Coincidence Point Results in Hausdorff Rectangular Metric Spaces with an Application to Lebesgue Integral Function

[1]CHUANPIT MUNGKALA, [1]ANANTACHAI PADCHAROEN, [2]PAKEETA SUKPRASERT

[1]Department of Mathematics, Faculty of Science and Technology
Rambhai Barni Rajabhat University, Chanthaburi 22000, THAILAND
[2]Department of Mathematics and Computer Science
Faculty of Science and Technology
Rajamangala University of Technology Thanyaburi (RMUTT)
Thanyaburi, Pathumthani 12110, THAILAND
[1]chuanpit.t@rbru.ac.th, [2]pakeeta_s@rmutt.ac.th

Abstract: -In this paper, we were able to produce certain coincidence point results for g-nondecreasing self-mappings fulfilling certain rational type contractions in a Hausdorff rectangular metric space utilizing $\mathcal{C}$-functions and generalized (ϑ, φ)-contractive mappings obeying an admissibility-type assumption.

Key-Words: Rational contractions, Rectangular metric space, Lebesgue integral function

Received: September 21, 2021. Revised: May 17, 2022. Accepted: June 20, 2022. Published: July 14, 2022.

1 Introduction

The Banach contraction argument [1] is a significant development in fixed point theory. In a variety of ways, it has already been generalized and expanded. In the literature, we encountered several innovative types of metric spaces, such as the one proposed by Branciari [2] and demonstrated an analogue of the Banach contraction principle in a rectangle metric space by replacing the triangle inequality with a weaker hypothesis termed quadrilateral inequality. Many authors then investigated fixed point outcomes in these spaces. More information on fixed point theorems in rectangular metric space is available here, see [3, 4, 5]

Samet et al. [6] developed the concept of α-ψ-contractive mapping in 2012, which is important because, unlike the Banach contraction principle, it does not require the contractive requirements to hold for every pair of points in the domain. It also takes into account the scenario of discontinuous mappings. As a result of these factors, there has been a tremendous increase in the literature dealing with fixed point problems using admissible mappings (see in [7, 8, 9]).

Most recently, two different generalizations of admissible mapping were given in which the author Ansari [7] used the idea of $\mathcal{C}$-class functions, whereas Budhia et al. [11] used a rectangular metric. In this paper, we prove coincidence and common fixed point theorems for two mappings in complete Hausdorff generalized metric spaces that meet a generalized (ϑ, ψ)-weakly contractive condition. Many known results in the literature are extended and generalized by the presented theorems.

2 Preliminaries

Definition 2.1. [2] *Let $\Phi \neq \emptyset$ be a set. A generalized metric (rectangular metric) is a function $\mu : \Phi \times \Phi \to [0, \infty)$, where the following conditions are fulfilled for all $v, \lambda, \alpha, \delta \in \Phi$ with $\alpha \neq \delta$ and $\alpha, \delta \notin \{v, \lambda\}$:*

(i) $\mu(v, \lambda) = 0$ *if and only if* $v = \lambda$;

(ii) $\mu(v, \lambda) = \mu(\lambda, v)$;

(iii) $\mu(v, \lambda) \leq \mu(v, \alpha) + \mu(\alpha, \delta) + \mu(\delta, \lambda)$ *(quadrilateral inequality).*

The pair (Φ, μ) is named as a generalized metric space (a rectangular metric space)

Definition 2.2. [2] *Let (Φ, μ) be a rectangular metric space, and let $\{v_n\}$ be a sequence in Φ.*

(i) *If $(v_n, v) \to 0$ as $n \to \infty$, then $\{v_n\}$ is called rectangular metric space convergent to a limit v.*

(ii) *If for every $\epsilon > 0$, there exists $n(\epsilon) \in \mathbb{N}$ such that $\mu(v_i, v_j) < \epsilon$ for all $i > j > n(\epsilon)$, then $\{v_n\}$ is called a rectangular metric space Cauchy sequence in rectangular metric space.*

(iii) *A rectangular metric space (Φ, μ) is called complete if every rectangular metric space Cauchy sequence is rectangular metric space convergent.*

Definition 2.3. [7] *A $\mathcal{C}$-function $\mathcal{F} : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a continuous function such that for all $v, \lambda \in [0, \infty)$:*

(i) $\mathcal{F}(v, \lambda) \leq v$;

(ii) $\mathcal{F}(v, \lambda) = v$ *implies that either* $v = 0$ *or* $\lambda = 0$.

The letter $\mathcal{C}$ will denote the class of all $\mathcal{C}$-functions.

Definition 2.4. [8] *Let $\omega, \kappa : \Phi \times \Phi \to [0, \infty)$ be two mappings. A map $\mathcal{M} : \Phi \to \Phi$ is said to be ω-admissible with respect to κ if $\omega(\mathcal{M}v, \mathcal{M}\lambda) > \kappa(\mathcal{M}v, \mathcal{M}\lambda)$ whenever $\omega(v, \lambda) > \kappa(v, \lambda)$ for all $v, \lambda \in \Phi$. If $\kappa(v, \lambda) = 1$ for all $v, \lambda \in \Phi$, then $\mathcal{M}$ is called an ω-admissible mapping.*

Definition 2.5. [10] *A nondecreasing continuous function $\vartheta : [0, \infty) \to [0, \infty)$ is called an altering distance function if $\vartheta(t) = 0$ if and only if $t = 0$.*

We denote by Ψ the class of altering distance functions.

Lemma 2.6. [11] *Let (Φ, μ) be a complete rectangular metric space and $\{v_n\}$ be a sequence in Φ such that $lim_{n\to\infty}\mu(v_n, v_{n+1}) = 0 = \lim_{n0\infty} \mu(v_n, v_{n+2})$ and $v_n \neq v_m$ for all positive integers $n \neq m$. If $\{v_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ in $\mathbb{N}$ with $m_k > n_k > k$ with $\mu(v_{m_k}, v_{n_k}) > \epsilon, \mu(v_{m_k-1}, v_{n_k}) < \epsilon$ so that the following hold:*

- (i) $\lim_{k\to\infty} \mu(v_{m_k-1}, v_{n_k+1}) = \epsilon$;
- (ii) $\lim_{k\to\infty} \mu(v_{m_k}, v_{n_k}) = \epsilon$;
- (iii) $\lim_{k\to\infty} \mu(v_{m_k-1}, v_{n_k}) = \epsilon$;
- (iv) $\lim_{k\to\infty} \mu(v_{m_k}, v_{n_k-1}) = \epsilon$;
- (v) $\lim_{k\to\infty} \mu(v_{m_k+1}, v_{n_k+1}) = \epsilon$;

Definition 2.7. [3] *Let g and $\mathcal{M}$ be self-mappings of a nonempty set.*

- (i) *A point $\xi \in \Phi$ is said to be a common fixed point of g and $\mathcal{M}$ if $\xi = g\xi = \mathcal{M}\xi$.*
- (ii) *A point $\xi \in \Phi$ is called a coincidence point of g and $\mathcal{M}$ if $g\xi = \mathcal{M}\xi$. And if $\eta = g\xi = \mathcal{M}\xi$, then η is said to be a point of coincidence of g and $\mathcal{M}$.*
- (iii) *The mappings $g, \mathcal{M} : \Phi \to \Phi$ are said to be weakly compatible if they commute at their coincidence point that is, $g\mathcal{M}\xi = \mathcal{M}g\xi$ whenever $g\xi = \mathcal{M}\xi$.*

Lemma 2.8. [3] *Let Φ be a nonempty set. Suppose that the mappings $g, \mathcal{M} : \Phi \to \Phi$ have a unique coincidence point ϱ in Φ. If g and $\mathcal{M}$ are weakly compatible, then g and $\mathcal{M}$ have a unique common fixed point.*

3 Main Result

Theorem 3.1. *Let (Φ, μ) be a Hausdorff rectangular metric space and $\mathcal{M} : \Phi \to \Phi$ be an ω-admissible mapping with respect to κ and let $g, \mathcal{M} : \Phi \to \Phi$ be two self maps such that $\mathcal{M}(\Phi) \subseteq g(\Phi)$ and $(g\Phi, \mu)$ is a complete rectangular metric space. Suppose there exist $\mathcal{F} \in \mathcal{C}$ and $\vartheta, \psi \in \Psi$ such that, for $v, \lambda \in \Phi$,*

$$\begin{aligned}&\omega(v, \lambda) > \kappa(v, \lambda) \Rightarrow\\&\vartheta(\mu(\mathcal{M}v, \mathcal{M}\lambda)) \leq \mathcal{F}(\vartheta(\Delta(v, \lambda)), \psi(\Delta(v, \lambda))),\end{aligned} \tag{1}$$

where

$$\begin{aligned}&\Delta(v, \lambda)\\&= \max\Big\{\mu(gv, g\lambda), \frac{\mu(gv, \mathcal{M}v) + \mu(g\lambda, \mathcal{M}\lambda)}{2},\\&\frac{\mu(gv, \mathcal{M}v) + \mu(g\lambda, \mathcal{M}v)}{2}, \frac{\mu(g\lambda, \mathcal{M}\lambda)(1 + \mu(gv, \mathcal{M}v))}{1 + \mu(gv, g\lambda)},\\&\frac{\mu(gv, \mathcal{M}v)(1 + \mu(g\lambda, \mathcal{M}\lambda))}{1 + \mu(\mathcal{M}v, \mathcal{M}\lambda)}\Big\}.\end{aligned}$$

Assume that

- (i) *the pair $(g, \mathcal{M})$ is ω-admissible regarding to the function κ;*
- (ii) *there exists $v_0 \in \Phi$ so that $\omega(v_0, gv_0) \geq \kappa(v_0, gv_0)$ and $\omega(v_0, \mathcal{M}v_0) \geq \kappa(v_0, \mathcal{M}v_0)$;*
- (iii) *g and $\mathcal{M}$ are continuous.*

Then g and $\mathcal{M}$ have a unique coincidence point in Φ. Moreover, if g and $\mathcal{M}$ are weakly compatible, then g and $\mathcal{M}$ have a unique common fixed point.

Proof. First, we shall show the existence of g and $\mathcal{M}$ coincidence point. By induction we get

Consider v_0 is an arbitrary point. Since $\mathcal{M}(\Phi) \subseteq g(\Phi)$, we create two iterative sequences in Φ, $\{v_n\}$ and $\{\lambda_n\}$, as follows:

$$\lambda_n = gv_{n+1} = \mathcal{M}v_n, \quad \text{for all } n = 0, 1, 2, \ldots. \tag{2}$$

If $\lambda_k = \lambda_{k-1}$ for some $k \in \mathbb{N}$, then $gv_k = \lambda_k = \lambda_{k-1} = \mathcal{M}v_k$ and g and $\mathcal{M}$ are have a point of coincidence.

Assume additionally that $\lambda_n \neq \lambda_{n-1}$ for every $n \in \mathbb{N}$. By letting $v = v_n$, $\lambda = v_{n+1}$ into (1) and condition (2), we get

$$\omega(v_n, v_{n+1}) > \kappa(v_n, v_{n+1}) \tag{3}$$

and

$$\begin{aligned}&\vartheta(\mu(\lambda_n, \lambda_{n+1}))\\&= \vartheta(\mu(\mathcal{M}v_n, \mathcal{M}v_{n+1}))\\&\leq \mathcal{F}(\vartheta(\Delta(v_n, v_{n+1})), \psi(\Delta(v_n, v_{n+1}))),\end{aligned} \tag{4}$$

where

$$\begin{aligned}
&\Delta(v_n, v_{n+1})\\
&= \max\Big\{\mu(gv_n, gv_{n+1}),\\
&\frac{\mu(gv_n, \mathcal{M}v_n) + \mu(gv_{n+1}, \mathcal{M}v_{n+1})}{2},\\
&\frac{\mu(gv_n, \mathcal{M}v_n) + \mu(gv_{n+1}, \mathcal{M}v_n)}{2},\\
&\frac{\mu(gv_{n+1}, \mathcal{M}v_{n+1})(1 + \mu(gv_n, \mathcal{M}v_n))}{1 + \mu(gv_n, gv_{n+1})},\\
&\frac{\mu(gv_n, \mathcal{M}v_n)(1 + \mu(gv_{n+1}, \mathcal{M}v_{n+1}))}{1 + \mu(\mathcal{M}v_n, \mathcal{M}v_{n+1})}\Big\}\\
&= \max\Big\{\mu(\lambda_{n-1}, \lambda_n), \frac{\mu(\lambda_{n-1}, \lambda_n) + \mu(\lambda_n, \lambda_{n+1})}{2},\\
&\frac{\mu(\lambda_{n-1}, \lambda_n) + \mu(\lambda_n, \lambda_n)}{2},\\
&\frac{\mu(\lambda_n, \lambda_{n+1})(1 + \mu(\lambda_{n-1}, \lambda_n))}{1 + \mu(\lambda_{n-1}, \lambda_n)}\\
&\frac{\mu(\lambda_{n-1}, \lambda_n)(1 + \mu(\lambda_n, \lambda_{n+1}))}{1 + \mu(\lambda_n, \lambda_{n+1})}\Big\}\\
&= \max\{\mu(\lambda_{n-1}, \lambda_n), \mu(\lambda_n, \lambda_{n+1})\}.
\end{aligned}$$

If $\Delta(v_n, v_{n+1}) = \mu(\lambda_n, \lambda_{n+1})$ for some $n \in \mathbb{N}$, then from (4),

$$\begin{aligned}
&\vartheta(\mu(\lambda_n, \lambda_{n+1}))\\
&= \vartheta(\mu(\mathcal{M}v_n, \mathcal{M}v_{n+1}))\\
&\le \mathcal{F}(\vartheta(\mu(\lambda_n, \lambda_{n+1})), \psi(\mu(\lambda_n, \lambda_{n+1})))\\
&\le \vartheta(\mu(\lambda_n, \lambda_{n+1})).
\end{aligned} \tag{5}$$

Using Definition 2.3, $\vartheta(\mu(\lambda_n, \lambda_{n+1})) = 0$ or $\psi(\mu(\lambda_n, \lambda_{n+1})) = 0$. So $\mu(\lambda_n, \lambda_{n+1}) = 0$, which is a contradiction. Consequently, $\Delta(v_n, v_{n+1}) = \mu(\lambda_{n-1}, \lambda_n)$ for every $n \in \mathbb{N}$. From (4) we get

$$\begin{aligned}
&\vartheta(\mu(\lambda_n, \lambda_{n+1}))\\
&\le \mathcal{F}(\vartheta(\mu(\lambda_{n-1}, \lambda_n)), \psi(\mu(\lambda_{n-1}, \lambda_n)))\\
&\le \vartheta(\mu(\lambda_{n-1}, \lambda_n)).
\end{aligned} \tag{6}$$

Since ϑ is nondecreasing,

$$\mu(\lambda_n, \lambda_{n+1}) \le \mu(\lambda_{n-1}, \lambda_n). \tag{7}$$

Thus, $\{\mu(\lambda_n, \lambda_{n+1})\}$ is a nonincreasing sequence of positive real numbers, so there exists $\phi \ge 0$ such that the limit

$$\lim_{n\to\infty} \mu(\lambda_n, \lambda_{n+1}) = \phi.$$

Also,

$$\lim_{n\to\infty} \mu(\lambda_{n-1}, \lambda_n) = \phi.$$

From $\mathcal{F}, \vartheta$ and ψ are continuous, we have

$$\begin{aligned}
&\lim_{n\to\infty} \vartheta(\mu(\lambda_n, \lambda_{n+1}))\\
&\le \lim_{n\to\infty} \mathcal{F}(\vartheta(\mu(\lambda_{n-1}, \lambda_n)), \psi(\mu(\lambda_{n-1}, \lambda_n)))\\
&= \mathcal{F}(\lim_{n\to\infty} \vartheta(\mu(\lambda_{n-1}, \lambda_n)), \lim_{n\to\infty} \psi(\mu(\lambda_{n-1}, \lambda_n))).
\end{aligned}$$

Hence,

$$\vartheta(\phi) \le \mathcal{F}(\vartheta(\phi), \psi(\mu(\phi))) \le \vartheta(\phi).$$

Again, using Definition 2.3 we get $\phi = 0$, that is,

$$\lim_{n\to\infty} \mu(\lambda_n, \lambda_{n+1}) = 0.$$

Now we will show whether $\mu(\lambda_n, \lambda_{n+2}) \to 0$ as $n \to \infty$. Utilizing (7) we have

$$\begin{aligned}
&\vartheta(\mu(\lambda_n, \lambda_{n+2}))\\
&= \vartheta(\mu(\mathcal{M}v_n, \mathcal{M}v_{n+2}))\\
&\le \mathcal{F}(\vartheta(\Delta(v_n, v_{n+2})), \psi(\Delta(v_n, v_{n+2})))\\
&\le \vartheta(\Delta(v_n, v_{n+2})).
\end{aligned} \tag{8}$$

Hence,

$$\vartheta(\mu(\lambda_n, \lambda_{n+2})) \le \vartheta(\Delta(v_n, v_{n+2})).$$

Since ϑ is an altering distance, we have

$$\mu(\lambda_n, \lambda_{n+2}) \le \Delta(v_n, v_{n+2}).$$

where

$$\begin{aligned}
&\Delta(v_n, v_{n+2})\\
&= \max\Big\{\mu(gv_n, gv_{n+2}),\\
&\frac{\mu(gv_n, \mathcal{M}v_n) + \mu(gv_{n+2}, \mathcal{M}v_{n+2})}{2},\\
&\frac{\mu(gv_n, \mathcal{M}v_n) + \mu(gv_{n+2}, \mathcal{M}v_n)}{2},\\
&\frac{\mu(gv_{n+2}, \mathcal{M}v_{n+2})(1 + \mu(gv_n, \mathcal{M}v_n))}{1 + \mu(gv_n, gv_{n+2})}\\
&\frac{\mu(gv_n, \mathcal{M}v_n)(1 + \mu(gv_{n+2}, \mathcal{M}v_{n+2}))}{1 + \mu(\mathcal{M}v_n, \mathcal{M}v_{n+2})}\Big\}
\end{aligned}$$

$$= \max \Big\{ \mu(\lambda_{n-1}, \lambda_{n+1}), \frac{\mu(\lambda_{n-1}, \lambda_n) + \mu(\lambda_{n+1}, \lambda_{n+2})}{2}, \frac{\mu(\lambda_{n-1}, \lambda_n) + \mu(\lambda_{n+1}, \lambda_n)}{2}, \frac{\mu(\lambda_{n+1}, \lambda_{n+2})(1 + \mu(\lambda_{n-1}, \lambda_n))}{1 + \mu(\lambda_{n-1}, \lambda_{n+1})} \frac{\mu(\lambda_{n-1}, \lambda_n)(1 + \mu(\lambda_{n+1}, \lambda_{n+2}))}{1 + \mu(\lambda_n, \lambda_{n+2})} \Big\}$$
$$\le \max \Big\{ \mu(\lambda_{n-1}, \lambda_{n+1}), \mu(\lambda_{n-1}, \lambda_n) + \mu(\lambda_{n+1}, \lambda_{n+2}), \mu(\lambda_{n-1}, \lambda_n) + \mu(\lambda_{n+1}, \lambda_n), \mu(\lambda_{n+1}, \lambda_{n+2})(1 + \mu(\lambda_{n-1}, \lambda_n)) \mu(\lambda_{n-1}, \lambda_n)(1 + \mu(\lambda_{n+1}, \lambda_{n+2})) \Big\}.$$

That will be seen $\lim_{n\to\infty} \Delta(v_n, v_{n+2}) = \lim_{n\to\infty} \mu(\lambda_{n-1}, \lambda_{n+1})$. From 8 and taking $n \to \infty$, we obtain

$$\begin{aligned} &\vartheta(\lim_{n\to\infty} \mu(\lambda_n, \lambda_{n+2})) \\ &\le \mathcal{F}(\vartheta(\lim_{n\to\infty} \mu(\lambda_{n-1}, \lambda_{n+1})), \psi(\lim_{n\to\infty} \mu(\lambda_{n-1}, \lambda_{n+1}))) \\ &\le \vartheta(\lim_{n\to\infty} \mu(\lambda_{n-1}, \lambda_{n+1})). \end{aligned} \tag{9}$$

Hence, the sequence $\{\mu(\lambda_n, \lambda_{n+2})\}$ is non-increasing and bounded below. Therefore, the sequence $\{\mu(\lambda_n, \lambda_{n+2})\}$ converges to a number, $\varpi \ge 0$. Taking limit as $n \to \infty$ in (8), we get

$$\vartheta(\varpi) \le \mathcal{F}(\vartheta(\varpi), \psi(\mu(\varpi))) \le \vartheta(\varpi).$$

Using Definition 2.3 we get $\varpi = 0$, that is,

$$\lim_{n\to\infty} \mu(\lambda_n, \lambda_{n+2}) = 0. \tag{10}$$

Assume that $\lambda_n \ne \lambda_m$ for all $m \ne n$ and demonstrate that $\{\lambda_n\}$ is an rectangular metric spaces Cauchy sequence. If feasible, make $\{\lambda_n\}$ not a Cauchy sequence, according to Lemma 2.6, there exists $\epsilon > 0$ such that we may identify the subsequences $\{\lambda_{m_k}\}$ and $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ with $m_k > n_k > k$ such that

$$\lim_{k\to\infty} \mu(\lambda_{m_k}, \lambda_{n_k}) = \lim_{k\to\infty} \mu(\lambda_{m_k-1}, \lambda_{n_k-1}) = \epsilon. \tag{11}$$

We now substitute $v = v_{n_k}$ and $\lambda = v_{m_k}$ in (1). Consider

$$\begin{aligned} &\vartheta(\lambda_{n_k}, \lambda_{m_k}) \\ &= \vartheta(\mu(\mathcal{M}v_{n_k}, \mathcal{M}v_{m_k})) \\ &\le \mathcal{F}(\vartheta(\Delta(v_{n_k}, v_{m_k})), \psi(\Delta(v_{n_k}, v_{m_k}))), \end{aligned} \tag{12}$$

where

$$\begin{aligned} &\Delta(v_{n_k}, v_{m_k}) \\ &= \max \Big\{ \mu(gv_{n_k}, gv_{m_k}), \frac{\mu(gv_{n_k}, \mathcal{M}v_{n_k}) + \mu(gv_{m_k}, \mathcal{M}v_{m_k})}{2}, \frac{\mu(gv_{n_k}, \mathcal{M}v_{n_k}) + \mu(gv_{m_k}, \mathcal{M}v_{n_k})}{2}, \\ &\quad \frac{\mu(gv_{m_k}, \mathcal{M}v_{m_k})(1 + \mu(gv_{n_k}, \mathcal{M}v_{n_k}))}{1 + \mu(gv_{n_k}, gv_{m_k})}, \frac{\mu(gv_{n_k}, \mathcal{M}v_{n_k})(1 + \mu(gv_{m_k}, \mathcal{M}v_{m_k}))}{1 + \mu(\mathcal{M}v_{n_k}, \mathcal{M}v_{m_k})} \Big\} \\ &= \max \Big\{ \mu(\lambda_{n_k-1}, \lambda_{m_k-1}), \frac{\mu(\lambda_{n_k-1}, \lambda_{n_k}) + \mu(\lambda_{m_k-1}, \lambda_{m_k})}{2}, \frac{\mu(\lambda_{n_k-1}, \lambda_{n_k}) + \mu(\lambda_{m_k-1}, \lambda_{n_k})}{2}, \\ &\quad \frac{\mu(\lambda_{m_k-1}, \lambda_{m_k})(1 + \mu(\lambda_{n_k-1}, \lambda v_{n_k}))}{1 + \mu(\lambda_{n_k-1}, \lambda_{m_k-1})}, \frac{\mu(\lambda_{n_k-1}, \lambda_{n_k})(1 + \mu(\lambda_{m_k-1}, \lambda v_{m_k}))}{1 + \mu(\lambda_{n_k}, \lambda_{m_k})} \Big\}. \end{aligned}$$

Then

$$\lim_{k\to\infty} \Delta(v_{n_k}, v_{m_k}) = \epsilon. \tag{13}$$

From condition (2), we get

$$\omega(v_{n_k}, v_{m_k}) \ge \kappa(v_{n_k}, v_{m_k}). \tag{14}$$

Using (12) and (12), we get

$$\vartheta(\epsilon) \le \mathcal{F}(\vartheta(\epsilon), \psi(\epsilon)) \le \vartheta(\epsilon).$$

This implies that $\vartheta(\epsilon) = 0$ or $\psi(\epsilon)) = 0$, thus, $\epsilon = 0$, but this is a contradiction with the fact $\epsilon > 0$. Thus, $\{\lambda_n\}$ is a rectangular metric space Cauchy sequence. Since $(g(\Phi), \Phi)$ is rectangular metric space complete, there exists $\eta \in g(\Phi)$ such that $\lambda_n \to \eta$ as $n \to \infty$. Let $\xi \in \Phi$ be such that $g\xi = \eta$. Then

$$\lim_{n\to\infty} \lambda_n = g\xi. \tag{15}$$

We shall prove that $\mathcal{M}\xi = g\xi$. Applying the inequality (18), with $v = v_n$, and $\lambda = \xi$ we obtain

$$\omega(v_n, \xi) > \kappa(v_n, \xi).$$

This implies that

$$\begin{aligned} &\vartheta(\mu(\lambda_n, \mathcal{M}\xi)) \\ &= \vartheta(\mu(\mathcal{M}v_n, \mathcal{M}\xi)) \\ &\le \mathcal{F}(\vartheta(\Delta(v_n, \xi)), \psi(\Delta(v_n, \xi))), \end{aligned} \tag{16}$$

where

$$\begin{aligned}&\Delta(v_n,\xi)\\&=\max\Big\{\mu(gv_n,g\xi),\frac{\mu(gv_n,\mathcal{M}v_n)+\mu(g\xi,\mathcal{M}\xi)}{2},\\&\frac{\mu(gv_n,\mathcal{M}v_n)+\mu(g\xi,\mathcal{M}v_n)}{2},\\&\frac{\mu(g\xi,\mathcal{M}\xi)(1+\mu(gv_n,\mathcal{M}v_n))}{1+\mu(gv_n,g\xi)},\\&\frac{\mu(gv_n,\mathcal{M}v_n)(1+\mu(g\xi,\mathcal{M}\xi))}{1+\mu(\mathcal{M}v_n,\mathcal{M}\xi)}\Big\}\\&=\max\Big\{\mu(\lambda_{n-1},g\xi),\frac{\mu(\lambda_{n-1},\lambda_n)+\mu(g\xi,\mathcal{M}\xi)}{2},\\&\frac{\mu(\lambda_{n-1},\lambda_n)+\mu(g\xi,\lambda_n)}{2},\\&\frac{\mu(g\xi,\mathcal{M}\xi)(1+\mu(\lambda_{n-1},\lambda_n))}{1+\mu(\lambda_{n-1},g\xi)},\\&\frac{\mu(\lambda_{n-1},\lambda_n)(1+\mu(g\xi,\mathcal{M}\xi))}{1+\mu(\lambda_n,\mathcal{M}\xi)}\Big\}.\end{aligned}$$

Since Φ is Hausdorff, $\{\lambda_n\}\to\eta$ where $n\to\infty$, we deduce that

$$\lim_{n\to\infty}\Delta(v_n,\xi)=\mu(g\xi,\mathcal{M}\xi). \tag{17}$$

Taking limit as $n\to\infty$ in (16),

$$\begin{aligned}&\vartheta(\mu(g\xi,\mathcal{M}\xi))\\&\leq\mathcal{F}(\vartheta(\mu(g\xi,\mathcal{M}\xi)),\psi(\mu(g\xi,\mathcal{M}\xi)))\\&\leq\vartheta(\mu(g\xi,\mathcal{M}\xi).)\end{aligned}$$

Consequently, we get $\vartheta(\mu(g\xi,\mathcal{M}\xi)) = 0$ or $\psi(\mu(g\xi,\mathcal{M}\xi)) = 0$ hence $\mu(g\xi,\mathcal{M}\xi)) = 0$, that is, $g\xi=\mathcal{M}\xi$. Thus we proved that $\eta=g\xi=\mathcal{M}\xi$ and so η is a point of coincidence of g and $\mathcal{M}$.

Now, we show that if the coincidence point of g and $\mathcal{M}$ exists, it is unique. Let η_1 and η_2 be the g and $\mathcal{M}$ coincidence points. Thus, there exists some $v,\lambda\in\Phi$ such that $\eta_1=\mathcal{M}v=gv$ and $\eta_2=\mathcal{M}\lambda=g\lambda$. We can deduce from (1) that

$$\begin{aligned}&\omega(v,\lambda)>\kappa(v,\lambda)\Rightarrow\\&\vartheta(\mu(\eta_1,\eta_2))=\vartheta(\mu(\mathcal{M}v,\mathcal{M}\lambda))\\&\leq\mathcal{F}(\vartheta(\Delta(v,\lambda)),\psi(\Delta(v,\lambda))),\end{aligned} \tag{18}$$

where

$$\begin{aligned}&\Delta(v,\lambda)\\&=\max\Big\{\mu(gv,g\lambda),\frac{\mu(gv,\mathcal{M}v)+\mu(g\lambda,\mathcal{M}\lambda)}{2},\\&\frac{\mu(gv,\mathcal{M}v)+\mu(g\lambda,\mathcal{M}v)}{2},\\&\frac{\mu(g\lambda,\mathcal{M}\lambda)(1+\mu(gv,\mathcal{M}v))}{1+\mu(gv,g\lambda)},\\&\frac{\mu(gv,\mathcal{M}v)(1+\mu(g\lambda,\mathcal{M}\lambda))}{1+\mu(\mathcal{M}v,\mathcal{M}\lambda)}\Big\}\\&=\max\Big\{\mu(\eta_1,\eta_2),\frac{\mu(\eta_1,\eta_1)+\mu(\eta_2,\eta_2)}{2},\\&\frac{\mu(\eta_1,\eta_1)+\mu(\eta_2,\eta_1)}{2},\frac{\mu(\eta_2,\eta_2)(1+\mu(\eta_1,\eta_1))}{1+\mu(\eta_1,\eta_2)},\\&\frac{\mu(\eta_1,\eta_1)(1+\mu(\eta_2,\eta_2))}{1+\mu(\eta_1,\eta_2)}\Big\}\\&=\mu(\eta_1,\eta_2).\end{aligned}$$

As a result, we deduce that $\eta_1=\eta_2$ by (18).

Recall that g and $\mathcal{M}$ are only weakly compatible. By Lemma 2.8, the point η is the unique common fixed point of g and $\mathcal{M}$ since it is the unique coincidence point of g and $\mathcal{M}$. □

Corollary 3.2. *Let (Φ,μ) be a Hausdorff rectangular metric space and $\mathcal{M}:\Phi\to\Phi$ be an ω-admissible mapping with respect to κ and let $g,\mathcal{M}:\Phi\to\Phi$ be two self maps such that $\mathcal{M}(\Phi)\subseteq g(\Phi)$ and $(g\Phi,\mu)$ is a complete rectangular metric space. Suppose there exist $\mathcal{F}\in\mathcal{C}$ and $\vartheta,\psi\in\Psi$ such that, for $v,\lambda\in\Phi$,*

$$\begin{aligned}&\omega(v,\lambda)>\kappa(v,\lambda)\Rightarrow\\&\vartheta(\mu(\mathcal{M}v,\mathcal{M}\lambda))\leq\vartheta(\Delta(v,\lambda))-\psi(\Delta(v,\lambda)),\end{aligned}$$

where

$$\begin{aligned}&\Delta(v,\lambda)\\&=\max\Big\{\mu(gv,g\lambda),\frac{\mu(gv,\mathcal{M}v)+\mu(g\lambda,\mathcal{M}\lambda)}{2},\\&\frac{\mu(gv,\mathcal{M}v)+\mu(g\lambda,\mathcal{M}v)}{2},\frac{\mu(g\lambda,\mathcal{M}\lambda)(1+\mu(gv,\mathcal{M}v))}{1+\mu(gv,g\lambda)},\\&\frac{\mu(gv,\mathcal{M}v)(1+\mu(g\lambda,\mathcal{M}\lambda))}{1+\mu(\mathcal{M}v,\mathcal{M}\lambda)}\Big\}.\end{aligned}$$

Assume that

(i) *the pair $(g,\mathcal{M})$ is ω-admissible regarding to the function κ;*

(ii) *there exists $v_0\in\Phi$ so that $\omega(v_0,gv_0)\geq\kappa(v_0,gv_0)$ and $\omega(v_0,\mathcal{M}v_0)\geq\kappa(v_0,\mathcal{M}v_0)$;*

(iii) *g and $\mathcal{M}$ are continuous.*

Then g and $\mathcal{M}$ have a unique coincidence point in Φ. Moreover, if g and $\mathcal{M}$ are weakly compatible, then g and $\mathcal{M}$ have a unique common fixed point.

Proof. Let $\mathcal{F}(v,\lambda) = v - \lambda$ in Theorem 3.1, we have following corollary 3.2. □

Corollary 3.3. *Let (Φ,μ) be a Hausdorff rectangular metric space and $\mathcal{M} : \Phi \to \Phi$ be an ω-admissible mapping with respect to κ and let $g, \mathcal{M} : \Phi \to \Phi$ be two self maps such that $\mathcal{M}(\Phi) \subseteq g(\Phi)$ and $(g\Phi,\mu)$ is a complete rectangular metric space. Suppose $\psi \in \Psi$ and $v,\lambda \in \Phi$, such that*

$$\omega(v,\lambda) > \kappa(v,\lambda) \Rightarrow \mu(\mathcal{M}v,\mathcal{M}\lambda) \leq \Delta(v,\lambda) - \psi(\Delta(v,\lambda)),$$

where

$$\Delta(v,\lambda) = \max\Big\{\mu(gv,g\lambda), \frac{\mu(gv,\mathcal{M}v)+\mu(g\lambda,\mathcal{M}\lambda)}{2}, \frac{\mu(gv,\mathcal{M}v)+\mu(g\lambda,\mathcal{M}v)}{2}, \frac{\mu(g\lambda,\mathcal{M}\lambda)(1+\mu(gv,\mathcal{M}v))}{1+\mu(gv,g\lambda)}, \frac{\mu(gv,\mathcal{M}v)(1+\mu(g\lambda,\mathcal{M}\lambda))}{1+\mu(\mathcal{M}v,\mathcal{M}\lambda)}\Big\}.$$

Assume that

- (i) *the pair $(g,\mathcal{M})$ is ω-admissible regarding to the function κ;*
- (ii) *there exists $v_0 \in \Phi$ so that $\omega(v_0,gv_0) \geq \kappa(v_0,gv_0)$ and $\omega(v_0,\mathcal{M}v_0) \geq \kappa(v_0,\mathcal{M}v_0)$;*
- (iii) *g and $\mathcal{M}$ are continuous.*

Then g and $\mathcal{M}$ have a unique coincidence point in Φ. Moreover, if g and $\mathcal{M}$ are weakly compatible, then g and $\mathcal{M}$ have a unique common fixed point.

Proof. Let $\vartheta(t) = t$ in Corollary 3.2, we have following corollary 3.3. □

Corollary 3.4. *Let (Φ,μ) be a Hausdorff rectangular metric space and $\mathcal{M} : \Phi \to \Phi$ be an ω-admissible mapping with respect to κ and let $g, \mathcal{M} : \Phi \to \Phi$ be two self maps such that $\mathcal{M}(\Phi) \subseteq g(\Phi)$ and $(g\Phi,\mu)$ is a complete rectangular metric space. For all $v,\lambda \in \Phi$ and $0 < k < 1$ such that such that*

$$\omega(v,\lambda) > \kappa(v,\lambda) \Rightarrow \mu(\mathcal{M}v,\mathcal{M}\lambda) \leq k\Delta(v,\lambda),$$

where

$$\Delta(v,\lambda) = \max\Big\{\mu(gv,g\lambda), \frac{\mu(gv,\mathcal{M}v)+\mu(g\lambda,\mathcal{M}\lambda)}{2}, \frac{\mu(gv,\mathcal{M}v)+\mu(g\lambda,\mathcal{M}v)}{2}, \frac{\mu(g\lambda,\mathcal{M}\lambda)(1+\mu(gv,\mathcal{M}v))}{1+\mu(gv,g\lambda)}, \frac{\mu(gv,\mathcal{M}v)(1+\mu(g\lambda,\mathcal{M}\lambda))}{1+\mu(\mathcal{M}v,\mathcal{M}\lambda)}\Big\}.$$

Assume that

- (i) *the pair $(g,\mathcal{M})$ is ω-admissible regarding to the function κ;*
- (ii) *there exists $v_0 \in \Phi$ so that $\omega(v_0,gv_0) \geq \kappa(v_0,gv_0)$ and $\omega(v_0,\mathcal{M}v_0) \geq \kappa(v_0,\mathcal{M}v_0)$;*
- (iii) *g and $\mathcal{M}$ are continuous.*

Then g and $\mathcal{M}$ have a unique coincidence point in Φ. Moreover, if g and $\mathcal{M}$ are weakly compatible, then g and $\mathcal{M}$ have a unique common fixed point.

Proof. Let $\psi(t) = (1-k)(t)$ for $0 < k < 1$ in Corollary 3.3, we have following corollary 3.4. □

4 Applications

Definition 4.1. *Let Υ be the class of functions $\chi : [0,\infty) \to [0,\infty)$ satisfying the following*

(1) χ is Lebesgue integrable function on each compact subset of $[0,\infty)$;

(2) $\int_0^{\epsilon} \chi(t)dt > 0$ for any $\epsilon > 0$.

Theorem 4.2. *Let (Φ,μ) be a Hausdorff rectangular metric space and let $g, \mathcal{M} : \Phi \to \Phi$ be two self maps such that $\mathcal{M}(\Phi) \subseteq g(\Phi)$ and $(g\Phi,\mu)$ is a complete rectangular metric space and that the following condition holds:*

$$\int_0^{\mu(\mathcal{M}v,\mathcal{M}\lambda)} \chi(t)dt \leq \int_0^{\Delta(v,\lambda)} \chi(t)dt - \int_0^{\Delta(v,\lambda)} \varphi(t)dt \tag{19}$$

for all $v,\lambda \in \Phi$ and $\chi,\varphi \in \Upsilon$, such that g and $\mathcal{M}$ satisfy inequality (1)*, where*

$$\Delta(v,\lambda) = \max\Big\{\mu(gv,g\lambda), \frac{\mu(gv,\mathcal{M}v)+\mu(g\lambda,\mathcal{M}\lambda)}{2}, \frac{\mu(gv,\mathcal{M}v)+\mu(g\lambda,\mathcal{M}v)}{2}, \frac{\mu(g\lambda,\mathcal{M}\lambda)(1+\mu(gv,\mathcal{M}v))}{1+\mu(gv,g\lambda)}, \frac{\mu(gv,\mathcal{M}v)(1+\mu(g\lambda,\mathcal{M}\lambda))}{1+\mu(\mathcal{M}v,\mathcal{M}\lambda)}\Big\}.$$

Then g and $\mathcal{M}$ have a unique coincidence point.

Proof. Let $\vartheta(t) = \int_0^t \chi(\tau)d\tau$ and $\psi(t) = \int_0^t \varphi(\tau)d\tau$. Then, $\vartheta,\ \psi \in \Psi$. Thus, by Theorem 3.1, g and $\mathcal{M}$ have a unique coincide fixed point. □

Theorem 4.3. *Let (Φ, μ) be a Hausdorff rectangular metric space and let $g, \mathcal{M} : \Phi \to \Phi$ be two self maps such that $\mathcal{M}(\Phi) \subseteq g(\Phi)$ and $(g\Phi, \mu)$ is a complete rectangular metric space and that the following condition holds:*

$$\int_0^{\mu(\mathcal{M}v,\mathcal{M}\lambda)} \chi(t)dt \leq \kappa \int_0^{\Delta(v,\lambda)} \chi(t)dt \qquad (20)$$

for all $v, \lambda \in \Phi$, $\chi \in \Upsilon$ and $0 \leq \kappa < 1$ such that g and $\mathcal{M}$ satisfy inequality (1), where

$$\begin{aligned}&\Delta(v, \lambda)\\ &= \max\Big\{\mu(gv, g\lambda), \frac{\mu(gv, \mathcal{M}v) + \mu(g\lambda, \mathcal{M}\lambda)}{2},\\ &\quad \frac{\mu(gv, \mathcal{M}v) + \mu(g\lambda, \mathcal{M}v)}{2},\\ &\quad \frac{\mu(g\lambda, \mathcal{M}\lambda)(1 + \mu(gv, \mathcal{M}v))}{1 + \mu(gv, g\lambda)},\\ &\quad \frac{\mu(gv, \mathcal{M}v)(1 + \mu(g\lambda, \mathcal{M}\lambda))}{1 + \mu(\mathcal{M}v, \mathcal{M}\lambda)}\Big\}.\end{aligned}$$

Then g and $\mathcal{M}$ have a unique coincidence point.

Proof. Let $f(t) = \chi(t) - \kappa\chi(t)$ Thus, by Theorem 3.1, g and $\mathcal{M}$ have a unique coincide fixed point. □

5 Acknowledgments

Firstly, Chuanpit Mungkala (chuanpit.t@rbru.ac.th) and Anantachai Padcharoen (anantachai.p@rbru.ac.th) were financially supported by the Research and Development Institute of Rambhaibarni Rajabhat University. Finally, Pakeeta Sukprasert (pakeeta_s@rmutt.ac.th) was financially supported by Rajamangala University of Technology Thanyaburi (RMUTT).

References:

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fundamenta Mathematicae, vol. 3, pp. 133-181, 1922.

[2] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math., 57(1-2):31-37, 2000.

[3] M. Arshad, J. Ahmad, E Karapinar, Some common fixed point results in rectangular metric spaces, International Journal of Analysis Volume 2013, Article ID 307234, 7 pages.

[4] N. Souayah, H. Aydi, T. Abdeljawad, and N. Mlaiki, "Best proximity point theorems on rectangular metric spaces endowed with a graph, Axioms, vol. 8, no. 1, p. 17, 2019.

[5] H. Aydi, N. Taş, N. Y. Özgür, and N. Mlaiki, "Fixed-discs in rectangular metric spaces," Symmetry, vol. 11, no. 2, p. 294, 2019.

[6] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anal., Theory Methods Appl., 75(4):2154-2165, 2012.

[7] A.H. Ansari, Note on ϕ-ψ-contractive type mappings and related fixed point, in The 2nd regional conference on mathematics and applications, Tonekabon, Iran, September, 2014, Payame Noor Univ., 2014, pp. 377-380.

[8] P. Salimi, A. Latif, N. Hussain, Modified α-ψ-contractive mappings with applications, Fixed Point Theory Appl., 2013(1):151, 2013.

[9] E. Karapınar, Discussion on contractions on generalized metric spaces, in Abstr. Appl. Anal., Volume 2014, 2014.

[10] M.S. Khan, M. Swaleh, S. Sessa, Fixed point theorems by altering distances between the points, Bull. Aust. Math. Soc., 30(1):1-9, 1984.

[11] L. Budhia, H. Aydi, A.H. Ansari, D. Gopal, Some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations, Nonlinear Analysis: Modelling and Control, Vol. 25, No. 4, 580-597.