathbb{P} \times \mathbb{P}$. The ATPS|PM algorithm is formally described as a recursive *process model* (PM) *ATPS*|PM in *Real-Time Process Algebra* (RTPA) [28], which is a form of *Intelligent Mathematics* (IM) [5] that enables readers to empirically test the twin prime theorem.

$$\textbf{ATPS}|\text{PM}(<\textbf{I}:: n_e^{\max}|\mathbb{N}_e \subset \mathbb{N}>; <\textbf{O}:: \mathbb{P}_\tau^{n_\tau|\mathbb{N}}|\Xi = \underset{n_\tau=1}{\overset{n_e^{\max}|\mathbb{N}_e}{R}}[(p_{\tau^-}^{n_e}, p_{\tau^+}^{n_e})|\mathbb{P}_\tau>;$$

$$<\textbf{H}:: \mathbb{P}_\tau|\Xi \subset \mathbb{P}|\Xi \times \mathbb{P}|\Xi = \underset{n_e/2|\mathbb{N}_e=4}{\overset{\infty^-}{R}}[(p_{\tau^-}^{n_e/2}, p_{\tau^+}^{n_e/2})|\mathbb{P}_\tau, \rho(p_{\tau^-}^{n_e/2}, p_{\tau^+}^{n_e/2})=1]>) \triangleq$$

$$\{\rightarrow \text{�}\ n_e^{\max}|\mathbb{N}_e \geq 8 \quad // \ n_e \geq 4$$

$$\rightarrow n_\tau|\mathbb{N} := 0$$

$$\rightarrow \underset{n_e/2|\mathbb{N}_e=4}{\overset{n_e^{\max}|\mathbb{N}_e}{R}}(\rightarrow p_{\tau^-}^{n_e/2}|\mathbb{N}_o = \frac{n_e|\mathbb{N}_e}{2} - 1$$

$$\rightarrow p_{\tau^+}^{n_e/2}|\mathbb{N}_o = \frac{n_e|\mathbb{N}_e}{2} + 1$$

$$\rightarrow (\text{�} \bigvee_{m|\mathbb{N}=2\ \wedge\ m\in\mathbb{P}}^{\left\lfloor\sqrt{p_{\tau^-}^{n_e/2}|\mathbb{N}_o}\right\rfloor} [p_{\tau^-}^{n_e/2}|\mathbb{N}_o \equiv 0 \ (\text{mod}\, m|\mathbb{N}) \vee p_{\tau^+}^{n_e}|\mathbb{N}_o \equiv 0 \ (\text{mod}\, m|\mathbb{N})] \quad // \text{ Primality test fail}$$

$$\rightarrow \varnothing$$

$$|\sim \qquad // \text{ The } n_\tau\text{+1th new } \mathbb{P}_\tau \text{ is found}$$

$$\rightarrow n_\tau|\mathbb{N} := n_\tau|\mathbb{N} + 1$$

$$\rightarrow Print("n_\tau|\mathbb{N}: \ (p_{\tau^-}^{n_e/2}|\mathbb{P}, p_{\tau^+}^{n_e/2}|\mathbb{P})|\mathbb{P}_\tau")$$

$$)$$

$$)$$

$$\}$$

Fig. 1. The algorithm of twin-prime sieve (ATPS)

The ATPS algorithm is a computational expression of the mathematical models according to Theorems 2. The input (*I*) of *ATPS*|PM is the maximum expected pivotal $n_e^{\max}|\mathbb{N}_e$ in the type of even numbers ($|\mathbb{N}_e$). The output (*O*) of *ATPS*|PM is a set (Ξ) of valid twin primes adjacent to each potential $n_e/2$ in $\mathbb{P}_\tau^{n_\tau|\mathbb{N}}|\Xi$. The *Hyperstructure* (*H*) denotes underpinning *Structure Models* (SMs) to be operated by the algorithm. *ATPS*|PM is implemented by a recursive process in the loop $\underset{n_e/2|\mathbb{N}_e=4}{\overset{n_e^{\max}|\mathbb{N}_e}{R}}(...)$ after the upper limit for an expected scope of iterations is validated by the if-then-(else) structure (�). It then determines if each potential pair of twin primes $(p_{\tau^-}^{n_e/2}, p_{\tau^+}^{n_e/2} \mid p_{\tau^+}^{n_e/2} - p_{\tau^-}^{n_e/2} \equiv 2)$ is valid according to the primality test criteria (Eq. 2). Once both the necessary and sufficient conditions are satisfied, the algorithm displays the n_τth twin pair otherwise it skips ($\rightarrow \varnothing$) the current iteration. Either outcome leads to the next iteration until the algorithm reaches $n_e^{\max}|\mathbb{N}_e$.

The ATPS algorithm provides a computational simulation for visualizing the recursive distribution pattern of twin primes in $\mathbb{P}_\tau \subset \mathbb{P}_\mu \subset \mathbb{P} \times \mathbb{P}$. It may be implemented in MATLAB or any programming language.

In order to visualize the proven twin prime theorem, a set of numerical experiments has been designed and implemented based on Algorithm 1, which provides empirical evidence for demonstrating the infinitive distribution of twin primes in $\mathbb{P}_\tau \subset \mathbb{P} \times \mathbb{P}$. The time complexity of the *ATPS*|PM algorithm is $O(n_e^{\max} \bullet \left\lfloor\sqrt{n_e^{\max}}\right\rfloor) \simeq O((n_e^{\max})^{\frac{3}{2}})$. The space requirement for *ATPS*|PM is constrained by the memory size of the underpinning computer. Therefore, for extremely large set of twin prime detections over 100,000,000, parallel computing facilities are required for supporting the algorithm.

Experiment 1. Applying Algorithm 1 in MATLAB, a set of experimental results has been obtained as shown in Figures 2 in the Cartesian space $\mathbb{P}_\tau \subset \mathbb{P}_\mu \subset \mathbb{P} \times \mathbb{P}$. Figure 2 demonstrates the trends of detected twin primes in the first 26 pairs of twin primes within the scope of $4 \leq N_e/2 \leq 52$ based on the twin-prime theorem. In Figure 2, the first three curves show those of $(p_{\tau^+}^{n_e/2}, \frac{n_e}{2}, p_{\tau^-}^{n_e/2})$, respectively. Both $p_{\tau^+}^{n_e/2}$ and $p_{\tau^-}^{n_e/2}$ are very close along the curve of $n_e/2$ because their half interval $k = p_{\tau^+}^{n_e/2} - \frac{n_e}{2} = \frac{n_e}{2} - p_{\tau^-}^{n_e/2} \equiv 1$. The fourth curve at the bottom of Figure 2 shows the gaps $g_{n_e^i} = p_\tau^{n_e^{i+1}/2} - p_\tau^{n_e^i/2} = n_e^{i+1} - n_e^i$ between each pair of validated twin primes in the infinite sequence. For example, $n_e^1/2 = 4,\ p_\tau^{n_e^1/2} = p_\tau^4 = (3,5)$. The density of twin primes d_τ among $\mathbb{P}_\tau^{500}$ is $d_\tau = \frac{\mathbb{P}_\tau^{500}}{n_e/2} = \frac{25}{500} = 5.00\%$. The maximum gap among the detected twin primes is 6 ● 12 = 72 at the position $n_e/2 = 24$.

Table 1. Statistical Distributions of Twin Primes based on the ATPS Algorithm

Scope (N)	500	10,000	100,000	1,000,000	10,000,000	100,000,000
Number of primes (N_p)	95	1,229	9,592	78,498	664,579	5,761,455
Number of pairs of twin primes (N_τ)	25	206	1,225	8,170	58,981	440,313
Density of pairs of twin primes ($d_\tau = N_\tau / N$)	5.00%	2.10%	1.23%	0.82%	0.59%	0.44%
Maximum gaps between twin primes (g_{max})	72	210	630	1,452	1,722	2,868
Position of maximum gap ($n_e/2 \in N$)	23	145	833	7,121	58,619	428,136

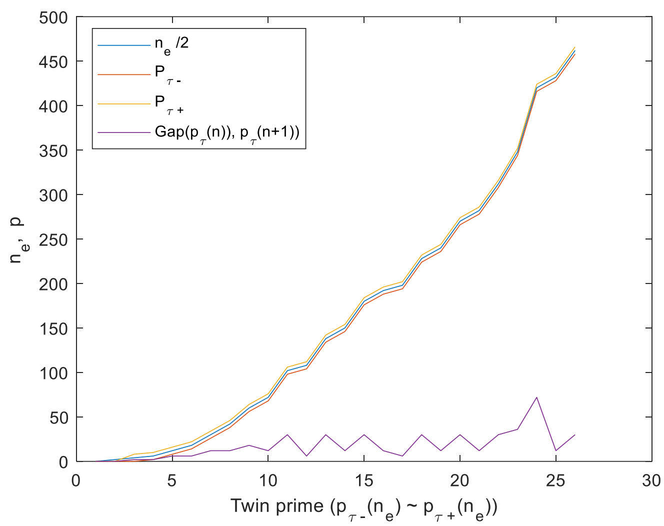

Fig. 2. Experimental results of infinite twin primes distribution in the space $\mathbb{P}_\tau \subset \mathbb{P} \times \mathbb{P}$

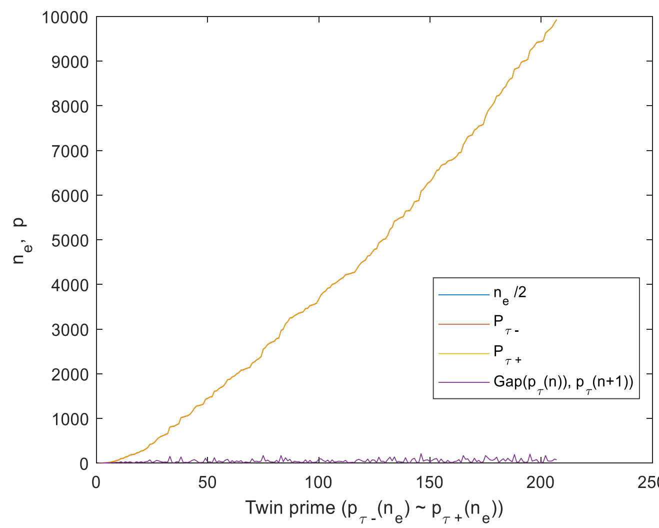

Fig. 3. Experimental results of infinite twin primes distribution in the space $\mathbb{P}_\tau \subset \mathbb{P} \times \mathbb{P}$

Experiment 2. Applying the ATPS algorithm for a larger set of twin primes in the scope of $4 \le n_e/2 \le 210$ revealing similar results as illustrated in Figure 3. There are 206 twin primes detected and the density of twin primes $d_\tau = \frac{\mathbb{P}_\tau^{10000}}{n_e/2} = \frac{210}{10000} = 2.10\%$. The maximum gap among the pairs of detected twin primes is 6 • 35 = 210 observed at $n_e/2 = 145$.

The inductive inference in the formal proof of Theorems 2 provides a rigorous methodology for dealing with the infinite twin prime problem. More large-scale testing based on Algorithm 1 have been conducted as summarized in Table 1 to support the twin prime theorem. The experimental data provide empirical evidence for Theorem 2 by demonstrating:

a) There is no tendency that the pairs of twin primes will disappear in the infinite sets of $\mathbb{N}$ constrained by $\mathbb{P}_\tau \subset \mathbb{P}_\mu \subset \mathbb{N}$ because $\lim_{n\to\infty} d_\tau n_e = \lim_{n\to\infty} n_e = \infty$.

b) There is no sign that the gaps g_{max} between the pairs of twin primes in $\mathbb{P}_\tau$ would irruptively jump to infinitive as shown in Figures 2 and 3.

c) The classes of N and N_τ are equivalent as shown in Table 1 where N_τ (Lemma 1) is monotonically growing along N such that $\lim_{n\to\infty} N_\tau = \infty \Rightarrow |\mathbb{P}_\tau| = \infty$.

The inductive inference towards the proof of Theorem 2 provides a rigorous methodology for dealing with the infinite twin prime problem. Large-scale testing based on Algorithm 1 have been conducted as summarized in Table 1 to support the proven twin prime theorem where the only limitation is computing speed and memory capacity. The experimental results have provided empirical evidence for confirming the twin prime theorem without exception. It demonstrates the ultimate power of human abstract inference underpinned by mathematical laws and formal analytic platforms.

Theorem 2, Algorithm 1, and associated experiments provide both formal proof and empirical verification of the infinity distribution of twin primes in $\mathbb{P}_\tau \subset \mathbb{P}_\mu \subset \mathbb{P} \times \mathbb{P}$. That is, $\lim_{n\to\infty} (|\mathbb{P}_\tau| \sim |\mathbb{P}_\mu| \sim |\mathbb{P}| \sim |\mathbb{N}|) = \infty$ based on Cantor's infinitive countability across the equivalent classes of sets in number theory [29]. The properties of mirror/twin primes and the theorem of mirror prime decomposition have also been applied to prove the Goldbach conjecture in my lab [18].

5. Conclusion

This work has presented a formal proof of the twin prime conjecture based on a novel mathematical model of two-dimensional mirror primes $\mathbb{P}_\mu \subset \mathbb{P} \times \mathbb{P}$ and their symmetric properties. A fundamental theorem of mirror-prime decomposition for arbitrary even numbers has been established towards the proof of the *twin prime conjecture*. By observing that the set of twin primes $\mathbb{P}_\tau$ with $p_\varepsilon^{n+1} - p_\varepsilon^n \equiv 2$ as a subset of mirror primes, this work has deduced the twin prime conjecture to a special case of the infinity of the recursive sequence of mirror prims $\mathbb{P}_\mu$:

$$\forall n_e \in \mathbb{N}_e \subset \mathbb{N} \text{ and } 1 \le (k \in N) \le \frac{n_e}{2} - 1:$$

$$\mathbb{P}_\tau = \mathop{R}_{n_e/2=4}^{\infty} [p_\tau^{n_e/2} = (p_{\tau^-}^{n_e/2}, p_{\tau^+}^{n_e/2}) \mid \rho(p_{\tau^-}^{n_e/2}) \wedge \rho(p_{\tau^+}^{n_e/2}) = 1] \subset \mathbb{P}_\mu$$

$$\text{since } |\mathbb{P}_\mu| = |\mathop{R}_{n_e/2=4}^{\infty} \{ \mathop{R}_{k=1}^{\frac{n_e}{2}-2} \begin{pmatrix} (p_{\mu^-}^{n_e/2} = \frac{n_e}{2} - k,\ p_{\mu^+}^{n_e/2} = \frac{n_e}{2} + k) \\ \mid \rho(p_{\mu^-}^{n_e/2}) \wedge \rho(p_{\mu^+}^{n_e/2}) = 1 \end{pmatrix} \}| = \infty,$$

so is $|\mathbb{P}_\tau| = |\mathop{R}_{n_e/2=4}^{\infty} (p_{\tau^-}^{n_e/2},\ p_{\tau^+}^{n_e/2}) \in \mathbb{P}_\mu| = \infty,\ \mathbb{P}_\tau \subset \mathbb{P}_\mu$, according to the MPD theorem and Cantor's equivalent counterability, such that there are infinitely many pairs of twin primes in $\mathbb{P}_\tau \subset \mathbb{P}_\mu \subset \mathbb{P} \times \mathbb{P}$.

The formal prove of the twin prime conjecture has been based on the discovery on the set of mirror primes $\mathbb{P}_\mu \subset \mathbb{P} \times \mathbb{P}$ and the establishment of the equivalent countability across $\lim_{n\to\infty} (|\mathbb{P}_\tau| \sim |\mathbb{P}_\mu| \sim |\mathbb{P}| \sim |\mathbb{N}|) = \infty$. Experiments using the algorithm for twin-prime sieve have visualized the proven twin-prime theorem and the infinitively recursive properties of twin primes among $\mathbb{P} \times \mathbb{P}$.

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