Some Identities for an Alternating Sum of Fibonacci and Lucus Numbers of Order *k*

SUKANYA SOMPROM, WAITAYA NIMNUAL, WATHCHARAPONG HONGTHONG
Department of Mathematics,
Faculty of Science and Technology,
Surindra Rajabhat University, Surin 32000,
THAILAND

Abstract: - In this paper, we defined $F_n^{(k)}$ be the Fibonacci of order k and $L_n^{(k)}$ be the Lucas number of order k. We presented some of their new identities as well as some results of relation for an alternating sum between Fibonacci and Lucas number of order k as follow;

$$\sum_{i=0}^{n}(-1)^i m^{n-1}\left(mL_{i+1}^{(k)}+\left((m-2)^2-4\right)F_i^{(k)}-2m^2F_{i-1}^{(k)}-m\sum_{j=3}^{k}jF_{i-j+2}^{(k)}\right)=(-1)^n m\left(F_{n+1}^{(k)}-2F_n^{(k)}\right).$$

Key-Words: - Alternating sums, Fibonacci number of order k, Lucas number of order k, Tribonacci number

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1 Introduction

In recent years, the Fibonacci and Lucas number are an ordinal number that is significant in mathematics. It has also gained widespread attention in applied mathematics, number theory, computers, etc. It can also be applied in other fields such as art, architecture, finance, etc. Moreover, it is also applied in financial. Investors use it to look at the price reversal of the asset, for example. In the application of art and architecture, the Golden Ratio of the Fibonacci sequence which is the theory that calculates the most beautiful proportions in the world is used in urban design. Even the Mona Lisa painting also uses the Golden Ratio. In nature, Fibonacci can also explain the spiral rotation of sunflower seeds or a cactus with a thorny arrangement corresponding to the Fibonacci sequence. The Fibonacci sequence was first presented in 1202 by Leonardo Fibonacci.

Let F_n be the Fibonacci number defined recursively as follows

$$F_n = F_{n-1} + F_{n-2} \quad ,$$

for $n \geq 2$ with $F_0 = 0$ and $F_1 = 1$. For example, the sequence of the Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, … .

Let L_n be the Lucas number defined recursively as follows

$$L_n = L_{n-1} + L_{n-2} \quad ,$$

for $n \geq 2$ with $L_0 = 2$ and $L_1 = 1$. For example, the sequence of the Lucas numbers are 2, 1, 3, 4, 7, 11, 18, 29, 47, … .

Researchers have been talking about the properties of Fibonacci and Lucas number for a long time. In 2016, Tom Edgar [1] produced some properties of an alternating sum Fibonacci-Lucas relations as demonstrated below:

$$k^{m+1}F_{m+1} = \sum_{i=0}^{m} k^i L_i + (k-2)\sum_{i=0}^{m+1} k^{i-1}F_i \quad ,$$

for $m, k \in \mathrm{N}$. In addition, there were also researchers who studied about alternating sum, Zvonko Cerin studied formulas the Lucas number for the odd and even terms and Emrah Kilic et al. showed some results of the Fibonacci and Lucas numbers from alternating Melham's sums, see [2, 3] for details.

Let F_n^k be the k-Fibonacci number defined recursively as follows

$$F_{n+1}^k = kF_n^k + F_{n-1}^k \quad ,$$

for $n \geq 1$ with $F_0^k = 0$ and $F_1^k = 1$.

Let L_n^k be the k-Lucas number defined recursively as follows

$$L_{n+1}^k = kL_n^k + L_{n-1}^k \quad ,$$

for $n \geq 1$ with $L_0^k = 2$ and $L_1^k = 1$.

The k-Fibonacci and k-Lucas numbers have been introduced on properties such as Yashwant K. Panwar et al. [4] studied the sums of odd and even terms of the k -Fibonacci numbers by using the Binet's formula. In 2014, Bijindra Singh et al. [5] studied the products of k -Fibonacci and k -Lucas numbers using the Binet's formulas. Moreover, There are people who are interested in studying the relationship of k -Fibonacci and k -Lucas numbers in various forms, see [6, 7] for details. In addition, there are still many researchers who are interested in studying and developing the (p,q) - Fibonacci and (p,q) -Lucas number. For example, in 2017, Alongkot S. and Mongkol T. [8] studied the product of (p,q) - Fibonacci and (p,q) -Lucas number by using the Binet' formulas to show their properties.

2 Preliminaries

In section 2, we introduced the fundamental Definitions of the Fibonacci and Lucas number of order k and Corollary.

Definition 1 For $n,k\in \mathrm{N}(k\neq 1)$, let $F_n^{(k)}$ be the Fibonacci number of order k defined recursively as follows:

$$F_n^{(k)}=\sum_{i=1}^{k}F_{n-j}^{(k)} ,$$

for $n\geq 2$ with $F_n^{(k)}=0$ for $-k+1\leq n\leq 0$ and $F_1^{(k)}=1$. In case of $k=2$ the equation is as follows:

$$F_n^{(2)}=F_{n-1}^{(2)}+F_{n-2}^{(2)} \text{ or } F_n=F_{n-1}+F_{n-2} ,$$

that is, the equation of the Fibonacci sequence.

$k\setminus n$	1	2	3	4	5	6	7	8	9
2	1	1	2	3	5	8	13	21	34
3	1	1	2	4	7	13	24	44	81
4	1	1	2	4	8	15	29	56	108
5	1	1	2	4	8	16	31	61	120
6	1	1	2	4	8	16	32	63	125
7	1	1	2	4	8	16	32	64	127
8	1	1	2	4	8	16	32	64	128

Table 1 Fibonacci numbers of order k

Definition 2 For $n,k\in \mathrm{N}(k\neq 1)$, let $L_n^{(k)}$ be the Lucas number of order k defined recursively as follows:

$$L_n^{(k)}=n+\sum_{j=1}^{n-1}L_{n-j}^{(k)} ,$$

for $2\leq n\leq k$ and

$$L_n^{(k)}=\sum_{j=1}^{k}L_{n-j}^{(k)} ,$$

for $n\geq k+1$ with $L_0^{(k)}=k$ and $L_1^{(k)}=1$. In case of $k=2$ the equation is as follows:

$$L_n^{(2)}=L_{n-1}^{(2)}+L_{n-2}^{(2)} \text{ or } L_n=L_{n-1}+L_{n-2} ,$$

that is, the equation of the Lucas sequence.

$k\setminus n$	1	2	3	4	5	6	7	8	9
2	1	3	4	7	11	18	29	47	76
3	1	3	7	11	21	39	71	131	241
4	1	3	7	15	26	51	99	191	367
5	1	3	7	15	31	57	113	223	439
6	1	3	7	15	31	63	120	239	475
7	1	3	7	15	31	63	127	247	493
8	1	3	7	15	31	63	127	255	502

Table 2 Lucas number of order k

Corollary 1 [9] The Lucas number of order k are expressed in terms of the Fibonacci numbers of order k by

$$L_n^{(k)}=\sum_{j=1}^{\min\{n,k\}}jF_{n-j+1}^{(k)} , \tag{1}$$

for $n,k\in \mathrm{N}(k\neq 1)$.

A number of researchers have studied the Fibonacci and Lucas numbers of order k using Corollary 1 in proving various Theorem and Lemmas, see for details [10].

In 2020, Spios D. Dafnis et al. [11] introduced some identities of Fibonacci and Lucas number of order k as follow:

$$\sum_{i=0}^{n}(-1)^i m^{n-i}\left(L_{i+1}^{(k)}+(m-2)F_i^{(k)}-\sum_{j=3}^{k}jF_{i-j+2}^{(k)}\right)$$
$$=(-1)^n F_{n+1}^{(k)} .$$

In this study, we presented some identities of the Fibonacci and Lucas number of order k defined by Definition 1, Definition 2, and the Corollary to show the theorem.

3 Main Theorem

In this section, we gave some theorem of the Fibonacci and Lucas number of order k. First, we provided a lemma used in the proofs of our results. Additionally, we presented corollary and example as follows.

Lemma 1 Let $F_n^{(k)}$ and $L_n^{(k)}$ be the Fibonacci and Lucas number of order k, respectively, then

$$L_{n+1}^{(k)}=F_{n+1}^{(k)}+2F_n^{(k)}+\sum_{j=3}^{\min\{n,k\}}jF_{n-j+2}^{(k)} .$$

Proof The result is obvious with deriving the following by Corollary 1.

Then we have the following Theorem.

Theorem 1 Let $\left(F_n^{(k)}\right)_{n\geq 0}$ and $\left(L_n^{(k)}\right)_{n\geq 0}$ be the Fibonacci and Lucas number of order k. Then,

$$\sum_{i=0}^{n}(-1)^i m^{n-1}\Big(mL_{i+1}^{(k)}+\left((m-2)^2-4\right)F_i^{(k)} -2m^2F_{i-1}^{(k)}-m\sum_{j=3}^{k}jF_{i-j+2}^{(k)}\Big) =(-1)^n m\left(F_{n+1}^{(k)}-2F_n^{(k)}\right). \tag{2}$$

Proof By Lemma 1, consider

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^i m^{n-1}\Big(mL_{i+1}^{(k)}+\left((m-2)^2-4\right)F_i^{(k)} -2m^2F_{i-1}^{(k)}-m\sum_{j=3}^{k}jF_{i-j+2}^{(k)}\Big)\\
&=\sum_{i=0}^{n}(-1)^i m^{n-i}\Bigg(\Big(mF_{i+1}^{(k)}+2mF_i^{(k)}+m\sum_{j=3}^{k}jF_{i-j+2}^{(k)}\Big)\\
&\quad +(m-2)^2F_i^{(k)}-4F_i^{(k)}-2m^2F_{i-1}^{(k)}-m\sum_{j=3}^{k}jF_{i-j+2}^{(k)}\Bigg)\\
&=\sum_{i=0}^{n}(-1)^i m^{n-i}\Big(mF_{i+1}^{(k)}+2mF_i^{(k)}\\
&\quad +\left(m^2-4m+4\right)F_i^{(k)}-4F_i^{(k)}-2m^2F_{i-1}^{(k)}\Big)\\
&=\sum_{i=0}^{n}(-1)^i m^{n-i}\Big(mF_{i+1}^{(k)}+m^2F_i^{(k)}-2mF_i^{(k)}-2m^2F_{i-1}^{(k)}\Big)\\
&=m^n\left(mF_1^{(k)}\right)\\
&\quad +(-1)^1m^{n-1}\left(mF_2^{(k)}+m^2F_1^{(k)}-2mF_1^{(k)}-2m^2F_0^{(k)}\right)\\
&\quad +(-1)^2m^{n-2}\left(mF_3^{(k)}+m^2F_2^{(k)}-2mF_2^{(k)}-2m^2F_1^{(k)}\right)\\
&\quad +(-1)^3m^{n-3}\left(mF_4^{(k)}+m^2F_3^{(k)}-2mF_3^{(k)}-2m^2F_2^{(k)}\right)\\
&\quad \vdots\\
&\quad +(-1)^n m^{n-n}\left(mF_{n+1}^{(k)}+m^2F_n^{(k)}-2mF_n^{(k)}-2m^2F_{n-1}^{(k)}\right)\\
&=(-1)^n mF_{n+1}^{(k)}-(-1)^n 2mF_n^{(k)}\\
&=(-1)^n m\left(F_{n+1}^{(k)}-2F_n^{(k)}\right).
\end{aligned}$$

Thus, the proof is complete.

From equation (2) , we considered the Table 1 and Table 2, by choosing $k=7$ and $n=4$ as an example. It was found that on the left side of equation (2), it could be calculated as $4m$. And on the right, it could be calculated as $4m$ where both sides were equal. From the above theorems, we obtained the well-known identities for Fibonacci and Lucas of order k. For $k=2$, i.e. $F_n^{(2)}=F_n$, $L_n^{(2)}=L_n$, Theorem 1 reduces to new identities in the following corollary.

Corollary 2 Let F_n and L_n be the Fibonacci and Lucas number, respectively, then

$$\sum_{i=0}^{n}(-1)^i m^{n-1}\left(mL_{i+1}+\left((m-2)^2-4\right)F_i-2m^2F_{i-1}\right) =(-1)^n m\left(F_{n+1}-2F_n\right). \tag{3}$$

Moreover,

$$\sum_{i=0}^{n}(-1)^i 2^{n-1}\left(2L_{i+1}-4F_i-8F_{i-1}\right) =(-1)^n 2\left(F_{n+1}-2F_n\right), \tag{4}$$

for $m=2$.

Proof The proof is similar to Theorem 1 by fixing $k=2$ in equation (2) and fixed $m=2$ in equation (3), which implies Corollary 2.

From corollary 2 , we considered the Table 1 and Table, by choosing $n=5$ as an example. It was found that on the left side of equation (3), it could be calculated as $2m$. On the right side, it could be calculated as $2m$. It was found that both were equal. For $m=2, n=5$, it was found that on the left side of equation (4), it could be calculated as 4 . On the right side, it could be calculated as 4. It was found that both were equal.

Let T_n be the sequence of Tribonacci number defined recursively as follows

$$T_n=T_{n-1}+T_{n-2}+T_{n-3}$$

for $n\geq 3$ with $T_0=0$, $T_1=1$ and $T_2=1$. For example, the sequence of the Tribonacci number are 0, 1, 1, 2, 4, 7, 13, 24, 44, … .

Let C_n be the sequence of Lucas number of order 3 defined recursively as follows

$$C_n=C_{n-1}+C_{n-2}+C_{n-3}$$

for $n\geq 3$ with $C_0=3$, $C_1=1$ and $C_2=3$. For example, the sequence of the Lucas number of order 3 are 3, 1, 3, 7, 11, 21, 39, 71, 131, … .

Considering Theorem 1, we were able to find the relationship between Tribonacci number and Lucas number of order 3.

Lemma 2 For $n\in \mathrm{N}$, let T_n be the sequence of Tribonacci number and C_n be the sequence of Lucas number of order 3, then we have

$$mC_{n+1}=mT_{n+1}+2mT_n+3mT_{n-1},$$

for $m\geq 0$.

Proof We used Corollary 1 and fixed $k=3$, then the proof is completes.

Corollary 3 For $n \in \mathrm{N}$, let T_n be the sequence of Tribonacci number and C_n be the sequence of Lucas number of order 3, then

$$\sum_{i=0}^{n}(-1)^i m^{n-i}\left(mC_{i+1}+\left((m-2)^2-4\right)T_i-2m^2T_i-3mT_{i-1}\right)$$
$$=(-1)^n m\left(T_{n+1}-2T_n\right). \qquad (5)$$

Moreover,

$$\sum_{i=0}^{n}(-1)^i 2^{n-1}\left(2C_{i+1}-12T_i-6T_{i-1}\right)=(-1)^n 2\left(T_{n+1}-2T_n\right),$$

for $m=2$.

Proof By Lemma 2, consider

$$\sum_{i=0}^{n}(-1)^i m^{n-i}\left(mC_{i+1}+\left((m-2)^2-4\right)T_i-2m^2T_i-3mT_{i-1}\right)$$
$$=\sum_{i=0}^{n}(-1)^i m^{n-1}\left(\left(mT_{i+1}+2mT_i+3mT_{i-1}\right)\right.$$
$$\left.+\left((m-2)^2-4\right)T_i-2m^2T_{i-1}-3mT_{i-1}\right)$$
$$=\sum_{i=0}^{n}(-1)^i m^{n-1}\left(mT_{i+1}+m^2T_i-2mT_i-2m^2T_{i-1}\right)$$
$$=(-1)^0 m^n\left(mT_1+m^2T_0-2mT_0-2m^2T_{-1}\right)$$
$$+(-1)^1 m^{n-1}\left(mT_2+m^2T_1-2mT_1-2m^2T_0\right)$$
$$+(-1)^2 m^{n-2}\left(mT_3+m^2T_2-2mT_2-2m^2T_1\right)$$
$$\vdots$$
$$+(-1)^n m^0\left(mT_{n+1}+m^2T_n-2mT_n-2m^2T_{n-1}\right)$$
$$=(-1)^n mT_{n+1}-(-1)^n 2mT_n$$
$$=(-1)^n m\left(T_{n+1}-2T_n\right).$$

Then, the proof is complete in equation (5). Next we fixed $m=2$ in equation (5), then

$$\sum_{i=0}^{n}(-1)^i 2^{n-i}\left(2C_{i+1}+\left((2-2)^2-4\right)T_i-2(2)^2T_i-3(2)T_{i-1}\right)$$
$$=\sum_{i=0}^{n}(-1)^i 2^{n-i}\left(2C_{i+1}-12T_i-6T_{i-1}\right).$$

By Lemmma 2, we have

$$2C_{n+1}=2T_{n+1}+4T_n+6T_{n-1}.$$

Therefore,

$$\sum_{i=0}^{n}(-1)^i 2^{n-1}\left(2C_{i+1}-12T_i-6T_{i-1}\right)=(-1)^n 2\left(T_{n+1}-2T_n\right).$$

From corollary 3, we considered the Table 1 and Table 2, by choosing $m=3$ and $n=5$ as an example. It was found that on the left side of equation (5) it could be calculated as 3. And on the right side, it could be calculated as 3. It was found that both were equal.

4 Conclusion

In this study, we considered the Fibonacci and Lucas number of order k by using the definitions in section 2 to show that some identities concerning an alternating sum of the Fibonacci and Lucas number of order k. From the theorem we have studied, we found that in case of $k=2$, the relationship of identity is in the form of alternating sum of the Fibonacci and Lucas number. And in case of $k=3$, alternating sum of the Tribonacci and Lucas number of order k is the theorem that we study covering identity in the form of $k=2,3$. And other cases in the form presented, the result can be obtained according to the theorem we studied. Moreover, we presented Lemmas, corollaries and examples. For those who are interested, relationships and sequence properties can be found in the form of an alternating sum.

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