On Pairwise P-closed Spaces.

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Abstract: In this paper, we introduce the concept of pairwise P-closed spaces and obtain some of their properties. Furthermore, we generalize some results concerning P-closed spaces to pairwise P-closed. Eventually, we conclude that every p-paracompact, p.w.T_2 bitopological space is p-normal.

Key–Words: p.w.P-closed, p.w.P-space, p-paracompact, p-continuous function, p-homeomorphism.

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1 Introduction

Since ancient times (in 1963) the notion of bitopological spaces was introduced and studied by Kelly [5], who defined essential separation axioms such as pairwise Hausdorff, p.w regular, pairwise normal and generalized several standard results such as Urysohn's Lemma.

Since then several mathematicians have studied bitopological spaces and their properties. In 1986, Hdeib and Pareek introduced the notion of L-closed spaces to be the spaces in which Lindelöf sets are closed [6]. Recently, in 2019, Almohor studied p.w.L-closed spaces as a generalization of L-closed spaces [1]. In this paper, we define the pairwise P-closed spaces and derive the relationship between p.w.L-closed and p.w.P-closed spaces then find under what conditions they are equivalent. After that, we study when the p.w.P-closed space is P-space and vice-versa. Afterward, we verify that the p.w-closed spaces are invariant under the p-homeomorphism function and establish nice related results. However, we also conclude some results about the product topology of p.w.P-closed spaces. Eventually, we obtained some results on p.w.P-closed spaces with p-normality property and generalized the important result that every T_2 paracompact space is normal in sense of bitopological spaces.

2 Preliminaries:

Definition 1 (2.1, [*1*]) *In the bitopological space (X, τ_1, τ_2), if each τ_1-Lindelöf subset of X is a τ_2-closed and each τ_2-Lindelöf subset of X is a τ_1-closed, then it is called pairwise L-closed space, however, we will denote to such a space by p.w.L-closed space.*

Definition 2 (2.26, [*2*]) *The $\tau_1\tau_2$-open cover $\breve{U}$ is a cover for the bitopological space (X, τ_1, τ_2) such that $\breve{U} \subseteq \tau_1 \cup \tau_2$, while it is called a p-open cover if it has at least one nonempty τ_1-open proper set and one nonempty τ_2-open proper set of X.*

Definition 3 (2.5, [*2*]) *The p-Lindelöf (s-Lindelöf) space (X, τ_1, τ_2) is bitopological space such that for each p-open ($\tau_1\tau_2$-open) cover for it, there is a countable p-open ($\tau_1\tau_2$-open) subcover.*

Definition 4 (2.4, [*3*]) *The bitopological space (X, τ_1, τ_2) is called pairwise T_1 or shortly p.w.T_1 if for each $x = y$ two distinct points in X there is a τ_1-neighborhood U of x that does not contain y and a τ_2-neighborhood V of y that does not contain x.*

Definition 5 (2.6, [*3*]) *The bitopological space (X, τ_1, τ_2) is called pairwise T_2 (p-Hausdorff) or shortly p.w.T_2 if for each two distinct points $x = y$ in X there are disjoint τ_1-open neighborhood U of x and τ_2-open neighborhood V of y.*

Definition 6 [*4*] *The bitopological space (X, τ_1, τ_2) is called pairwise P-space or simply p.w.P-space if a countable intersection of τ_1-open subsets is a τ_2-open subset in X and a countable intersection of τ_2-open subsets is a τ_1-open subset in X.*

Definition 7 [*2*] *The bitopological space (X, τ_1, τ_2) is called Lindelöf (resp.compact) (resp.paracompact) space if it is both τ_1-Lindelöf (resp.τ_1-compact) (resp.τ_1-paracompact) and τ_2-Lindelöf (resp.τ_2-compact) (resp.τ_2-paracompact).*

Definition 8 [5] *Considering the bitopological space (X, τ_1, τ_2), then we mean by τ_1 is regular with respect to τ_2 if for each τ_1-closed F and $x \notin F$ there are disjoint τ_1-neighborhood U for x and τ_2-neighborhood V for F. Here we denote to such space by τ_1-regular w.r.t. τ_2.*

Definition 9 [5] *The bitopological space (X, τ_1, τ_2) is called p-regular if τ_1 is regular with respect to τ_2 and τ_2 is regular with respect to τ_1.*

Definition 10 (2.18 [3]) *The bitopological space (X, τ_1, τ_2) is called p-normal if for every two disjoint subsets in X where one of them is τ_1-closed subset F_1, and the other is τ_2-closed subset F_2, then there are two disjoint open subsets G_1 and G_2 such that G_1 is a τ_1-open containing F_2 and G_2 is τ_2-open containing F_1. In other words for a τ_1-closed subset F_1 and a τ_2-closed subset F_2 such that $F_1 \cap F_2 = \Phi$, there is a τ_1-open subset G_1 and a τ_2-open subset G_2 such that $F_1 \subseteq G_2$ and $F_2 \subseteq G_1$ satisfying $G_1 \cap G_2 = \Phi$.*

Definition 11 (2.24, [1])*The bitopological space (X, τ_1, τ_2) is called hereditary pairwise Lindelöf if every τ_1 or τ_2 subspace of X is Lindelöf.*

Definition 12 (3.1, [1])*Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces, and let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function, then f is called p-continuous function if the induced functions $f_1 : (X, \tau_1) \to (Y, \sigma_1)$ and $f_2 : (X, \tau_2) \to (Y, \sigma_2)$ are continuous.*

Definition 13 (3.3, [1])*Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces, and let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function, then f is called p-homeomorphism iff it is a bijection, p-continuous as well as f^{-1}.*

Definition 14 (2.45, [2])*Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces, and let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function, then f is called p-closed function if the induced functions $f_1 : (X, \tau_1) \to (Y, \sigma_1)$ and $f_2 : (X, \tau_2) \to (Y, \sigma_2)$ are closed.*

Proposition 15 (2.1 [6])*Let (X, τ_1, τ_2) be bitopological space such that every countable subset is closed, then every countable subset is discrete and every compact subset is finite.*

3 Pairwise P-closed spaces:

Definition 16 *In the bitopological space (X, τ_1, τ_2), if each τ_1-paracompact subset of X is τ_2-closed and each τ_2-paracompact subset of X is τ_1-closed, then it is called pairwise P-closed space, however, we will denote to such a space by p.w.P-closed space.*

Example 17 *Consider the bitopological space $(\mathbb{R}, \tau_{dis}, \tau_{dis})$ where $\mathbb{R}$ is real numbers and τ_{dis} is discrete topology. This bitopological space is p.w.P-closed since $\tau_1 = \tau_{dis} = \tau_2$, and any subset A of $(\mathbb{R}, \tau_{dis})$ is paracompact and closed, i.e every τ_{dis}-paracompact is τ_{dis}-closed.*

Example 18 *Consider the bitopological space $(\mathbb{R}, \tau_{coc}, \tau_u)$ where $\mathbb{R}$ is real numbers and τ_{coc}, τ_u are co-countable, usual topology respectively. Then This bitopological space is not p.w.P-closed, since any subset A in $\mathbb{R}$, is τ_{coc}-paracompact but it is not necessary τ_u-closed.*

Remark 19 *In the set of spaces of p.w.L-closed every countable set is closed but the converse like the case of single spaces is not true. Considering the irrational numbers with the relative usual topology, we can observe that it is not L-closed so not p.w.L-closed but every countable set has discrete topology and closed. This result is also true in p.w.P-closed.*

Proposition 20 *Let (X, τ_1, τ_2) be a p.w.P-closed space, then every countable subset of X is closed and every compact subset of X is finite.*

Proof 21 *Since every countable subset of X is τ_1 and τ_2-Lindelöf, so it will be τ_1 and τ_2-paracompact hence it is τ_2 and τ_1-closed, therefore X has discrete topologies see* (2.1 [6]).

Observe that in the case of a single topology every P-closed space is an L-closed space but the converse need not be true. This is also achieved in bitopological spaces. Every p.w.P-closed space is p.w.L-closed space but the converse need not be true. To show this let A be a τ_1-Lindelöf subset of X, so it is τ_1-paracompact, hence it is τ_2-closed. similarly for a τ_2-Lindelöf subset of X. For the converse, we introduce the following example.

Example 22 *Consider the bitopologicl space (X, τ_1, τ_2), such that $X = Y \cup \{x\}$, where Y is an uncountable set that has discrete topology and x has a co-countable neighborhood, this way we constructed a T_1 topology, τ_1 on X, while τ_2 is just a discrete topology. Let A be any τ_1-Lindelöf subset of X, then clearly it is τ_2-closed. On other hand, let B be a τ_2-Lindelöf subset of X, then B will be countable subset of X but this does not matter here. However, B is a τ_1-closed whether it contains x or* *not. Hence this bitopological space is p.w.L-closed. Eventually, we want to show this bitopological space* (X, τ_1, τ_2) *is not a p.w.P-closed space. Consider the subset* Y *of* X *which is* τ_2*-paracompact but not* τ_1*-closed since the complement of it is* $\{x\}$ *which is* τ_1*-closed but does not* τ_1*-open under the* T_1 *topology* (X, τ_1)*, So the result.*

Proposition 23 *The pairwise P-closeness is a hereditary property.*

Proof 24 *Let* $(Y, \tau_{Y_1}, \tau_{Y_2})$ *be a bitopological subspace of the p.w.P-closed bitopological space* (X, τ_1, τ_2)*, i.e* $Y \subseteq X$*,* τ_{Y_1} *and* τ_{Y_2} *are relatives topologies corresponding to* τ_1 *and* τ_2 *respectively. Let* F *be a* τ_{Y_1}*-paracompact subset of* Y*, so it is a* τ_1*-paracompact in* X *and then* τ_2*-closed subset of* X *because* X *is p.w.P-closed space. Hence* $F = F \cap Y$ *is* τ_{Y_2} *closed in* Y*. Similarly we can show that every* τ_{Y_2}*-paracompact subset of* Y *is* τ_{Y_1}*-paracompact subset of* Y*. Thus* Y *is p.w.P-closed subspace of* X*.*

In a bitopological space (X, τ_1, τ_2) when we say that every paracompact subset has a dense Lindelöf subspace iff every τ_1-paracompact subset of X has a dense τ_1-Lindelöf subspace of it and every τ_2-paracompact subset of X has a dense τ_2-Lindelöf subspace of it, i.e every τ_i-paracompact subset has a dense τ_i-Lindelöf subspace of it. Hence every τ_i-paracompact is τ_i-Lindelöf for $i = \{1, 2\}$.

Remark 25 *Every p.w.L-closed bitopological space* (X, τ_1, τ_2) *that has the property every* τ_i*-paracompact subset has a dense* τ_i*-Lindelöf subspace of it, is p.w.P-closed bitopological space.*

Proposition 26 *The p-Hausdorff, p.w.P-space bitopological space* (X, τ_1, τ_2) *that has the property every* τ_i*-paracompact subset has a dense* τ_i*-Lindelöf subspace of it, is a p.w.P-closed.*

Proof 27 *Since every p-Hausdorff, p.w.P-space bitopological space is p.w.L-closed see corollary (2.15 [1]), and every p.w.L-closed bitopological space with the property that every* τ_i*-paracompact subset of* X *has a dense* τ_i*-Lindelöf subspace of it, is p.w.P-closed. Hence the result.*

Proposition 28 *The paracompact p.w.P-closed bitopological space* (X, τ_1, τ_2) *is p.w.P-space.*

Proof 29 *Let* (X, τ_1, τ_2) *be paracompact p.w.P-closed space. Let* A *be a* τ_1*-*G_δ *set i.e* $A = \cap_{\alpha=1}^{\infty} U_\alpha$*, where* U_α *is a* τ_1*-open for each* α*. Then* A *is a* τ_2*-open subset of* X *since the complement of* A *is* τ_1*-*F_σ *set as the following show,* $X - A = X - \cap_{\alpha=1}^{\infty} U_\alpha = \cup_{\alpha=1}^{\infty}(X - U_\alpha)$*, so it is* τ_1*-paracompact set and hence* τ_2*-closed. Therefore* A *is a* τ_2*-open subset of* X*. By a similar argument we can show that any* τ_2*-*G_δ *set is a* τ_1*-open subset of* X*. So the result.*

Proposition 30 *The Lindelöf, p.w.P-closed space* (X, τ_1, τ_2) *is p.w.P-space.*

Proof 31 *Consider the Lindelöf, p.w.P-closed space* (X, τ_1, τ_2)*. Let* $A = \cap_{\alpha=1}^{\infty} U_\alpha$*, where* U_α *is a* τ_1*-open for each* α *i.e.* A *is a* τ_1*-*G_δ *set, then the complement of* A *is* τ_1*-*F_σ *set like in the previous proof, but we know that* F_σ *set in Lindelöf space is Lindelöf, hence the complement of* A *is* τ_1*-Lindelöf so* τ_1*-paracompact thus* τ_2*-closed. Hence* A *is* τ_2*-open. Similarly the other case.*

Proposition 32 *The* T_3*, paracompact, p.w.L-closed space* (X, τ_1, τ_2) *such that every* τ_i*-paracompact subset of* X *has a dense* τ_i*-Lindelöf subspace of it, is p.w.P-space.*

Proof 33 *Consider the* τ_i*-*G_δ *set* A*, then the complement of* A *is* τ_i*-*F_σ *in paracompact topology* (X, τ_i)*. So* A^c *will be a* τ_i*-paracompact subset in* (X, τ_i) *since the* F_σ*-set in paracompact space is paracompact itself. Therefore* A^c *has a dense* τ_i*-Lindelöf subspace of it and hence it is* τ_i*-Lindelöf in p.w.L-closed bitopological space so it is* τ_j*-closed. Thus* A *is* τ_j*-open for* $i, j \in \{1, 2\}, i \neq j$*.*

Corollary 34 *In the* T_3*, paracompact bitopological space* (X, τ_1, τ_2) *which satisfies that every* τ_i*-paracompact subset of* X *has a dense* τ_i*-Lindelöf subspace of it, the following are equivalent:*

1. (X, τ_1, τ_2) is p.w.P-space.
2. (X, τ_1, τ_2) is p.w.P-closed.

Proof 35 *Obvious from previous propositions.*

Definition 36 *The bitopological space* (X, τ_1, τ_2) *is called pairwise hereditary paracompact if every* τ_1 *or* τ_2 *subspace of* X *is paracompact. I will denote to such a bitopological space by hereditary paracompact space.*

Proposition 37 *For a hereditary paracompact bitopological space* (X, τ_1, τ_2)*, the following are equivalent:*

1. (X, τ_1, τ_2) is p.w.P-closed
2. (X, τ_1, τ_2) is discrete.

Proof 38 (1) $\to$ (2) *Suppose that* (X, τ_1, τ_2) *is p.w.P-closed space and let* A *be any subset of* X*, then* A *will be* τ_1 *and* τ_2*-paracompact. Hence it is* τ_1 *and* τ_2*-closed. Therefore* (X, τ_1, τ_2) *is discrete.*

(2) $\to$ (1) *Let* (X, τ_1, τ_2) *be discrete bitopological space, and let* A *be a* τ_i*-paracompact subset of* X*, then* A *will be* τ_j*-closed since* (X, τ_j) *is a discrete topology for* $i, j \in \{1, 2\}, i \neq j$*. Hence* (X, τ_1, τ_2) *is p.w.P-closed.*

Proposition 39 *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be p-closed, p-continuous, one-to-one function between the bitopological spaces* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *such that* (Y, σ_1, σ_2) *is p.w.P-closed, then* (X, τ_1, τ_2) *is p.w.P-closed as well as p.w.L-closed.*

Proof 40 *Let* F *be a* τ_i*-paracompact subset in* X*, then* $f(F)$ *is a* σ_i*-paracompact subset in* Y *because* f *is a p-closed, p continuous function. But* (Y, σ_1, σ_2) *is p.w.P-closed so* $f(F)$ *is* σ_j*-closed. Hence* $F = f^{-1}(f(F))$ *is* τ_j*-closed for* $i, j \in \{1, 2\}, i \neq j$*, consequently the result.*

Proposition 41 *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a p-continuous function between the bitopological spaces* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *such that* (Y, σ_1, σ_2) *is p.w.P-closed and* (X, τ_1, τ_2) *is p-Lindelöf, then* f *is a p-closed function.*

Proof 42 *Consider a* τ_i*-closed subset* A *of* X*, then* A *will be* τ_j*-Lindelöf subset of* X *because* X *is p-Lindelöf* (2.29[2])*, So* $f(A)$ *will be* σ_j*-Lindelöf since* f *is p-continuous and hence* σ_i*-closed for* $i, j \in \{1, 2\}, i \neq j$*. This show that* f *is a p-closed function, consequently it is a p-homeomorphism if it is a bijection function.*

Corollary 43 *The bijection, p-continuous function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *between the bitopological spaces* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *such that* (Y, σ_1, σ_2) *is p.w.P-closed and* (X, τ_1, τ_2) *is p-Lindelöf, is p-homeomorphism function.*

Proof 44 *It is obvious from the previous proposition.*

Proposition 45 *The pairwise P-closeness is a bitopological property.*

Proof 46 *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be a p-homeomorphism from the p.w.P-closed bitopological space* (X, τ_1, τ_2) *onto* (Y, σ_1, σ_2)*. We aim to verify that the image of* f *is also p.w.P-closed. Consider the* σ_i*-paracompact subset* B *in* Y*, then* $f^{-1}(B) = A$ *is* τ_i*-paracompact since* f^{-1} *is a p-closed and p-continuous.* X *is p.w.P-closed bitopological space this cause* A *is* τ_j*-closed and then* $f(A) = B$ *is* σ_j*-closed for* $i, j \in \{1, 2\}, i \neq j$*. Therefore* Y *is p.w.P-closed bitopological space and hence the result.*

Proposition 47 *Let* (X, τ_1, τ_2) *be a p.w.*T_3 *bitopological space such that every point in* X *has a p.w.P-closed neighborhood, then* X *will be a p.w.P-closed space.*

Proof 48 *Consider* $i, j \in \{1, 2\}, i \neq j$*. Let* F *be a* τ_i*-paracompact subset in* X*. Now we want to show that* F *is a* τ_j*-closed, for this purpose suppose that* $x \in cl_j(F)$ *i.e* x *belongs to the closure of* F *in* (X, τ_j) *but not in* F*. Then* x *has a p.w.P-closed neighborhood* U*. This* U *could be* τ_i*-open or* τ_j*-open. Suppose* U *is* τ_i*-open. By p.w. regularity of* X *there exists a* τ_i*-open neighborhood* H *such that* $x \in H \subseteq cl_j(H) \subseteq U$*. Since* $cl_j(H) \cap F$ *is a* τ_i*-paracompact subset of* U*, it is* τ_j*-closed subset of the p.w.P-closed neighborhood* U*, hence* $U - (cl_j(H) \cap F)$ *is* τ_j*-open neighborhood of* x*, but* $(U - (cl_j(H) \cap F)) \cap F = \Phi$ *which contradicts that* $x \in cl_j(F)$*. Consequently,* $x \in cl_j(F)$ *and hence* F *is a* τ_j*-closed subset in* X*.*

For the other case, let U *be a* τ_j*-open and we still assume the same constructions that* $x \in cl_j(F)$ *and* $x \notin F$*. So by the same argument* $\exists$ *a* τ_j*-open neighborhood* G *such that* $x \in G \subseteq cl_i(G) \subseteq U$*. And* $cl_i(G) \cap F$ *is a* τ_i*-paracompact subset in* U*, so it is* τ_j*-closed and then* $U - (cl_i(G) \cap F)$ *will be* τ_j*-open which yields to the same contradiction like in the first case. Hence* (X, τ_1, τ_2) *is p.w.P-closed space.*

Definition 49 *Consider the bitopological space* (X, τ_1, τ_2)*, then it is called pairwise almost paracompact if every* τ_i*-open cover* $\{U_\alpha, \alpha \in \Lambda\}$ *has an open (closed) locally finite refinement collection of it* $\{U^*_\gamma, \gamma \in \Gamma\}$ *such that* $X = \cup_{\gamma \in \Gamma} cl_j(U^*_\gamma)$ *for all* $i, j \in \{1, 2\} i \neq j$*.*

Definition 50 *Consider the bitopological space* (X, τ_1, τ_2)*, then it is called hereditarily pairwise almost paracompact if every subspace of* X *is pairwise almost paracompact.*

Proposition 51 *For the pairwise P-closed space* (X, τ_1, τ_2)*, the following are equivalent:*

1. X *is hereditary pairwise almost lindelöf.*
2. X *is a hereditary lindelöf.*
3. X *is countable discrete.*

Proof 52 *Since every pairwise P-closed space* (X, τ_1, τ_2) *is pairwise L-closed, the results arise immediately from proposition* (2.30[*1*])*.*

Proposition 53 *For the p-regular, pairwise P-closed space* (X, τ_1, τ_2)*, the following are equivalent:*

1. X *is hereditary pairwise almost paracompact.*

2. *X is hereditary paracompact.*

3. *X is discrete space.*

Proof 54 $(1) \to (2)$ *Suppose the bitopological space (X, τ_1, τ_2) is hereditary pairwise almost paracompact. Let A be a subspace of X, so A is pairwise almost paracompact. Let $\cup_{\alpha\in\Lambda}U_\alpha$ be a τ_{A_i}-open cover of A in A. By the hereditary p-regular property we can find another τ_{A_i}-open cover $\cup_{\alpha\in\Lambda}V_\alpha$ of A such that $V_\alpha \subseteq cl_j(V_\alpha) \subseteq U_\alpha$ for each $\alpha \in \Lambda$, and $i, j \in \{1, 2\} i \neq j$. Consider this cover $\cup_{\alpha\in\Lambda}V_\alpha$ then there is a locally finite refinement collection of it $\{V^*_\beta\}_{\beta\in\Gamma}$ such that $A = \cup_{\beta\in\Gamma}cl_jV^*_\beta$ since A is pairwise almost paracompact. Therefore A has locally finite refinement collection $\{cl_jV^*_\beta\}_{\beta\in\Gamma}$ of $\{U_\alpha\}_{\alpha\in\Lambda}$. Hence A is τ_{A_i}-paracompact, i.e. A is paracompact, and then consequently X is hereditary paracompact.*

$(2) \to (3)$ *Suppose the bitopological space (X, τ_1, τ_2) is hereditary paracompact and consider any subset A of X, then A is τ_{A_i}-paracompact for $i \in \{1, 2\}$ and then it will be τ_{A_j}-closed for $j \in \{1, 2\}$ i.e A is τ_i-paracompact and then τ_j-closed. Hence X is discrete space.*

$(3) \to (1)$ *Suppose X is discrete space so paracompact space. Let A be any subspace of X, then A is τ_i-closed for $i \in \{1, 2\}$ and then paracompact subspace of X. Consider the τ_{A_i}-open cover of A in A, $\cup_{\alpha\in\Lambda}U_\alpha$. Then A has a locally finite refinement collection of it, that is $\{V_\beta\}_{\beta\in\Gamma}$ i.e $A = \cup_{\beta\in\Gamma}V_\beta$ so $A = cl_jA = cl_j(\cup_{\beta\in\Gamma}V_\beta) = (\cup_{\beta\in\Gamma}cl_jV_\beta)$, hence X is hereditary pairwise almost paracompact.*

4 On Product of Pairwise P-closed spaces:

The product of two paracompact topological spaces is not necessary paracompact space. In general, the product of two paracompact bitopological spaces also need not be paracompact as the following example shows.

Example 55 *Let $X = \mathbb{R} \times [0, 1]$, the product of real numbers with closed interval $[0, 1]$. Consider the topologies τ_1 and τ_2 that generated by the bases $B_1 = \{[x, y), x < y, x, y \in X\}$, and $B_2 = \{(x, y], x < y, x, y \in [0, 1]\}$ respectively, where $<$ is lexicographical order on X. Then (X, τ_1, τ_2) is Lindelöf hence paracompact. However, the product of bitopological spaces $(X\times X, \tau_1\times\tau_1, \tau_2\times\tau_2)$ is neither Lindelöf nor paracompact. Actually $(X \times X, \tau_1 \times \tau_1)$ is not paracompact because it can not be normal, since it does not achieve Jone's lemma that states " If the topological space X contains a dense set D and a closed discrete subspace L such that $|L| \geq 2^{|D|}$, where $|.|$ denote to the cardinality, then X is not normal space. Observe that $D = (\mathbb{Q} \times ([0, 1] \cap \mathbb{Q}))^2$ is dense set in $(X \times X, \tau_1 \times \tau_1)$, where $\mathbb{Q}$ is rational numbers and $L = \{(x, y) : x = -y, x \in \mathbb{R}\} \times \{0\}$ is closed subset in $(X \times X, \tau_1 \times \tau_1)$ which satisfy that $|L| \geq 2^{|D|}$. The next proposition shows us when the product of p.w.P-closed spaces is p.w.P-closed space.*

Proposition 56 *Let (X, τ_1, τ_2) and (Y, η_1, η_2) be two T_3, p-regular, p.w.P-closed spaces such that every $(\tau_i\times\eta_i)$-paracompact subset of $X\times Y$ has a $(\tau_i\times\eta_i)$-dense, $(\tau_i \times \eta_i)$-Lindelöf subset for $i \in \{1, 2\}$, then $(X \times Y, \tau_1 \times \eta_1, \tau_2 \times \eta_2)$ is p.w.P-closed.*

Proof 57 *Let P be a $(\tau_1 \times \eta_1)$-paracompact subset of the product bitopological space $X \times Y$ and let $(x_o, y_o) \notin P$. Our goal is to verify P is a $(\tau_2 \times \eta_2)$-closed set. Observe that P is a $(\tau_1 \times \eta_1)$-Lindelöf subset. Now $(x_o, y_o) \notin ((\{x_o\} \times Y) \cap P)$. Evidently, $(\{x_o\} \times Y)$ is $(\tau_1 \times \eta_1)$-closed in $X \times Y$ this follows that $((\{x_o\} \times Y) \cap P)$ is $(\tau_1\times\eta_1)$-closed in $(\tau_1\times\eta_1)$-paracompact,$(\tau_1 \times \eta_1)$-Lindelöf subspace P, so it is $(\tau_1\times\eta_1)$-paracompact,$(\tau_1\times\eta_1)$-Lindelöf and isomorphic for some η_1-paracompact, η_1-Lindelöf subset in p.w.P-closed space Y, therefore it is η_2-closed does not contain y_o. As Y is p-regular, then there is an η_2-open neighborhood V of y_o that satisfies $((X \times cl_{\eta_1}(V)) \cap ((\{x_o\} \times Y) \cap P) = \Phi$. Then $\pi_X((X \times cl_{\eta_1}(V)) \cap P)$ is τ_2-closed in the p.w.P-closed space X, since it is the image of $(\tau_1\times\eta_1)$-Lindelöf, $(\tau_1\times\eta_1)$-paracompact subset under the continuous function π_X. Now $[X-\pi_X((X\times cl_{\eta_1}(V))\cap P)]\times[Y\cap(X\times V)]$ is an $(\tau_2\times\eta_2)$-open neighborhood of (x_o, y_o), which is disjoint from P, therefore P is closed in $X\times Y$, hence the result. The same argument we need to verify any $(\tau_2 \times \eta_2)$-paracompact subset of the product bitopological space $(X \times Y, \tau_1 \times \eta_1, \tau_2 \times \eta_2)$ is $(\tau_1 \times \eta_1)$-closed. Consequently $(X \times Y, \tau_1 \times \eta_1, \tau_2 \times \eta_2)$ is p.w.P-closed space.*

Corollary 58 *Consider the T_3, p-regular, p.w.P-closed spaces (X_i, τ_i, η_i), $i \in \{1, 2, ..., n\}$, such that every $(\Pi^n_{i=1}\tau_i)$-paracompact subset of $\Pi^n_{i=1}X_i$ has a $(\Pi^n_{i=1}\tau_i)$-dense, $(\Pi^n_{i=1}\tau_i)$-Lindelöf subset, and every $\Pi^n_{i=1}\eta_i$-paracompact subset of $\Pi^n_{i=1}X_i$ has a $(\Pi^n_{i=1}\eta_i)$-dense, $(\Pi^n_{i=1}\eta_i)$-Lindelöf subset, then the finite product $(\Pi^n_{i=1}X_i\ , \Pi^n_{i=1}\tau_i, \Pi^n_{i=1}\eta_i)$ is p.w.P-closed.*

Proof 59 *One can show this result by induction on i and use the previous proposition.*

No direct relationship between p-paracompact spaces and p.w.P-closed spaces but because both of them give to us the p-normality of the space under some condition which is a useful property, we will derive these results here.

Proposition 60 *A p-regular p.w.P-closed space is p-normal.*

Proof 61 *Since a p.w.P-closed bitopological space is p.w.L-closed bitopological space and a p-regular p.w.L-closed is p-normal see proposition* $(2.23[1])$.

It is known that every paracompact T_2 is normal in a single case. Through the following results, we see that the results can be obtained in bitopological spaces.

Definition 62 *The p-paracompact space* (X, τ_1, τ_2) *is a bitopological space such that for each p-open cover for* X*, there is a locally finite p-open (p-closed) (even neither p-open nor p-closed) cover refinement of it.*

Proposition 63 *Every p-paracompact, p.w.*T_2 *bitopological space is p-regular.*

Initially, we need to introduce the following lemma.

If the bitopological space (X, τ_1, τ_2) is p.w.T_2, then it satisfies that for every distinct two points x and y in X there is a τ_1-open neighborhood U for x is such that its closure under τ_2 does not contain y. In other words for $x \neq y$, $\exists$ a τ_1-open neighborhood U such that $x \in U$ and $y \notin cl_2(U)$.

Proof 64 *As a consequence of the p.w.*T_2 *property on* X*, we have for any two distinct points* $x \neq y$ *a* τ_1*-open neighborhood* U *for* x *and a* τ_2*-open neighborhood* V *for* y *such that* $U \cap V = \Phi$*, so* $U \subseteq V^c$ *the complement of* V*. Therefore* $cl_2(U) \subseteq V^c$*, and hence* $y \notin cl_2(U)$.

One can show by the same argument also $x \notin cl_1(V)$ for some τ_2-open neighborhood V for y.

Remark 65 *Observe that every* τ_1*-closed (*τ_2*-closed) subset in p-paracompact bitopologial space* (X, τ_1, τ_2) *is* τ_2*-paracompact (*τ_1*-paracompact). To show that consider a* τ_1*-closed subset* A *and choose an arbitrary* τ_2*-open cover for it, that is* $\cup_\alpha V_\alpha$*, so* $\cup_\alpha (V_\alpha \cap A)$ *forms a cover for* A *with relative* τ_2*-topology. Then the complement of* A*,* A^c *with* $\cup_\alpha V_\alpha$ *forms a p-open cover for* X*, and this cover has an open locally finite refinement cover for* X *since* X *is a p-paracompact bitopological space, hence* A *has a* τ_2 *open locally finite refinement cover of* $\cup_\alpha V_\alpha$*, and then of* $\cup_\alpha (V_\alpha \cap A)$*, i.e* A *is* τ_2*-paracompact.*

Now we can show the previous proposition.

Proof 66 *Consider any* τ_2*-closed set* F*. Let* $x \notin F$*, so for each* $y \in F$ *there is a* τ_1*-open neighborhood* U_{x_y} *for* y *such that its closure under* τ_2 *does not contain* x*, i.e* $\exists$ *a* τ_1*-open neighborhood* U_{x_y} *such that* $y \in U_{x_y}$ *and* $x \notin cl_2(U_{x_y})$*. Continue in this way then we get* $\forall y \in F$*,* $\exists$ *a* τ_1*-open neighborhood* U_{x_y} *such that* $y \in U_{x_y}$ *and* $x \notin cl_2(U_{x_y})$*, therefore* $F \subseteq \cup_{y \in B} U_{x_y}$*, so* $F \subseteq \cup_{y \in B} U^*_{x_y}$ *where* $\{U^*_{x_y} : y \in F\}$ *is* τ_1*-open locally finite refinement cover of* $\{U_{x_y} : y \in F\}$*, and* $x \notin \cup_{y \in B} cl_2(U^*_{x_y})$*. But* $\cup_{y \in B} cl_2(U^*_{x_y}) = cl_2(\cup_{y \in B} U^*_{x_y})$ *since* $\{U^*_{x_y} : y \in F\}$ *is locally finite set. Consequently* $x \notin cl_2(\cup_{y \in B} U^*_{x_y})$ *and then* $x \in (cl_2(\cup_{y \in B} U^*_{x_y}))^c$*, the complement set which is* τ_2*-open, i.e there are disjoint* τ_1*-open set contains* F *that is* $(\cup_{y \in B} U^*_{x_y})$ *and* τ_2*-open set* $(cl_2(\cup_{y \in B} U^*_{x_y}))^c$*. Hence the result.*

Finally we can conclude the following result.

Proposition 67 *Every p-paracompact, p.w.*T_2 *bitopological space is p-normal.*

Proof 68 *Consider the disjoint* τ_1*-closed* A *and* τ_2*-closed* B*. Then For each* $x \in A$ *there exists disjoint* τ_2*-open neighborhood* V_x *and* τ_1*-open neighborhood* U_x *such that* $x \in V_x$*, and* $B \subseteq U_x$*, hence we have* $A \subseteq \cup_{x \in A} V_x$*. Because* A *is* τ_2*-paracompact, then* $A \subseteq \cup_{x \in A} V^*_x$*, where* $\{V^*_x : x \in A\}$ *is a locally finite refinement set of* $\{V_x\}$*. Choose any* τ_1*-open neighborhood* U_x *containing* B *and take the* τ_2*-open neighborhood* $(\cup_{x \in A} V^*_x) \backslash cl_2(U_x)$*. Therefore* (X, τ_1, τ_2) *is p-normal.*

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