Continuum Wavelets and Distributions

YURI K. DEM'YANOVICH, OLGA N. IVANTSOVA, ALEKSANDRA Y. PONOMAREVA
Parallel Algorithm Department
Saint Petersburg State University
University nab.,7/9, St. Petersburg
RUSSIA

Abstract: - The purpose of this work is to obtain a wavelet expansion of information flows, which are distribution flows (in the terminology of Schwartz). The concept of completeness is introduced for a family of abstract functions. Using the mentioned families, nested spaces of distribution flows are constructed. The projection of the enclosing space onto the nested space generates a wavelet expansion. Decomposition and reconstruction formulas for the above expansion are derived. These formulas can be used for wavelet expansion of the original information flow coming from the analog device. This approach is preferable to the approach in which the analog flow is converted into a discrete numerical flow using quantization and digitization. The fact is that quantization and digitization lead to significant loss of information and distortion. This paper also considers the wavelet expansion of a discrete flow of distributions using the Haar type functions.

Key-Words: - wavelets, information flows, distributions, calibration relations

Received: September 24, 2021. Revised: May 21, 2022. Accepted: June 23, 2022. Published: July 14, 2022.

1 Introduction

The processing of numerical information flows with classical and non-classical wavelets have been studied in a large number of works. Research wavelet decompositions for flows of a more complex nature (flows of matrices, p-adic numbers, etc.) were mainly based on the theory of non-classical wavelets.

Wavelet decomposition is one of the main means of the processing of numerical information flows. Let us give several examples of the application of these expansions in technology and medicine. In research [1] the separate models for signal de-noising with different ratio signal/noise were designed. The discrete wavelet decompositions were used. The result was applied to the computerized analysis of Lung Sound.

Paper [2] is devoted to the damage severity quantification of the brain by using a wavelet packet. The proposed technique shows significant benefit in compressing spatio-spectral patterns of multichannel signals in just a unified visual representation.

The timely and high-quality maintenance of electrical networks is a prerequisite for their trouble-free operation.

In work [3], complex wavelets are used for creating an efficient algorithm for such processing. The proposed algorithm achieves higher accuracy with reduced training time in the classification of events than compared to the reported event classification methods. To date, there are several studies on the theory of wavelets, among which deserve special mention works by I. Daubechies [4], C. Chui [5], S. Mallat [6], Yu. N. Subbotin and Chernykh [7], I. Ya. Novikov, V. Yu. Protasov and M. A. Skopina [8]. Research in this field also includes a series of modern works. Cubic wavelets with two zero moments are obtained in work [9]. Five-diagonal splitting for cubic splines with six zero moments on the segment was obtained in [10]. Paper [11] deals with structural issues concerning wavelet frames and their dual frames. In paper [12] the authors define the wavelet multiplier and Landau-Pollak-Slepian operators on the Hilbert space. In paper [13] the wavelet optimized finite difference B-spline polynomial chaos method is proposed. The method is applied to the solution of stochastic partial differential equations. In paper [14] the authors propose a highly efficient and accurate valuation method for exotic-style options based on the novel Shannon Wavelet Inverse Fourier Technique (SWIFT).

These studies mainly reflect the classic approach to wavelets, which is based on various variants of the Fourier transform, applied to the multiple-scale ratio to obtain a scaling function and ultimately wavelet decomposition. However, the practice of processing numeric flows required expanding the framework of the classical theory.

W.Sveldens constructed a lifting scheme for an area that is not invariant relative to the shift. The concept of non-stationary wavelets, introduced by I.Ya. Novikov, also led to the expansion framework of the mentioned theory. The need to significantly speed up computations was faced with great theoretical difficulties that arose on the path of the development of the classical approach to the wavelet expansions (see [15] – [27]). In paper [18] the authors propose an algorithm with a high level of confidentiality while maintaining high image quality. Paper [19] presents a powerful, fast and reliable signal analysis method based on the massively parallel continuous wavelet transform algorithm. The nonlinear wavelet estimates of the spectral densities for non-Gaussian linear processes are considered in paper [20]. The paper [21] presents an efficient algorithm based on the Galerkin method using biorthogonal Hermite multiwavelets with cubic splines. The authors of paper [22] propose an effective approach to obtaining approximate solutions of linear and nonlinear two-dimensional Volterra integro-differential equations. with usage of two-dimensional wavelets. In [23], to solve the problems of low contrast and fuzzy boundary in the traditional wavelet transform, a threshold function is proposed. Paper [24] presents a new structure for a single-pixel image using compression probing in shift-invariant spaces by using the sparsity property of the wavelet representation. In [25] case studies of typical nonlinear de-noising problems in various domains are conducted. Study [26] focused on the classification of Electroencephalography signal. The study aims to make a classification with fast response and high-performance rate. Paper [27] proposes and discusses a new Electroencephalography de-noising technique, based on a combination of wavelet transforms and conventional filters.

The listed works show the wide use of wavelets in various fields of human activity. They apply to physics, chemistry, biology and medicine. In most cases these are the results of a large number of measurements at some points in space and at certain points in time. In fact the mentioned measurements are neither a point nor instantaneous. This fact, long noticed, led to the theory of Schwartz distributions. Along with value streams ordinary functions should also be considered distribution flows. In this regard, the use of distributions is more natural, since such an approach reflects the idea of a trial function. Mentioned flows of distributions can be continuous or discrete. In this and in another case, their wavelet decomposition is important, allowing the more efficient use of computer and communication resources.

The purpose of this work is to study information flows associated with certain trajectories in distribution spaces. Elements of the spaces are the mentioned trajectories, whose parameters take the values from a set of non-zero Lebesgue measure. For these trajectories (also called the families of distributions) the concept of completeness is introduced. The complete family is used to build a space of distributions. The criterion of embedding of the mentioned spaces is discussed.

The projection of the enclosing space on an embedded space generates a wavelet decomposition. It is shown that from the considered continuum case, we can pass to a discrete case. As a result of the transition, we obtain spaces of the Haar-type functions. In this case, the mentioned embedding criterion becomes calibration ratios.

2 Generating Function

Let $\mathcal{M}$, $\mathcal{K}$ be measurable sets of non-zero Lebesgue measure on real axis. Let $\mathcal{K}\subset\mathcal{M}\subset \Theta$, where Θ is an open set of the real axis. Consider linear space $K = K(\Theta)$ of basic functions (in this case we assume that $K(\Theta)$ is the standard linear space of main functions). Thus the space $K(\Theta)$ consists of all infinitely differentiable and compactly supported functions $\mathrm{v}(\theta)$, $\theta \epsilon\Theta$, i.e. such that $\operatorname{supp}\mathrm{v} \subset \Theta$.

The space of distributions (the space dual to $K(\Theta)$) denote $K' = K'(\Theta)$. The relevant duality is denoted by sharp brackets, namely, the result action of the distribution $f_0\epsilon K'$ on the main function $\mathrm{v}\epsilon K$ is denoted by $< f_0,\mathrm{v} >$.

Let $c_{(x)}$ be a family of distributions from the space K', where x is a family parameter, $x\epsilon\mathcal{M}$. A family $c = \{c_{(x)} \mid x\epsilon\mathcal{M}\}$ of this kind is called *a trajectory in K'* (or an abstract function with values in K').The expression $c_{(x)}$ is called *the trajectory component*. For the record of trajectory components it is sometimes convenient to use square brackets, setting $[c]_x = c_{(x)}$. For the main function $\mathrm{v}\,\epsilon K$, the expreson $\psi_{c,\mathrm{v}} = \psi_{c,\mathrm{v}}(x) =< c_{(x)},\mathrm{v} >$ is an ordinary function of the argument x defined on measurable set $\mathcal{M}$.

Let $p > 1$, $q = 1 - p^{-1^{-1}}$. Consider the set all trajectories c with property

$$\psi_{c,\mathrm{v}}\epsilon L_q(\mathcal{M}) \;\; \forall\, \mathrm{v}\epsilon K. \qquad (1)$$

We denote this set by $\mathcal{L}_q$.

Lemma 1. *The following statements are true.*

1. The set (1) is not empty.

2. There are trajectories $c = \{c_{(x)} \mid x\epsilon\mathcal{M}\}$, which do not lie in the set $\mathcal{L}_q(\mathcal{M})$.

3. The set $\mathcal{L}_q(\mathcal{M})$ is a linear space.

Proof. 1. Consider the trajectory $c = \{c_{(x)} \mid x\epsilon\mathcal{M}\}$, where $c_{(x)} = \delta_x$ is the delta-function at the point $x\epsilon\mathcal{M}$. In this case $\psi_{c,\mathrm{v}} = \psi_{c,\mathrm{v}}(x) =< c_{(x)}, \mathrm{v} >= \mathrm{v}(x)$ is continuous function in $\mathcal{M}$, so $\psi_{c,\mathrm{v}}\epsilon L_q(\mathcal{M})$. So, it is established that the set $\mathcal{L}_q(\mathcal{M})$ is not empty.

2. Let x_0 be a point of an open interval, contained in $\mathcal{M}$. Consider the trajectory $c = \{c_{(x)} \mid x\epsilon\mathcal{M}\}$, where $c_{(x)} = (x - x_0)^{-\gamma}\delta_x, \gamma > 0$. Then $\psi_{c,\mathrm{v}}(x) = < c_{(x)}, \mathrm{v} > = (x - x_0)^{\gamma}\mathrm{v}(x)$. Remaining in set K, choose a main function v such that $\mathrm{v}(x_0) \neq 0$ and choose γ so that $\gamma q > 1$. In this case, the function $\psi_{c,\mathrm{v}}(x)$ does not belong to space $L_q(\mathcal{M})$. The second part of the lemma is proved.

3. If c and d are two elements of the set $\mathcal{L}_q(\mathcal{M})$, then by definition the functions $\psi_{c,\mathrm{v}}(x)$ and $\psi_{d,\mathrm{v}}(x)$ lie in the space $L_q(\mathcal{M})$. We have $\psi_{\lambda c+\mu\epsilon d,\mathrm{v}}(x) =< \lambda c + \mu d, \mathrm{v} >= \lambda < c, \mathrm{v} > +\mu < d, \mathrm{v} >= \lambda\psi_{c,\mathrm{v}}(x) + \psi_{d,\mathrm{v}}(x)$.
It follows that $\psi_{\lambda c+\mu d,\mathrm{v}}\epsilon L_q(\mathcal{M})$. The third part of the lemma is proved.

It is obvious that $Cl\{\psi_{c,\mathrm{v}} \mid c\epsilon\mathcal{L}_q(\mathcal{M}), \mathrm{v}\epsilon K\} = L_q(\mathcal{M})$. Therefore, for $w\epsilon L_q(\mathcal{M})$, $c\epsilon\mathcal{L}_q(\mathcal{M})$ we can discuss the integral

$$\int_{x\epsilon\mathcal{M}} < c_{(x)}, \mathrm{v} > w(x)dx. \quad (2)$$

Let us introduce the notation

$$\Omega^* = \mathcal{L}_q(\mathcal{M}), \Omega = L_p(\mathcal{M}), \quad (3)$$

$$(d, w)_\Omega = \int_{x\epsilon\mathcal{M}} d(x)\, w(x)dx \quad (4)$$

$\forall\, d\epsilon L_q(\mathcal{M})\ \ \forall\, w\epsilon L_p(\mathcal{M})$.

Since for $c\epsilon\mathcal{L}_q(\mathcal{M})$ and $\mathrm{v}\epsilon K$ function $\psi_{c,\mathrm{v}}(x) =< c_{(x)}, \mathrm{v} >$ lies in the space $L_q(\mathcal{M})$, so from (3) -- (4) for expressions (2) we have

$$(< c_{(x)}, \mathrm{v} >, w)_\Omega = \\ = \int_{x\epsilon\mathcal{M}} < c_{(x)}, \mathrm{v} > w(x)dx < +\infty\ \forall\ \mathrm{v}\epsilon K. \quad (5)$$

In what follows, we will sometimes use a shorter notation without mention of the main function $\mathrm{v}\epsilon K$. For example, conditions (5) can be written in the form

$$(c_{(x)}, w)_\Omega = \int_{x\epsilon\mathcal{M}} c_{(x)}\, w(x)dx < +\infty\,. \quad (6)$$

Here and below, the presence of the main function $\mathrm{v}\epsilon K$ is implied (see (5) -- (6)).

In what follows, we will need the notion of a complete abstract function.

In this connection, we first introduce the concept of a complete family of mappings.

Definition 1. *Let H be a linear topological space, H^* be the dual space, T be a non-empty set of parameters t. For every fixed $t\epsilon T$ we consider a mapping Γ: $T \to H$. The family of the mappings $\{\Gamma(t)\}_{t\epsilon T}$ is called complete in H if for $d\epsilon H^*$ the condition $< d, \Gamma(t) >\equiv 0\ \ \forall t\epsilon T$ follows the equality $d = 0$, i.e.*

$$< d, \Gamma(t) >\equiv 0\ \ \forall t\epsilon T\ \ => d = 0.$$

In particular, if T is a set of numbers, then $\Gamma(t)$ is called an abstract function with values in H (or a trajectory in H). In this case, if family $\{\Gamma(t)\}_{t\epsilon T}$ is complete, then it is called *the complete abstract function in H* (or *the complete trajectory in H*).

Let T be some set on the real axis, $T \subset \mathbb{R}^1$.

Consider the complete trajectory $\{\omega(t)\}_{t\epsilon T}$ in the space $L_p(\mathcal{M})$. In this way $\omega(t) = \{\omega_{(x)}(t) \big|\ x\epsilon\ \mathcal{M}\}$ and

1) for every fixed $t\epsilon T$ the function $\omega_{(x)}(t)$ of argument $x\epsilon\ \mathcal{M}$ is an element of space $L_p(\mathcal{M})$, $\omega_{(x)}(\cdot)\epsilon L_p(\mathcal{M})$,

2) the relation

$$(d, \omega(t))_\Omega \equiv 0\ \forall t\epsilon T, d\epsilon L_q(\mathcal{M})\ \ => d = 0. \quad (7)$$

An example of a complete trajectory in $L_p(\mathcal{M})$ for the case when $\mathcal{M}$=(0,1), T={0,1,2,…} is the family $\omega_{(x)}(t) = x^t$.

Consider the linear space Ʋ defined by the relation

$$Ʋ = \{\upsilon \mid \upsilon = \upsilon(\mathrm{t}) = (c, \omega(t))_\Omega = \\ = \int_{x\epsilon\mathcal{M}} c_{(x)}\ \ \omega_{(x)}(t)dx\ \forall c\epsilon\Omega^*, t\epsilon T\}. \quad (8)$$

Note that according to the accepted notation, the formula $(c, \omega(t))_\Omega$ contains the main function implicitly, so that the mentioned formula is equivalent to formula $(< c, \mathrm{v} >, \omega(t))_\Omega$. Thus the space Ʋ consists of distributions.

Lemma 2. *For any element υ of the space Ʋ there is a unique family distributions c, $c\epsilon\mathcal{L}_q(\mathcal{M})$, such that $\upsilon(t) = (c, \omega(t))_\Omega\ \ \forall t\epsilon T$.*

Proof. We will prove by contradiction. Suppose there is an element υ, $\upsilon\epsilon$Ʋ, which has two representations

$$\upsilon(t) = (\bar{c}, \omega(t))_\Omega = (\bar{\bar{c}}, \omega(t))_\Omega\ \ \forall t\epsilon T, \quad (9)$$

where $\bar{c}$ and $\bar{\bar{c}}$ are two families from the space $\mathcal{L}_q(\mathcal{M})$. By (9) the identity follows

$$(< \bar{c} - \bar{\bar{c}}\ , \mathrm{v} >, \omega(t))_\Omega = 0\ \ \forall t\epsilon T. \quad (10)$$

Introducing the notation $d(x) =< \bar{c}_{(x)} - \bar{\bar{c}}_{(x)}, \mathrm{v} >$ from (10) we get relation (7), so that $d = 0$. So the families $\bar{c}$ and $\bar{\bar{c}}$ are the same.

The resulting contradiction proves the assertion. This concludes the proof.

3 Embedded Space

Here we consider an analog of previous construction with replacement of the set $\mathcal{M}$ by the set $\mathcal{K}$,

$\boldsymbol{\mathcal{K}}\subset\boldsymbol{\mathcal{M}}\subset \Theta$. In particular, $\widetilde{\Omega} = L_p(\boldsymbol{\mathcal{K}})$, $\widetilde{\Omega}^* = \mathcal{L}_q(\boldsymbol{\mathcal{K}})$, $(\tilde{d}, \widetilde{w})_{\widetilde{\Omega}} =$

$$\int_{y\epsilon\boldsymbol{\mathcal{K}}} \tilde{d}(y)\, \widetilde{w}(y)dy\ \forall \tilde{d}\ \epsilon L_q(\boldsymbol{\mathcal{K}}), \forall \widetilde{w}\ \epsilon L_p(\boldsymbol{\mathcal{K}}). \quad (11)$$

Let $\mathfrak{R}$ be a linear operation from $L_p(\boldsymbol{\mathcal{M}})$ to $L_p(\boldsymbol{\mathcal{K}})$. For a distribution $\tilde{a}\epsilon\mathcal{L}_q(\boldsymbol{\mathcal{K}})$ taking into account the previous agreement for $\psi_{\tilde{a},\mathrm{v}}(y) = < \tilde{a}_{(y)}, \mathrm{v} > \forall \mathrm{v}\epsilon K$ we have $\psi_{\tilde{a},\mathrm{v}}\ \epsilon L_q(\boldsymbol{\mathcal{K}})$.

Using (11), for $w\epsilon L_p(\boldsymbol{\mathcal{M}})$ we have

$$(\psi_{\tilde{a},\mathrm{v}}, \mathfrak{R}\mathrm{w})_{\widetilde{\Omega}} = (\mathfrak{R}^*\psi_{\tilde{a},\mathrm{v}}, w)_\Omega <=>$$
$$(< \tilde{a}_{(y)}, \mathrm{v} >, \mathfrak{R}\mathrm{w})_{\widetilde{\Omega}} = (\mathfrak{R}^* < \tilde{a}_{(y)}, \mathrm{v} >, w)_\Omega \quad (12)$$

Formulas of the form (12) will sometimes be written in the form

$$(\tilde{a}, \mathfrak{R}\mathrm{w})_{\widetilde{\Omega}} = (\mathfrak{R}^*\tilde{a}, w)_\Omega\,. \quad (13)$$

Suppose $\widetilde{\omega}(t) = \{\widetilde{\omega}_{(y)}(t) \mid y\epsilon\boldsymbol{\mathcal{K}}\}, t\epsilon T\}$ is a complete family in $L_p(\boldsymbol{\mathcal{K}})$. Consider the linear space $\widetilde{\mho}$ defined by the relation

$$\widetilde{\mho} = \{\tilde{\upsilon} \mid \tilde{\upsilon} = \tilde{\upsilon}(\mathrm{t}) = (\tilde{a}, \widetilde{\omega}(t))_{\widetilde{\Omega}} =$$
$$= \int_{y\epsilon\boldsymbol{\mathcal{K}}} \tilde{a}_{(y)}\ \widetilde{\omega}_{(y)}(t)dy\ \forall\tilde{a}\epsilon\widetilde{\Omega}^*, t\epsilon T\}. \quad (14)$$

Note that according to the accepted notation, the formula $(\tilde{a}, \widetilde{\omega}(t))_{\widetilde{\Omega}}$ contains the main function implicitly, so that the mentioned formula is equivalent to formula $(< \tilde{a}, \mathrm{v} >, \widetilde{\omega}(t))_{\widetilde{\Omega}}$, $\mathrm{v}\epsilon K$. Thus the space $\widetilde{\mho}$ consists of distributions.

Consider the linear operation $\mathcal{P}$, which acts from space Ω into the space $\widetilde{\Omega}$,

$$\mathcal{P}: \Omega\ \to\ \widetilde{\Omega}\ <=>\ \mathcal{P}c = \tilde{c}\quad \forall c\epsilon\Omega,\ \tilde{c}\ \epsilon\ \widetilde{\Omega}. \quad (15)$$

Let's suppose that

$$\widetilde{\omega}(t) = \mathcal{P}\omega(t)\quad \forall t\epsilon T\,. \quad (16)$$

Theorem 1. *If the relations (15) -- (16) are right, then the space (14) is contained in the space (8),*

$$\widetilde{\mho} \subset \mho. \quad (17)$$

Proof. According to formula (14), for the element $\tilde{\upsilon}\epsilon\widetilde{\mho}$ a fair representation is

$$\tilde{\upsilon}(t) = (\tilde{a}, \widetilde{\omega}(t))_{\widetilde{\Omega}} <=>\ < \tilde{\upsilon}(t), \mathrm{v} >=$$
$$= (< \tilde{a}_{(\cdot)}, \mathrm{v} >, \widetilde{\omega}_{(\cdot)}(t))_{\widetilde{\Omega}}\quad \forall \mathrm{v}\epsilon K\quad \forall\tilde{a}\epsilon\widetilde{\Omega}^*. \quad (18)$$

Using representation (16) in (18), we have

$$< \tilde{\upsilon}, \mathrm{v} >= (< \tilde{a}_{(\cdot)}, \mathrm{v} >, \mathcal{P}\omega_{(\cdot)}(t))_{\widetilde{\Omega}} =$$
$$= (\mathcal{P}^* < \tilde{a}_{(\cdot)}, \mathrm{v} >, \omega_{(\cdot)}(t))_\Omega\quad \forall \mathrm{v}\epsilon K.$$

In view of the obvious relationship

$$\mathcal{P}^*: L_q(\boldsymbol{\mathcal{M}})\ \leftarrow\ L_q(\boldsymbol{\mathcal{K}})$$

we get

$$< \tilde{\upsilon}, \mathrm{v} >= (< c_{(\cdot)}, \mathrm{v} >, \omega_{(\cdot)}(t))_\Omega\quad \forall \mathrm{v}\epsilon K. \quad (19)$$

where $< c, \mathrm{v} > = \mathcal{P}^* < \tilde{a}_{(\cdot)}, \mathrm{v} >$ belongs to the space $L_q(\boldsymbol{\mathcal{M}})$.

From the definition (8) of the space $\mho$ it is clear that the distribution (19) is an element of this space, $\tilde{\upsilon}\ \epsilon\mho$. Formula (17) has been established.

This completes the proof.

4 Wavelet Decomposition

Let condition (16) be satisfied. According to Theorem 1 the space $\widetilde{\mho}$ is embedded in the space $\mho$, i.e. relation (17) holds.

Consider the projection operation P_0 of the space $\mho$ onto the space $\widetilde{\mho}$,

$$P_0:\quad \mho\ \to \widetilde{\mho}. \quad (20)$$

According to Lemma 2, for the element $\upsilon\epsilon\mho$ there are unique elements $c\epsilon\Omega^*$ and $\tilde{a}\epsilon\widetilde{\Omega}^*$ such that the next representations hold

$$\upsilon(\mathrm{t}) = (c, \omega(t))_\Omega,\qquad \tilde{\upsilon}(\mathrm{t}) = (\tilde{a}, \widetilde{\omega}(t))_{\widetilde{\Omega}},$$
$$P_0\upsilon = P_0[(c, \omega(\cdot))_\Omega] = (\tilde{a}, \widetilde{\omega}(\cdot))_{\widetilde{\Omega}}\,. \quad (21)$$

Thus the element $\tilde{a}\epsilon\widetilde{\Omega}^*$ is uniquely defined by the element $c\epsilon\Omega^*$. Appropriate map $c \to \tilde{a}$ is denoted by $\boldsymbol{Q}$,

$$\tilde{a} = \boldsymbol{Q}c. \quad (22)$$

It is easy to see that $\boldsymbol{Q}$ is the linear operation acting from the space Ω^* into the space $\widetilde{\Omega}^*$,

$$\boldsymbol{Q}:\ \Omega^*\ \to \widetilde{\Omega}^*.$$

From (21) -- (22) it follows, that operation $\boldsymbol{Q}$ is defined by the operation P_0 according to the formula

$$P_0[(c, \omega(\cdot))_\Omega](\mathrm{t}) \equiv$$
$$\equiv (\boldsymbol{Q}c, \widetilde{\omega}(t))_{\widetilde{\Omega}}\ \ \forall c\ \epsilon\Omega^*\ \forall t\ \epsilon T. \quad (23)$$

Theorem 2. *For any element* $\upsilon\epsilon\mho$, $\upsilon = \upsilon(t) = (c, \omega(t))_\Omega\ \ \forall t\ \epsilon T, c\ \epsilon\Omega^*$, *the ratio*

$$P_0\upsilon(t) = (\mathcal{P}^*\boldsymbol{Q}c, \omega(t))_\Omega, <=> < P_0\upsilon(t), v >$$
$$= (\mathcal{P}^* < \boldsymbol{Q}c, v >, \omega(t))_\Omega$$

(24)

is right.

Proof. Since formula (23) is equivalent to formula

$$P_0[(< c, \mathrm{v} >, \omega(\cdot))_\Omega](\mathrm{t}) \equiv (< Qc, \mathrm{v} >, \widetilde{\omega}(t))_{\widetilde{\Omega}}$$
$$\forall c\ \epsilon\Omega^*\ \forall t\ \epsilon T\quad \forall \mathrm{v}\ \epsilon K, \quad (25)$$

then, taking into account relation (16), from (25) we find

$$P_0[(< c, \mathrm{v} >, \omega(\cdot))_\Omega](\mathrm{t}) \equiv (< Qc, \mathrm{v} >, \widetilde{\omega}(t))_{\widetilde{\Omega}}$$
$$\equiv (< Qc, \mathrm{v} >, \mathcal{P}\omega(t)\,)_{\widetilde{\Omega}}\ \ \equiv (\mathcal{P}^* < Qc, \mathrm{v} >, \omega(t)\,)_\Omega$$
$$\forall c\ \epsilon\Omega^*\ \forall t\ \epsilon T\quad \forall \mathrm{v}\ \epsilon K. \quad (26)$$

Using the notation adopted in (13) -- (14), we see that relation (26) leads to equality (24).

This completes the proof.

Let us introduce the operation $Q_0 = \mathcal{I} - P_0$, where $\mathcal{I}$ is the identity operation in $\mho$. As a result of projection (20) we obtain the direct sum

$$\mho = \widetilde{\mho} + W,$$

where $\widetilde{\mho} = P_0\mho$, $W = Q_0\mho$.

Consider $c\ \epsilon\Omega^*$. Let's put

$$b = c - \mathcal{P}^*\boldsymbol{Q}c\ \ <=> < b, \mathrm{v} > -\mathcal{P}^* < Qc, \mathrm{v} > \quad (27)$$

$\forall \mathrm{v}\epsilon K$.

Theorem 3. *For* $\upsilon\epsilon\mho$ *relations*

$$Q_0\upsilon = \upsilon - P_0\upsilon = (b, \omega)_\Omega\,, \quad (28)$$
$$c = \mathcal{P}^*\tilde{a} + b \quad (29)$$

are fulfilled. Here $\tilde{a} = \boldsymbol{Q}c$.

Proof. From (24) -- (27) we have

$$< Q_0 \upsilon, \mathrm{v} >=< \upsilon, v > - < P_0 \upsilon, \mathrm{v} >=$$
$$(< c, \mathrm{v} >, \omega)_\Omega - (\text{Ƥ}^* < \boldsymbol{Q}\upsilon, \mathrm{v} >, \omega)_\Omega = (\mathrm{b}, \omega)_\Omega.$$

Thus, relation (28) is valid. From formulas (22) and (27) we obtain relation (29).

This concludes the proof.

The element $c \epsilon \Omega$ is the *initial flow*, the element $\tilde{a}$ is the *main flow* and the element b is the *wavelet flow*. Formulas (22), (27) are called *decomposition formulas*, and formulas (29) is called *reconstruction formulas*.

We introduce a linear operation

$$\widehat{\boldsymbol{Q}} : L_q(\boldsymbol{\mathcal{M}}) \to L_q(\boldsymbol{\mathcal{K}})$$

by formula

$$\widehat{\boldsymbol{Q}} < c, \mathrm{v} > \equiv < \boldsymbol{Q} c, \mathrm{v} >. \qquad (30)$$

Theorem 4. *For the operation defined by formula (23) to be the projection operation P_0 of the space U onto the space $\widetilde{\mathrm{U}}$ it is necessary and sufficient to have*

$$\widehat{\boldsymbol{Q}} \text{Ƥ}^* = I.$$

(31)

Here I is the identical operation in the space $L_q(\boldsymbol{\mathcal{M}})$.

Proof. *Necessity.* Let P_0 be a projection operation onto the space $\widetilde{\mathrm{U}}$. Then the idempotency condition is satisfied: $P_0^2 = P_0$. In other words, on elements of the space $\widetilde{\mathrm{U}}$ the operation P_0 acts as the identical operation,

$$P_0[(< \tilde{a}, \mathrm{v} >, \omega(\cdot))_\Omega] \equiv (< \tilde{a}, \mathrm{v} >, \widetilde{\omega})_{\widetilde{\Omega}} \qquad (32)$$
$$\forall \mathrm{v} \epsilon K \quad \forall \tilde{a} \epsilon \widetilde{\Omega}^*.$$

On the other hand, by the definition of the operation P_0 we have

$$P_0[(< c, \mathrm{v} >, \omega(\cdot))_\Omega] \equiv (< Qc, \mathrm{v} >, \widetilde{\omega})_{\widetilde{\Omega}} \qquad (33)$$
$$\forall \mathrm{v} \epsilon K \quad \forall c \epsilon \Omega^*.$$

Using the definition of the operation $\widehat{\boldsymbol{Q}}$ (see formula (30)) from (33) we obtain

$$P_0[(< c, \mathrm{v} >, \omega(\cdot))_\Omega] \equiv (\widehat{\boldsymbol{Q}} < c, \mathrm{v} >, \widetilde{\omega})_{\widetilde{\Omega}} \qquad (34)$$
$$\forall \mathrm{v} \epsilon K \quad \forall c \epsilon \Omega^*.$$

Setting $< c, \mathrm{v} >= \text{Ƥ}^* < \tilde{a}, \mathrm{v} >$, by (34) we find

$$P_0[(\text{Ƥ}^* < \tilde{a}, \mathrm{v} >, \omega(\cdot))_\Omega] \equiv$$
$$\equiv (\widehat{\boldsymbol{Q}} \text{Ƥ}^* < \tilde{a}, \mathrm{v} >, \widetilde{\omega})_{\widetilde{\Omega}} \quad \forall \mathrm{v} \epsilon K \quad \forall \tilde{a} \epsilon \widetilde{\Omega}^*. \qquad (35)$$

The obvious transformation of the left side of relation (35) gives us the formula

$$P_0[(< \tilde{a}, \mathrm{v} >, \widetilde{\omega}(\cdot))_{\widetilde{\Omega}}] \equiv$$
$$\equiv (\widehat{\boldsymbol{Q}} \text{Ƥ}^* < \tilde{a}, \mathrm{v} >, \widetilde{\omega})_{\widetilde{\Omega}} \quad \forall \mathrm{v} \epsilon K \quad \forall \tilde{a} \epsilon \widetilde{\Omega}^*. \qquad (36)$$

Using property (16) on the left side of formula (36), we get the equality of the left sides of relations (32) and (36). Therefore identity

$$(< \tilde{a}, \mathrm{v} >, \widetilde{\omega})_{\widetilde{\Omega}} \equiv$$
$$\equiv (\widehat{\boldsymbol{Q}} \text{Ƥ}^* < \tilde{a}, \mathrm{v} >, \widetilde{\omega})_{\widetilde{\Omega}} \quad \forall \mathrm{v} \epsilon K \quad \forall \tilde{a} \epsilon \widetilde{\Omega}^* \qquad (37)$$

is right. In view of the completeness of the family $\widetilde{\omega}(t)$ we derive relation (31) by formula (37).

The necessity has been proven.

Sufficiency. Assume that relation (31) holds. The definition of the P_0 operation given by formula (23), shows that for an element $\upsilon \epsilon \mathrm{U}$ we have $P_0 \upsilon \epsilon \widetilde{\mathrm{U}}$. Notice, that in view of the notation (30), formula (23) is equivalent to formula (34).

Let $\tilde{\upsilon}$ be an arbitrary element of the space $\widetilde{\mathrm{U}}$. In (30) we take $< c, \mathrm{v} >= \text{Ƥ}^* < \tilde{a}, \mathrm{v} >$. As a result, we get

$$P_0[(\text{Ƥ}^* < \tilde{a}, \mathrm{v} >, \omega(\cdot))_\Omega] \equiv$$
$$\equiv (\widehat{\boldsymbol{Q}} \text{Ƥ}^* < \tilde{a}, \mathrm{v} >, \widetilde{\omega})_{\widetilde{\Omega}} \quad \forall \mathrm{v} \epsilon K \quad \forall \tilde{a} \epsilon \widetilde{\Omega}^*. \qquad (38)$$

In view of assumption (31), from relation (38) we easily find

$$P_0[(< \tilde{a}, \mathrm{v} >, \widetilde{\omega}(\cdot))_{\widetilde{\Omega}}] \equiv$$
$$\equiv (\widehat{\boldsymbol{Q}} \text{Ƥ}^* < \tilde{a}, \mathrm{v} >, \widetilde{\omega})_{\widetilde{\Omega}} \quad \forall \mathrm{v} \epsilon K \quad \forall \tilde{a} \epsilon \widetilde{\Omega}^*. \qquad (39)$$

It follows from (39) that the operation P_0 is idempotent. The sufficiency of relation (31) has been established.

This concludes the proof.

5 Integral Operation Case

Here we give an illustration of the previous situations where Ƥ is an integral operation.

Let $U(y, x)$ be a function of two arguments $x \epsilon \boldsymbol{\mathcal{M}}, y \epsilon \boldsymbol{\mathcal{K}}$ such that the integral operation Ƥ with kernel $U(y, x)$,

$$\text{Ƥ}: g(y) = \int_{x \epsilon \boldsymbol{\mathcal{M}}} U(y, x) w(x) dx, \qquad (40)$$

maps the space $L_p(\boldsymbol{\mathcal{M}})$ to the space $L_p(\boldsymbol{\mathcal{K}})$,

$$\text{Ƥ}: L_p(\boldsymbol{\mathcal{M}}) \to L_p(\boldsymbol{\mathcal{K}}), \text{Ƥ}^*: L_q(\boldsymbol{\mathcal{M}}) \leftarrow L_q(\boldsymbol{\mathcal{K}}). \; (41)$$

Consider two abstract functions

$$\omega(t) = \{\omega_{(x)}(t) \mid x \epsilon \boldsymbol{\mathcal{M}}\}, \; \widetilde{\omega}(t) = \{\widetilde{\omega}_{(y)}(t) \mid y \epsilon \boldsymbol{\mathcal{K}}\},$$

which are complete in the spaces $L_p(\boldsymbol{\mathcal{M}})$ and $L_p(\boldsymbol{\mathcal{K}})$, respectively.

Let's suppose that

$$\widetilde{\omega}(t) = \text{Ƥ}\omega(t) \qquad (42)$$

Theorem 5. *If relation (42) holds, then*

$$\widetilde{\mathrm{U}} \subset \mathrm{U}. \qquad (43)$$

Proof. According to formula (14), for the element $\tilde{\upsilon} \epsilon \widetilde{\mathrm{U}}$ the representation is true

$$\tilde{\upsilon} = (\tilde{a}, \widetilde{\omega})_{\widetilde{\Omega}} <=> < \tilde{\upsilon}, \mathrm{v} >= (< \tilde{a}_{(\cdot)}, \mathrm{v} >, \widetilde{\omega}_{(\cdot)})_{\widetilde{\Omega}}$$
$$\forall \mathrm{v} \epsilon K \quad \forall \tilde{a} \epsilon \widetilde{\Omega}^*. \qquad (44)$$

Relation (44) is equivalent to the formula

$$< \tilde{\upsilon}(t), \mathrm{v} >= \int_{y \epsilon \boldsymbol{\mathcal{K}}} < \tilde{a}_{(y)}, \mathrm{v} > \widetilde{\omega}_{(y)}(t) dy \qquad (45)$$
$$\forall \mathrm{v} \epsilon K, \; \tilde{a} \epsilon \widetilde{\Omega}^*.$$

In view of formula (40), condition (42) can be rewritten as

$$\widetilde{\omega}_{(y)}(t) = \int_{x \epsilon \boldsymbol{\mathcal{M}}} U(y, x) \, \omega_{(x)}(t) dx. \qquad (46)$$

Using representation (46) in relation (45), we have

$< \tilde{\upsilon}(t), \mathrm{v} >=$

$$= \int_{y\epsilon\mathcal{K}} < \tilde{a}_{(y)}, \mathrm{v} > \int_{x\epsilon\mathcal{M}} U(y,x)\, \omega_{(x)}(t) dxdy \quad (47)$$
$$\forall \mathrm{v}\epsilon K, \ \ \tilde{a}\epsilon\widetilde{\Omega}^*.$$

Rearranging the order of integration in (47) leads to the formula

$< \tilde{\upsilon}(t), \mathrm{v} >=$
$$= \int_{x\epsilon\mathcal{M}} \left[\int_{y\epsilon\mathcal{K}} U(y,x) < \tilde{a}_{(y)}, \mathrm{v} > \quad dy \right] \omega_{(x)}(t) dx$$
$$\forall \mathrm{v}\epsilon K, \ \ \tilde{a}\epsilon\widetilde{\Omega}^*. \qquad (48)$$

According to condition (41), the expression in square brackets is an element of the space $L_q\,(\mathcal{M})$. Thus, in accordance with formula (8) relation (48) is a representation for element of the space $\mho$.

This completes the proof.

6 Space of the Haar Type

Let $\mathcal{M} = \Theta$ be an interval (α, β). Consider a grid

$$X: ... < x_{-1} < x_0 < x_1 < x_2 < \cdots \qquad (49)$$
$$\lim_{j\to-\infty} x_j = \alpha, \quad \lim_{j\to-\infty} x_j = \beta. \qquad (50)$$

Let's put $\omega(t) = \{\omega_{(x)}(t) \mid x\epsilon\mathcal{M}\}$, where $\omega_{(x)}(t)$ is defined by grid (49) -- (50),

$$\omega_{(x)}(t) = \begin{cases} \frac{1}{x_{j+1}-x_j} & for\ x, t\epsilon[x_j, x_{j+1}) \\ 0 & otherwise. \end{cases} \qquad (51)$$

If t is fixed in the interval Θ, then there is $j\epsilon\mathbb{Z}$ so that $t\epsilon[x_j, x_{j+1})$. When so fixed t the expression $\omega_{(x)}(t)$ is piecewise constant function of the argument $x\epsilon\mathcal{M}$. This the function is equal to the constant $(x_{j+1} - x_j)^{-1}$ for $x\epsilon[x_j, x_{j+1})$ and equals zero for $x\epsilon\mathcal{M}\backslash[x_j, x_{j+1})$.

Thus, for every fixed $t\epsilon\,\Theta$ it is obvious that implication $\omega_{(\cdot)}(t)\epsilon L_q\,(\mathcal{M})$ is correct.

Let $C^{-1}(X)$ be the space of piecewise constant functions, which are constants on each interval $[x_j, x_{j+1})$, $j\epsilon\mathbb{Z}$. For functions $\bar{u}$, $\bar{g}$ from the space $C^{-1}(X)$ we introduce the notation $||\bar{u}||_p = (\sum_{j\epsilon\mathbb{Z}} |\,\bar{u}(x_j)\,|^p)^{1/p}$, $||\bar{g}||_q = (\sum_{j\epsilon\mathbb{Z}} |\,\bar{g}(x_j)\,|^q)^{1/q}$.

Consider dual spaces $\bar{l}_p = \{\bar{u} \mid \bar{u}\epsilon C^{-1}(X), ||\bar{u}||_p < +\infty\}$ and $\bar{l}_q = \{\bar{g} \mid \bar{g}\epsilon C^{-1}(X), ||\bar{g}||_q < +\infty\}$.

Relevant duality can be defined by the formula $< \bar{g}, \bar{u} > = \sum_{j\epsilon\mathbb{Z}} \bar{g}(x_j)\, \bar{u}(x_j) \ \forall \bar{g}\,\epsilon\, \bar{l}_q \quad \forall \bar{u}\,\epsilon\, \bar{l}_p$.

It is easy to check that $\omega_{(x)}(t)$ is a complete trajectory in $\bar{l}_p$. We introduce the notation $\omega_j(t) = \omega_{(x)}(t)$ for $x\epsilon[x_j, x_{j+1})$. From (51) we get

$$\omega_j(t) = \begin{cases} \dfrac{1}{x_{j+1} - x_j} & for\ t\epsilon[x_j, x_{j+1}) \\ 0 & otherwise \end{cases} \qquad (52)$$

By definition we put $\Omega = \bar{l}_p$, $\Omega^* = \mathcal{L}_q(\mathcal{M})$.
For $c\epsilon\Omega^*$ we have

$$(< c, \mathrm{v} >, \omega(t))_\Omega =$$
$$= \int_{x\epsilon\mathcal{M}} < c_{(x)}, \mathrm{v} > \omega_{(x)}(t) dx \ \forall \mathrm{v}\epsilon K. \quad (53)$$

By (53) we reduce

$$(< c, \mathrm{v} >, \omega(t))_\Omega = \int_\alpha^\beta < c_{(x)}, \mathrm{v} > \omega_{(x)}(t)\, dx =$$
$$= \sum_{j\epsilon\mathbb{Z}} \int_{x_j}^{x_{j+1}} < c_{(x)}, \mathrm{v} > \omega_{(x)}(t)\, dx =$$
$$= \sum_{j\epsilon\mathbb{Z}} < c_j, \mathrm{v} > \omega_j(t) \quad \forall \mathrm{v}\epsilon K, \qquad (54)$$

where

$$< c_j, \mathrm{v} >= \int_{x_j}^{x_{j+1}} < c_{(x)}, \mathrm{v} > dx, c\epsilon\mathcal{L}_q(\mathcal{M}). \quad (55)$$

The convergence of the series and integrals appearing here is obvious.

Consider the linear space $\mho$ of trajectories in the distribution space K' by setting

$$\mho = \{\upsilon \mid \upsilon = \upsilon(t) = (c, \omega(t))_\Omega \quad \forall c\epsilon\Omega^*\}. \qquad (56)$$

In view of formulas (54) -- (56) we have

$$\mho = \{\upsilon \mid \upsilon = \upsilon(t), < \upsilon(t), \mathrm{v} >=$$
$$= \sum_{j\epsilon\mathbb{Z}} < c_j, \mathrm{v} > \omega_j(t) \quad \forall c_j\epsilon K'\}. \qquad (57)$$

Denote by S the linear space of locally summable functions,

$$S = \{w \mid w = \mathrm{w(t)} = \sum_{j\epsilon\mathbb{Z}} \ \vec{c}_j\ \omega_j(t) \ \ \forall \vec{c}\epsilon l_q\}. \qquad (58)$$

Let us assume that the condition

$$sup_{j\epsilon\mathbb{Z}}(x_{j+1} - x_j) = M < +\infty \qquad (59)$$

is right. For a finite interval (α, β) condition (59) is always satisfied.

Theorem 6. *Under condition (59) family* $\{\Phi_v\}_{v\epsilon K}$ *linear homomorphisms of the space* $\mho$ *into the space S exists.*

Proof. By Holder's inequality, we have

$$|< c_j, \mathrm{v} >|^q \le$$
$$(x_{j+1} - x_j)^{q/p} \int_{x_j}^{x_{j+1}} |< c_{(x)}, \mathrm{v} >|^q dx.$$

Hence we have

$$\sum_{j\epsilon\mathbb{Z}} |< c_j, \mathrm{v} >|^q \le M^{q/p} \int_\alpha^\beta |< c_{(x)}, \mathrm{v} >|^q dx.$$

In view of the condition $c\epsilon\mathcal{L}_q(\mathcal{M})$ we can see the sum $\sum_{j\epsilon\mathbb{Z}} |< c_j, \mathrm{v} >|^q$ is finite. So we have

$$c(v) = \{< c_j, \mathrm{v} >\}\epsilon l_q. \qquad (61)$$

Consider the mapping Φ_v of the space $\mho$ into the space S by matching the element $\upsilon\,\epsilon\,\mho$, $< \upsilon(t), \mathrm{v} >= \sum_{j\epsilon\mathbb{Z}} < c_j, \mathrm{v} > \omega_j(t)$ an element

$$w = w(t) = \sum_{j\epsilon\mathbb{Z}} \vec{c}_j \omega_j(t), \qquad (62)$$

where for a fixed $\mathrm{v}\epsilon K$ the expressions $\vec{c}_j =< c_j, v >$ are numbers. From relations (57), (59) and (61) - (62) follows the implication $w\epsilon S$. The linearity of the mapping Φ_v is obvious. This completes the proof.

7 Embedding of the Haar Type Spaces

Next, we assume that

$$\boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{M}} = (\alpha, \beta) . \qquad (63)$$

Consider the function $U(x, y)$ given by the formula

$$U(x,y) = \begin{cases} 1 & if\ x, y \epsilon (\tilde{x}_s, \tilde{x}_{s+1}) \\ 0 & othewise, \end{cases} \qquad (64)$$

where

$$(\tilde{x}_s, \tilde{x}_{s+1}) = \\ = \begin{cases} (x_{2s}, x_{2s+2}) & for\ s \geq 0 \\ (x_{-s(s-1)/2}, x_{-s(s+1)/2}) & for\ s < 0. \end{cases} \qquad (65)$$

We define the linear operator Ƥ by the relations

$$\text{Ƥ}: \Omega \ \rightarrow \ \widetilde{\Omega} \ <=> \forall w \epsilon \Omega \ \ \widetilde{w} = \text{Ƥ}\mathrm{w}, \ \ \widetilde{w} \epsilon \widetilde{\Omega} \ <=> \\ <=> \ \widetilde{w}(y) = \int_{x \epsilon \boldsymbol{\mathcal{M}}} U(x,y) \boldsymbol{w}(x)\, dx. \qquad (66)$$

By (63) -- (66) we get

$$\widetilde{w}(y) = \int_{\tilde{x}_s}^{\tilde{x}_{s+1}} w(x) dx \ \ \forall y \epsilon (\tilde{x}_s, \tilde{x}_{s+1}). \qquad (67)$$

We substitute the function $w(x) = \omega_{(x)}(t)$ (see formula (51)) in (67). We put $\widetilde{w}(y) = \widetilde{\omega}_{(y)}(t)$, $y \epsilon (\tilde{x}_s, \tilde{x}_{s+1})$. For $s \geq 0$ we have

$$\widetilde{\omega}_{(y)}(t) = \int_{x_{2s}}^{x_{2s+1}} \omega_{(x)}(t) dx + \\ + \int_{x_{2s+1}}^{x_{2s+2}} \omega_{(x)}(t) dx \quad \forall y \epsilon (x_{2s}, x_{2s+2}). \, (68)$$

If $y \epsilon [x_{2s}, x_{2s+2})$, then in the first integral values x and t are in the same the same grid interval, namely, in the interval $[x_{2s}, x_{2s+1})$. According to formula (51) in this integral the integrand is equal to $(x_{2s+1} - x_{2s})^{-1}$. The second integral (68) under these conditions is equal to zero. Thus,

$$\widetilde{\omega}_{(y)}(t) = 1 \ \ \forall y \epsilon [x_{2s}, x_{2s+2}) \ \forall t \ \epsilon (x_{2s}, x_{2s+1}). \, (69)$$

Similarly, we find

$$\widetilde{\omega}_{(y)}(t) = 1 \forall y \epsilon [x_{2s}, x_{2s+2}), t \ \epsilon (x_{2s+1}, x_{2s+2}). \, (70)$$

It is easy to see that

$$\widetilde{\omega}_{(y)}(t) = 0 \ \ \forall t \ \epsilon (\alpha, \beta) \backslash [x_{2s}, x_{2s+1}). \qquad (71)$$

From (69) -- (71) it follows that for $s \geq 0$ function value $\widetilde{\omega}_{(y)}(t)$ on the interval $[\tilde{x}_s, \tilde{x}_{s+1})$ is unit, and outside this interval its value is zero.

For s< 0 we have

$$\widetilde{\omega}_{(y)}(t) = \sum_{j=-s(s-1)/2}^{-s(s+1)/2-1} \int_{x_j}^{x_{j+1}} \omega_{(x)}(t) dx \qquad (72) \\ \forall y \epsilon (x_{2s}, x_{2s+2}).$$

If in (72) the variable t is in the interval$[x_i, x_{i+1})$, then the integral of functions $\omega_{(x)}(t)$ over the mentioned interval is left in the last sum. In view of formula (51), the result of integration turns out to be equal to one. Thus, throughout the entire interval $[x_s, x_{s+1})$ function $\widetilde{\omega}_{(y)}(t)$ is equal to unit, and outside this interval it is equal to zero,

$$\widetilde{\omega}_{(y)}(t) = \begin{cases} 1\, for \ \ y, t \epsilon [\tilde{x}_s, \tilde{x}_{s+1}) \\ \quad 0\ otherwise. \end{cases} \qquad (73)$$

Introducing the notation $\widetilde{\omega}_s(t) = \widetilde{\omega}_{(y)}(t)$ for $y \epsilon [\tilde{x}_s, \tilde{x}_{s+1})$, from (51) and (73) we obtain the calibration relations

$$\widetilde{\omega}_s(t) = \sum_{[x_j, x_{j+1}) \subset [\tilde{x}_s, \tilde{x}_{s+1})} (x_{j+1} - x_j)\, \omega_j(t) \quad (74) \\ \forall s \epsilon \mathbb{Z} \quad \forall t \epsilon (\alpha, \beta).$$

Consider the linear space $\widetilde{\text{Ʊ}}$ trajectories in the distribution space K' by setting

$$\widetilde{\text{Ʊ}} = \{ \tilde{\text{ʊ}} \mid \tilde{\text{ʊ}} = \tilde{\text{ʊ}}(\mathrm{t}) = \left(\tilde{a}, \widetilde{\omega}(t)\right)_{\widetilde{\Omega}} \ \forall \tilde{a} \epsilon \ \widetilde{\Omega}^* \ \}. \quad (75)$$

By definition

$$(<\tilde{a}, v>, \widetilde{\omega}(t))_{\widetilde{\Omega}} = \int_{y \epsilon \boldsymbol{\mathcal{K}}} < \tilde{a}_{(y)}, \mathrm{v} > \widetilde{\omega}_{(y)}(t) dy \\ \forall \epsilon K. \qquad (76)$$

From (73) and (76) we have

$$(<\tilde{a}, \mathrm{v}>, \widetilde{\omega}(t))_{\widetilde{\Omega}} = \\ = \sum_{j \epsilon \mathbb{Z}} \int_{\tilde{x}_j}^{\tilde{x}_{j+1}} < \tilde{a}_{(y)}, \mathrm{v} > \widetilde{\omega}_{(y)}(t)\, dy = \\ = \sum_{j \epsilon \mathbb{Z}} < \tilde{a}_j, \mathrm{v} > \widetilde{\omega}_j(t), \qquad (77)$$

where

$$< \tilde{a}_j, \mathrm{v} >= \int_{\tilde{x}_j}^{\tilde{x}_{j+1}} < \tilde{a}_{(y)}, \mathrm{v} > dy \ \ \forall v \epsilon K.$$

The convergence of the series and integrals appearing in (77) is obvious.

Thus, the space (75) can be represented in the form

$$\widetilde{\text{Ʊ}} = \{ \tilde{\text{ʊ}} \mid \tilde{\text{ʊ}} = \sum_{j \epsilon \mathbb{Z}} \tilde{a}_j \widetilde{\omega}_j \quad \forall \tilde{a}_j \epsilon K' \ \} \qquad (78)$$

Let $\tilde{S}$ be the linear space of locally summable functions,

$$\tilde{S} = \{ \widetilde{w} \mid \widetilde{w} = \widetilde{w}(t) = \textstyle\sum_{j \epsilon \mathbb{Z}} \bar{a}_j \widetilde{\omega}_j \quad \forall \bar{a} \epsilon l_q \ \}, \quad (79)$$

where $\bar{a} = \{ \bar{a}_j \}_{j \epsilon \mathbb{Z}}$, $\bar{a}_j \epsilon \mathbb{R}^1$.

Let us assume that the condition

$$\sup_{j \epsilon \mathbb{Z}} \left(\tilde{x}_{j+1} - \tilde{x}_j \right) = \widetilde{M} <+\infty \qquad (80)$$

are right.

Theorem 7. *Under condition (80) the following statements are true:*

1. Linear spaces $\widetilde{\text{Ʊ}}$ *and* $\tilde{S}$ *are subspaces of the spaces* Ʊ *and* S *respectively.*

2. Under condition (80) there exists a family $\{\widetilde{\Phi}_v\}_{v \epsilon K}$ *linear homomorphisms of the space* $\widetilde{\text{Ʊ}}$ *into the space* $\tilde{S}$*.*

Proof. The first assertion follows from calibration relation (74). The proof of the second assertion is carried out similarly to the proof of Theorem 6.

8 Projection onto Embedded Space

In the space S, consider a new coordinate system ϖ_j, obtained from system (52) by multiplying the coordinate functions ω_j into constants $(x_{j+1}-x_j)$. We have

$$\varpi_j=(x_{j+1}-x_j)\omega_j,$$

$$\varpi_j(t)=\begin{cases}1 & for\ \ t\epsilon[x_j,x_{j+1}),\\ 0 & for\ \ t\epsilon(\alpha,\beta)[x_j,x_{j+1}).\end{cases} \qquad (81)$$

Calibration relation (74) takes the form

$$\widetilde{\omega}_s(t)=\sum_{j\epsilon\mathbb{Z}}p_{sj}\,\varpi_j(t)\ \ \forall s\epsilon\mathbb{Z}\ \ \forall t\epsilon(\alpha,\beta), \qquad (82)$$

where

$$p_{sj}=\begin{cases}1 & for\ (x_j,x_{j+1})\subset(\widetilde{x}_s,\widetilde{x}_{s+1})\\ 0 & for\ (x_j,x_{j+1})\cap(\widetilde{x}_s,\widetilde{x}_{s+1})=\emptyset.\end{cases} \qquad (83)$$

In the space $\tilde{S}$, consider the functionals g_i and $\tilde{g}_i$ given by the formulas

$$\ll g_i,u\gg=u(x_i),\qquad \ll \widetilde{g}_i,u\gg=u(\widetilde{x}_i)\quad \forall u\epsilon S. \ (84)$$

The properties of biorthogonality are easily verified

$$\ll g_i,\varpi_j\gg=\delta_{ij},\ \ \ll \widetilde{g}_i,\widetilde{\omega}_j\gg=\delta_{ij}\quad \forall i,j\epsilon\mathbb{Z}, \ (85)$$

where δ_{ij} is the Kronecker symbol.

Projection of $\bar{P}_0$ from S onto $\tilde{S}$ define by functionals (84) -- (85),

$$\bar{P}_0u=\sum_{i\epsilon\mathbb{Z}}\ll \widetilde{g}_i,u\gg\widetilde{\omega}_i\qquad \forall u\,\epsilon S\,. \qquad (86)$$

Let $\bar{Q}_0$ be the operator $I-\bar{P}_0$.

Operation (86) derives a wavelet decomposition of space S,

$$S=\widetilde{S}+W. \qquad (87)$$

Consider an element $u\epsilon S$ in basis $\{\varpi_j\}_{i\epsilon\mathbb{Z}}$ of the space S,

$$u=\sum_{i\epsilon\mathbb{Z}}\bar{c}_i\,\varpi_i \qquad (88)$$

$$\bar{c}_i=\ \ll g_i,u\gg. \qquad (89)$$

Let's be known the coefficients $\bar{a}_i$ and $\bar{b}_i$ of projections for the element u onto spaces $\tilde{S}$ and W in (87),

$$\bar{P}_0\mathrm{u}=\sum_{i\epsilon\mathbb{Z}}\bar{a}_i\,\widetilde{\omega}_i,\quad \bar{Q}_0\mathrm{u}=\sum_{i'\epsilon\mathbb{Z}}\bar{b}_{i'}\,\varpi_{i'}, \qquad (90)$$

where $\bar{a}_i=\ll \widetilde{g}_i,u\gg$, $\bar{b}_{i'}=\ll g_{i'},\bar{Q}_0u\gg$.

Let's express the numbers $\bar{c}_j$ through the numbers $\bar{a}_i$ and $\bar{b}_{i'}$. According to the formulas (82) and (90) we have the representation

$$u=\sum_{i\epsilon\mathbb{Z}}\bar{a}_i\,\widetilde{\omega}_i+\sum_{i'\epsilon\mathbb{Z}}\bar{b}_{i'}=\sum_{i'\epsilon\mathbb{Z}}\left(\sum_{i\epsilon\mathbb{Z}}\bar{a}_i\,p_{i,i'}+\bar{b}_{i'}\right)\varpi_{i'}. \qquad (91)$$

On the other hand, representation (88) -- (89) is valid. Equating the right parts of representations (88) and (91) taking into account the linear independence of the coordinate splines $\{\varpi_i\}_{i\epsilon\mathbb{Z}}$ leads to ratios

$$\bar{c}_j=\sum_{i\epsilon\mathbb{Z}}\bar{a}_i\,p_{i,j}+\bar{b}_j\quad \forall j\epsilon\mathbb{Z}. \qquad (92)$$

Relations (92) are *formulas of reconstruction*.

We introduce vectors

$\bar{\boldsymbol{a}}=(\dots,\bar{a}_{-2},\bar{a}_{-1},\bar{a}_0,\bar{a}_1,\bar{a}_2,\dots)$,
$\bar{\boldsymbol{b}}=(\dots,\bar{b}_{-2},\bar{b}_{-1},\bar{b}_0,\bar{b}_1,\bar{b}_2,\dots)$,
$\bar{\boldsymbol{c}}=(\dots,\bar{c}_{-2},\bar{c}_{-1},\bar{c}_0,\bar{c}_1,\bar{c}_2,\dots)$,

as well as the matrix $\mathrm{P}=(p_{sj})_{s,j\epsilon\mathbb{Z}}$, whose elements p_{sj} are given by formula (83).

The introduced notation allows us to write the formulas reconstruction (92) in the form

$$\bar{\boldsymbol{c}}=\mathrm{P}^T\bar{\boldsymbol{a}}+\bar{\boldsymbol{b}}. \qquad (93)$$

The vector $\bar{\boldsymbol{c}}$ is the *initial flow*, $\bar{\boldsymbol{a}}$ is *main flow*, and the vector $\bar{\boldsymbol{b}}$ is *wavelet flow*.

Consider $u\epsilon S$, $u=\sum_{s\epsilon\mathbb{Z}}\bar{c}_s\,\varpi_s$. Using equalities

$$\bar{a}_i=\ll \widetilde{g}_i,u\gg,$$

we have

$$\bar{a}_i=\sum_{s\epsilon\mathbb{Z}}\bar{c}_s\ll \widetilde{g}_i,\varpi_s\gg.$$

From (92) we successively derive

$$\bar{b}_j=\bar{c}_j-\sum_{i\epsilon\mathbb{Z}}p_{i,j}\sum_{s\epsilon\mathbb{Z}}\bar{c}_s\ll \widetilde{g}_i,\varpi_s\gg.$$

Using the notation

$$q_{i,s}=\ll \widetilde{g}_i,\varpi_s\gg, \qquad (94)$$

we get

$$\bar{a}_i=\sum_{s\epsilon\mathbb{Z}}q_{i,s}\bar{c}_s, \qquad (95)$$

$$\bar{b}_j=\bar{c}_j-\sum_{s\epsilon\mathbb{Z}}\bar{c}_s\sum_{i\epsilon\mathbb{Z}}p_{i,j}\,q_{i,s}. \qquad (96)$$

Formulas (95) -- (96) are called *formulas decomposition*.

Introducing the matrix $\boldsymbol{Q}=(q_{i,s})_{i,s\epsilon\mathbb{Z}}$ we rewrite the decomposition formulas in the form

$$\bar{\boldsymbol{a}}=\boldsymbol{Q}\bar{\boldsymbol{c}},\qquad \bar{\boldsymbol{b}}=\bar{\boldsymbol{c}}-\mathrm{P}^T\boldsymbol{Q}\bar{\boldsymbol{c}}. \qquad (97)$$

Note that the constants $p_{s,j}$ are calculated from formulas (65) and (83), while the numbers $q_{i,s}$ are determined by formulas (85), (94).

Referring to the projection of the space $\mho$ on $\widetilde{\mho}$, note that their structure is similar the structure of the spaces S and $\tilde{S}$, respectively.

According to formulas (57) and (58) we have

$$\mho=\{\upsilon\ |\upsilon=\upsilon(t),\upsilon(t)=\sum_{j\epsilon\mathbb{Z}}c_j\omega_j(t)\},$$

$$S=\{w\ |\ w=\mathrm{w}(t)=\sum_{j\epsilon\mathbb{Z}}\ \vec{c}_j\ \omega_j(t)\ \}.$$

In the same way, the structures of the spaces (78) and (79) are similar,

$$\widetilde{\mho}=\{\tilde{\upsilon}\ |\ \tilde{\upsilon}=\ \sum_{j\epsilon\mathbb{Z}}\ \tilde{a}_j\widetilde{\omega}_j\},$$

$$\tilde{S}=\{\widetilde{w}\ |\ \widetilde{w}=\widetilde{w}(t)=\sum_{j\epsilon\mathbb{Z}}\ \bar{a}_j\widetilde{\omega}_j\}.$$

To construct a wavelet expansion in the case of spaces $\mho$ and $\widetilde{\mho}$ we just use obtained formulas for the mentioned decomposition of the spaces S and $\tilde{S}$.

For this purpose, we introduce the vectors

$\tilde{\boldsymbol{a}} = (\ldots, \tilde{a}_{-2}, \tilde{a}_{-1}, \tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \ldots)$,
$\boldsymbol{b} = (\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots)$,
$\boldsymbol{c} = (\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots)$,
whose components are elements of the space K'.

We assume that $\tilde{\boldsymbol{a}} \epsilon \widetilde{\mho}$, and $\boldsymbol{b}, \boldsymbol{c} \epsilon \mho$.

Theorem 8. *The wavelet expansion for spaces* $\widetilde{\mho}$⍰$\mho$ *consists of decomposition formulas*

$$\tilde{\boldsymbol{a}} = \boldsymbol{Q}\boldsymbol{c}, \qquad \boldsymbol{b} = \boldsymbol{c} - \mathrm{P}^T \boldsymbol{Q} \bar{\boldsymbol{c}},$$

and reconstruction formulas

$$\boldsymbol{c} = \mathrm{P}^T \tilde{\boldsymbol{a}} + \boldsymbol{b}.$$

The element $\boldsymbol{c}$ is called *the initial distribution flow*, element $\tilde{\boldsymbol{a}}$ is called *basic distribution flow*, and $\boldsymbol{b}$ is called *wavelet flow of distributions*.

9 Discussion

In this paper, we first consider the space K of basic functions (i.e., infinitely differentiable compactly supported functions) and the space K' of distributions.

Then we discuss the set $\mathcal{L}_q(\boldsymbol{\mathcal{M}})$ of trajectories in the K' whose action on any basic function lies in the space $L_q(\boldsymbol{\mathcal{M}})$, $q \epsilon (1, +\infty)$. It is proved that $\mathcal{L}_q(\boldsymbol{\mathcal{M}})$ is not empty, and it is a linear space. For example, a trajectory of the distributions δ_x, $x \epsilon \boldsymbol{\mathcal{M}}$, belongs to $\mathcal{L}_q(\boldsymbol{\mathcal{M}})$.

In addition, we consider abstract functions of real variable $t \epsilon T$ with values in the space $L_p(\boldsymbol{\mathcal{M}})$, $p = (1 - q^{-1})^{-1}$.

The notion of *a complete abstract function* in $L_p(\boldsymbol{\mathcal{M}})$ is introduced. Let $\omega_{(x)}(t)$ be a complete abstract function in $L_p(\boldsymbol{\mathcal{M}})$. An example of a complete function is $\omega_{(x)}(t) = x^t$ for $\boldsymbol{\mathcal{M}}$=(1,2), $t \epsilon T = \{0,1,2, \ldots\}$. We consider a space S of distribution trajectories which are generated by the function $\omega_{(x)}(t)$. The space $\tilde{S}$, embedded in the space S is constructed similarly. A projection operation P_0 of the space S onto $\tilde{S}$ generates wavelet decomposition of the space S. From here decomposition and reconstruction formulas for distribution flows are obtained. These formulas can be used for wavelet decompositions of the information flows coming from analog devices. This approach is preferable to situations where the analog flow turns into a discrete numerical flow with the help of quantization and digitization. The point is that quantization and digitization lead to significant loss of information and to distortion. Therefore, it is preferable to carry out the wavelet decomposition of the original analog flow. However in certain cases it is required to process discrete flows of distributions. This is not difficult to achieve using special generating function options. One of these options is also presented in this work. In the case under consideration, we arrive at the Haar-type coordinate functions. As a result we obtain the decomposition and reconstruction formulas corresponding to this case for discrete flow of distributions.

An important issue is the adaptive choice of embedded space and projection operation for it. In the case of ordinary functions, this problem is solved by the appropriate consolidation of the initial divisions (see [15] – [17]). Such a choice can be made by the use of local approximation properties for functions in one or another metric space. For distribution flows, similar issues are to be considered in the future.

10 Conclusion

In this paper, a new approach to the construction of wavelet expansions is considered. This approach allows us to consider discrete and continuous flows of distributions. Such flows arise in many physical problems. As it is known, point actions considered in theory, in fact actually do not exist. In this regard, the use of distributions is more natural, since such an approach reflects the idea of a trial function. Mentioned flows of distributions can be continuous or discrete. In that and in another case, their wavelet decomposition is important, allowing the more efficient use of computer and communication resources. This work shows the possibility of wavelet decomposition of both continuous and discrete flow of distributions. In the future, it is planned to use the distributions to study the spaces of dipoles and the conditions for their embedding. We suppose to obtain a wavelet decomposition of the dipole spaces.

References:

[1] Mozhde Firoozi Pouyani, Mansour Vali, MohammadAmin Ghasemi, Lung sound signal denoising using discrete wavelet transform and artificial neural network, Biomedical Signal Processing and Control. Vol. 72, 2022, article id.103329.

[2] Fakharian P., Naderpour, H. Damage Severity Quantification Using Wavelet Packet Transform and Peak Picking Method, Practice Periodical on Structural Design and Construction., Vol. 27(1), 2022, article id. 04021063.

[3] Ramalingappa, L., Manjunatha, A. Power quality event classification using complex wavelets phasor models and customized convolution neural network, International Journal of Electrical and Computer Engineering.Vol. 12(1), 2022, pp. 22-31.

[4] Daubechies I. Ten Lectures on Wavelets, CBMS-NSR, Series in Appl. Math.-SIAM,1992.
[5] Chui C.K. An Introduction to Wavelets.-N.Y.: Academic Press, 1992.
[6] Stephane Mallat. A Wavelet Tour of Signal Processing.-N.Y.: Academic Press, 1999.
[7] Yu.N. Subbotin, N. I. Chernykh, Interpolation wavelets in boundary value problems, Proc. Steklov Inst. Math. (Suppl.). Vol. 300, 2018, pp. 172-183.
[8] Novikov I.Ya.,Protasov V.Yu., Skopina M.A., Wavelet Theory. AMS, Translations Mathematical Monographs. V. 239 (2011).
[9] Boris Shumilov. Shifted Cubic Spline Wavelets with Two Vanishing Moments on the Interval and a Splitting Algorithm, WSEAS Transactions on Systems, Vol. 19, 2020, pp.149-158.
[10] Boris Shumilov. On Five-Diagonal Splitting for Cubic Spline Wavelets with Six Vanishing Moments on a Segment, WSEAS Transactions on Information Science and Applications. Vol. 17, 2020, pp. 156-165.
[11] Ana Benavente, Ole Christensen, Marzieh Hasannasab, Hong Oh Kim, Rea Young Kim, Federico D.Kovac, Approximately dual pairs of wavelet frames, Journal of Mathematical Analysis and Applications. Vol. 507(2), 2022, article id.125841.
[12] Abdullah, M.K., Kamyabi-Gol, R.A., On wavelet multiplier and Landau-Pollak-Slepian operators on L2(G, H), Journal of Pseudo-Differential Operators and Applications. Vol. 10(2), 2019, pp.257-267.
[13] Kaur, N., Goyal, K. An adaptive wavelet optimized finite difference B-spline polynomial chaos method for random partial differential equations, Applied Mathematics and Computation.Vol.415, 2022, article id.126738.
[14] Huang, C.-S., O'Hara, J.G., Mataramvura, S. Highly efficient Shannon wavelet-based pricing of power options under the double exponential jump framework with stochastic jump intensity and volatility, Applied Mathematics and Computation, Vol.414, February 2022, article id. 126669.
[15] Yuri K. Dem'yanovich, Tatjana A. Safonova,Andrey N. Terekhov, Approximation Functionals and Their Application, WSEAS Transactions on Mathematics. Vol. 20, 2021, pp. 489-495.
[16] Yuri K. Dem'yanovich. Approximation on manifold, WSEAS Transactions on Mathematics, 20, 2021, pp. 62-73.
[17] Yuri K. Dem'yanovich. Adaptive Haar Type Wavelets on Manifolds, Journal of Mathematical Sciences (United States), 251, 6, December 28,2020, pp.797-813.
[18] Ahmad, M.A., Elloumi, M., Samak, A.H., (...), Kaid, M.A., Iliopoulos, C. Hiding patients' medical reports using an enhanced wavelet steganography algorithm in DICOM images, Alexandria Engineering Journal, 61(12), 2022, ñ. 10577-10592.
[19] Ganjalizadeh, V., Meena, G.G., Wall, T.A., Hawkins, A.R., Schmidt, H. Fast custom wavelet analysis technique for single molecule detection and identification, Nature Communications, 13(1), 2022, 1035.
[20] Li, L., Zhang, B. Nonlinear wavelet-based estimation to spectral density for stationary non-Gaussian linear processes, Applied and Computational Harmonic Analysis, 60, 2022, ñ. 176-204.
[21] Ashpazzadeh, E., Chu, Y.-M., Hashemi, M.S., Moharrami, M., Inc, M. Hermite multiwavelets representation for the sparse solution of nonlinear Abel's integral equation, Applied Mathematics and Computation, 427, 2022,127-171.
[22] Behera, S., Saha Ray, S.Two-dimensional wavelets scheme for numerical solutions of linear and nonlinear Volterra integro-differential equations, Mathematics and Computers in Simulation, 198, 2022, ñ. 332-358.
[23] Wang, G., Guo, S., Han, L., (...), Song, X., Zhao, Z. Asymptomatic COVID-19 CT image denoising method based on wavelet transform combined with improved PSO, Biomedical Signal Processing and Control, 76, 2022 ,103-707.
[24] Vlasic, T., Sersic, D. Single-pixel compressive imaging in shift-invariant spaces via exact wavelet frames, Signal Processing: Image Communication, 105, 2022,116-702.
[25] Zhengmao Ye, Hang Yin, Yongmao Ye. Case Studies on Discrete Wavelet Denoising via Kernel based Nonlinear Component, Analysis, WSEAS Transactions on Computer Research, Volume 9, 2021.
[26] Seda Postalcioglu. Wavelet Transform Based Feature Extraction for EEG Signal Classification, WSEAS Transactions on Computers, Volume 20, 2021.
[27] Meryem Felja, Asmae Bencheqroune, Mohammed Karim, Ghita Bennis. Removing Artifacts From EEG Signal Using Wavelet Transform and Conventional Filters, WSEAS Transactions on Information Science and Applications, Volume 17, 2020.