

# On the Complexity of Determining Whether there is a Unique Hamiltonian Cycle or Path

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**Abstract:** The decision problems of the existence of a Hamiltonian cycle or of a Hamiltonian path in a given directed or undirected graph, and of the existence of a truth assignment satisfying a given Boolean formula  $\mathcal{C}$ , are well-known *NP*-complete problems. Here we study the problems of the *uniqueness* of a Hamiltonian cycle or path in an undirected, directed or oriented graph, and show that they have the same complexity, up to polynomials, as the problem U-SAT of the uniqueness of an assignment satisfying  $\mathcal{C}$ . As a consequence, these Hamiltonian problems are *NP*-hard and belong to the class *DP*, like U-SAT.

**Key-Words:** Graph Theory, Hamiltonian Cycle, Hamiltonian Path, Complexity Theory, *NP*-Hardness, class *DP*, Decision Problems, Polynomial Reduction, Uniqueness of Solution, Boolean Satisfiability Problems.

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## 1 Introduction

### 1.1 The Hamiltonian Cycle and Path Problems

We shall denote by  $G = (V, E)$  a finite, simple, undirected graph with vertex set  $V$  and edge set  $E$ , where an *edge* between  $x \in V$  and  $y \in V$  is indifferently denoted by  $xy$  or  $yx$ . The *order* of the graph is its number of vertices,  $|V|$ .

If  $V = \{v_1, v_2, \dots, v_n\}$ , a *Hamiltonian path*  $\mathcal{HP} = \langle v_{i_1} v_{i_2} \dots v_{i_n} \rangle$  is an ordering of all the vertices in  $V$ , such that  $v_{i_j} v_{i_{j+1}} \in E$  for all  $j$ ,  $1 \leq j \leq n - 1$ . The vertices  $v_{i_1}$  and  $v_{i_n}$  are called the *ends* of  $\mathcal{HP}$ . A *Hamiltonian cycle* is an ordering  $\mathcal{HC} = \langle v_{i_1} v_{i_2} \dots v_{i_n} (v_{i_1}) \rangle$  of all the vertices in  $V$ , such that  $v_{i_n} v_{i_1} \in E$  and  $v_{i_j} v_{i_{j+1}} \in E$  for all  $j$ ,  $1 \leq j \leq n - 1$ . Note that the same Hamiltonian cycle admits  $2n$  representations, e.g.,  $\langle v_{i_2} v_{i_3} \dots v_{i_n} v_{i_1} (v_{i_2}) \rangle$  or  $\langle v_{i_n} v_{i_{n-1}} \dots v_{i_2} v_{i_1} (v_{i_n}) \rangle$ .

A *directed graph*  $H = (X, A)$  is defined by its set  $X$  of vertices and its set  $A$  of directed edges, also called *arcs*, an arc being an ordered pair  $(x, y)$  of vertices; with this respect,  $(x, y)$  and  $(y, x)$  are two different arcs and may coexist. A directed graph is said to be *oriented* if it is antisymmetric, i.e., if we have, for any pair  $\{x, y\}$  of vertices, at most one of the two arcs  $(x, y)$  or  $(y, x)$ ; if  $(x, y) \in A$ , we say that  $y$  is the *out-neighbour* of  $x$ , and  $x$  is the *in-neighbour* of  $y$ , and we define the *in-degree* and *out-degree* of a vertex accordingly. The notions of *directed Hamiltonian cycle* and of *directed Hamiltonian path* are extended to a directed graph by considering the arcs  $(v_{i_n}, v_{i_1}) \in A$  and  $(v_{i_j}, v_{i_{j+1}}) \in A$  in the above defi-

nitions. When there is no ambiguity, we shall often drop the words “directed” and “Hamiltonian”.

The following six problems (stated as one) are well known, in graph theory as well as in complexity theory:

**Problem** HAMC / HAMP (Hamiltonian Cycle / Hamiltonian Path):

**Instance:** An undirected, directed or oriented graph.

**Question:** Does the graph admit a Hamiltonian cycle / Hamiltonian path?

As we shall see (Proposition 2), they have been known to be *NP*-complete for a long time. In this paper, we shall be interested in the following problems, and shall locate them in the complexity classes:

**Problem** U-HAMC[U] (Unique Hamiltonian Cycle in an Undirected graph):

**Instance:** An undirected graph  $G = (V, E)$ .

**Question:** Does  $G$  admit a *unique* Hamiltonian cycle?

**Problem** U-HAMP[U] (Unique Hamiltonian Path in an Undirected graph):

**Instance:** An undirected graph  $G = (V, E)$ .

**Question:** Does  $G$  admit a *unique* Hamiltonian path?

**Problem** U-HAMC[D] (Unique directed Hamiltonian Cycle in a Directed graph):

**Instance:** A directed graph  $H = (X, A)$ .

**Question:** Does  $H$  admit a *unique* directed Hamiltonian cycle?

**Problem** U-HAMP[D] (Unique directed Hamiltonian Path in a Directed graph):

**Instance:** A directed graph  $H = (X, A)$ .

**Question:** Does  $H$  admit a *unique* directed Hamiltonian path?

**Problem** U-HAMC[O] (Unique directed Hamiltonian Cycle in an Oriented graph):

**Instance:** An oriented graph  $H = (X, A)$ .

**Question:** Does  $H$  admit a *unique* directed Hamiltonian cycle?

**Problem** U-HAMP[O] (Unique directed Hamiltonian Path in an Oriented graph):

**Instance:** An oriented graph  $H = (X, A)$ .

**Question:** Does  $H$  admit a *unique* directed Hamiltonian path?

We shall prove in Section 2 that these problems have the same complexity, up to polynomials, as the problem of the uniqueness of a truth assignment satisfying a Boolean formula (U-SAT). As a consequence, all are *NP*-hard and belong to the class *DP*. The closely related problem Unique Optimal Travelling Salesman has been investigated in [13], see Remark 8.

In similar works, we reexamine some famous problems, from the viewpoint of uniqueness of solution: Vertex Cover and Dominating Set (as well as its generalization to domination within distance  $r$ ) [8],  $r$ -Identifying Code together with  $r$ -Locating-Dominating Code [9], and Graph Colouring and Boolean Satisfiability [10]. We shall re-use here results from [10], and modify a construction from [8].

In the sequel, we shall need the following tools, which constitute classical definitions related to graph theory or to Boolean satisfiability. A *vertex cover* in an undirected graph  $G$  is a subset of vertices  $V^* \subseteq V$  such that for every edge  $e = uv \in E$ ,  $V^* \cap \{u, v\} \neq \emptyset$ . We denote by  $\phi(G)$  the smallest cardinality of a vertex cover of  $G$ ; any vertex cover  $V^*$  with  $|V^*| = \phi(G)$  is said to be *optimal*.

Next we consider a set  $\mathcal{X}$  of  $n$  *Boolean variables*  $x_i$  and a set  $\mathcal{C}$  of  $m$  *clauses* ( $\mathcal{C}$  is also called a *Boolean formula*); each clause  $c_j$  contains  $\kappa_j$  *literals*, a literal being a variable  $x_i$  or its complement  $\bar{x}_i$ . A *truth assignment* for  $\mathcal{X}$  sets the variable  $x_i$  to TRUE, also denoted by T, and its complement to FALSE (or F), or *vice-versa*. A truth assignment is said to *satisfy* the clause  $c_j$  if  $c_j$  contains at least one true literal, and to satisfy the set of clauses  $\mathcal{C}$  if every clause contains at least one true literal. The following decision problems are classical problems in complexity.

**Problem** VC (Vertex Cover with bounded size):

**Instance:** An undirected graph  $G$  and an integer  $k$ .

**Question:** Does  $G$  admit a vertex cover of size at most  $k$ ?

**Problem** SAT (Satisfiability):

**Instance:** A set  $\mathcal{X}$  of variables, a collection  $\mathcal{C}$  of clauses over  $\mathcal{X}$ , each clause containing at least two

different literals.

**Question:** Is there a truth assignment for  $\mathcal{X}$  that satisfies  $\mathcal{C}$ ?

The following problem is stated for any fixed integer  $k \geq 2$ .

**Problem**  $k$ -SAT ( $k$ -Satisfiability):

**Instance:** A set  $\mathcal{X}$  of variables, a collection  $\mathcal{C}$  of clauses over  $\mathcal{X}$ , each clause containing exactly  $k$  different literals.

**Question:** Is there a truth assignment for  $\mathcal{X}$  that satisfies  $\mathcal{C}$ ?

**Problem** 1-3-SAT (One-in-Three Satisfiability):

**Instance:** A set  $\mathcal{X}$  of variables, a collection  $\mathcal{C}$  of clauses over  $\mathcal{X}$ , each clause containing exactly three different literals.

**Question:** Is there a truth assignment for  $\mathcal{X}$  such that each clause of  $\mathcal{C}$  contains *exactly one* true literal?

We shall say that a clause (respectively, a set of clauses) is *1-3-satisfied* by an assignment if this clause (respectively, every clause in the set) contains exactly one true literal. We shall also consider the following variants of the above problems:

U-VC (Unique Vertex Cover with bounded size),

U-SAT (Unique Satisfiability),

U- $k$ -SAT (Unique  $k$ -Satisfiability),

U-1-3-SAT (Unique One-in-Three Satisfiability).

They have the same instances as VC, SAT,  $k$ -SAT and 1-3-SAT respectively, but now the question is “Is there a *unique* vertex cover / truth assignment. . .?”.

We shall give in Propositions 3–7 what we need to know about the complexities of these problems.

## 1.2 Some Classes of Complexity

We refer the reader to, e.g., [1], [6], [11] or [14] for more on this topic. A *decision problem* is of the type “Given an instance  $I$  and a property  $\mathcal{PR}$  on  $I$ , is  $\mathcal{PR}$  true for  $I$ ?”, and has only two solutions, “yes” or “no”. The class  $P$  will denote the set of problems which can be solved by a *polynomial* (time) algorithm, and the class  $NP$  the set of problems which can be solved by a *nondeterministic polynomial* algorithm. A *polynomial reduction* from a decision problem  $\pi_1$  to a decision problem  $\pi_2$  is a polynomial transformation that maps any instance of  $\pi_1$  into an equivalent instance of  $\pi_2$ , that is, an instance of  $\pi_2$  admitting the same answer as the instance of  $\pi_1$ ; in this case, we shall write  $\pi_1 \leq_p \pi_2$ . Cook [4] proved that there is one problem in  $NP$ , namely SAT, to which every other problem in  $NP$  can be polynomially reduced. Thus, in a sense, SAT is the “hardest” problem inside  $NP$ . Other problems share this property in  $NP$  and are called *NP-complete* problems; their class is denoted by  $NP-C$ . The way to show that a decision problem  $\pi$  is *NP-complete* is, once it is proved

to be in  $NP$ , to choose some  $NP$ -complete problem  $\pi_1$  and to polynomially reduce it to  $\pi$ . From a practical viewpoint, the  $NP$ -completeness of a problem  $\pi$  implies that we do not know any polynomial algorithm solving  $\pi$ , and that, under the assumption  $P \neq NP$ , which is widely believed to be true, no such algorithm exists: the time required can grow exponentially with the size of the instance (when the instance is a graph, its size is polynomially linked to its order; for a Boolean formula, the size is polynomially linked to, e.g., the number of variables plus the number of clauses).

The *complement* of a decision problem, “Given  $I$  and  $\mathcal{PR}$ , is  $\mathcal{PR}$  true for  $I$ ?”, is “Given  $I$  and  $\mathcal{PR}$ , is  $\mathcal{PR}$  false for  $I$ ?”. The class  $co-NP$  (respectively,  $co-NP-C$ ) is the class of the problems which are the complement of a problem in  $NP$  (respectively,  $NP-C$ ).

For problems which are not necessarily decision problems, a *Turing reduction* from a problem  $\pi_1$  to a problem  $\pi_2$  is an algorithm  $\mathcal{A}$  that solves  $\pi_1$  using a (hypothetical) subprogram  $\mathcal{S}$  solving  $\pi_2$  such that, if  $\mathcal{S}$  were a polynomial algorithm for  $\pi_2$ , then  $\mathcal{A}$  would be a polynomial algorithm for  $\pi_1$ . Thus, in this sense,  $\pi_2$  is “at least as hard” as  $\pi_1$ . A problem  $\pi$  is *NP-hard* (respectively, *co-NP-hard*) if there is a Turing reduction from some  $NP$ -complete (respectively,  $co-NP$ -complete) problem to  $\pi$  [6, p. 113].

**Remark 1** Note that with these definitions, *NP-hard* and *co-NP-hard* coincide [6, p. 114].

The notion of completeness can of course be extended to classes other than  $NP$  or  $co-NP$ . Observe that  $NP$ -hardness is defined differently in [5] and [7]: there, a problem  $\pi$  is *NP-hard* if there is a *polynomial* reduction from some  $NP$ -complete problem to  $\pi$ ; this may lead to confusion (see Section 3).

We also introduce the classes  $P^{NP}$  (also known as  $\Delta_2$  in the hierarchy of classes) and  $L^{NP}$  (also denoted by  $P^{NP[O(\log n)]}$  or  $\Theta_2$ ), which contain the decision problems which can be solved by applying, with a number of calls which is polynomial (respectively, logarithmic) with respect to the size of the instance, a subprogram able to solve an appropriate problem in  $NP$  (usually, an  $NP$ -complete problem); and the class  $DP$  [15] (or  $DIF^P$  [2] or  $BH_2$  [11], [17], ...) as the class of languages (or problems)  $L$  such that there are two languages  $L_1 \in NP$  and  $L_2 \in co-NP$  satisfying  $L = L_1 \cap L_2$  (in Figure 1,  $DP-C$  and  $P^{NP}-C$  denote respectively the class of the  $DP$ -complete problems and the class of the  $P^{NP}$ -complete problems). This class is not to be confused with  $NP \cap co-NP$  (see the warning in, e.g., [14, p. 412]); actually,  $DP$  contains  $NP \cup co-NP$  and is contained in  $L^{NP}$ . See Figure 1.

Membership to  $P$ ,  $NP$ ,  $co-NP$ ,  $DP$ ,  $L^{NP}$  or  $P^{NP}$  gives an upper bound on the complexity of a problem (this problem is not more difficult than ...), whereas

a  $NP$ -hardness result gives a lower bound (this problem is at least as difficult as any problem belonging to  $NP$  or to  $co-NP$ ). Still, such results are conditional in the sense that we do not know whether or where the classes of complexity collapse.

We now consider some of the problems from Section 1.1.

**Proposition 2** [12], [6, pp. 56–60 and pp. 199–200] *The decision problems HAMC and HAMP, in an undirected, directed or oriented graph, are NP-complete.*  $\diamond$

The problems VC, SAT and 3-SAT are also three of the basic and most well-known  $NP$ -complete problems [4], [6, p. 39, p. 46, p. 190 and p. 259]. More generally,  $k$ -SAT is  $NP$ -complete for  $k \geq 3$  and polynomial for  $k = 2$ . The problem 1-3-SAT, which is obviously in  $NP$ , is also  $NP$ -complete [16, Lemma 3.5], [6, p. 259], [10, Rem. 3].

The following results will be used in the sequel.

**Proposition 3** [10] *For every integer  $k \geq 3$ , the decision problems U-SAT, U- $k$ -SAT and U-1-3-SAT have equivalent complexity, up to polynomials.*  $\diamond$

Using the previous proposition and results from [2] and [14, p. 415], it is rather simple to obtain the following two results.

**Proposition 4** *For every integer  $k \geq 3$ , the decision problems U-SAT, U- $k$ -SAT and U-1-3-SAT are co-NP-hard and thus NP-hard by Remark 1.*  $\diamond$

**Proposition 5** *For every integer  $k \geq 3$ , the decision problems U-SAT, U- $k$ -SAT and U-1-3-SAT belong to the class DP.*  $\diamond$

**Remark 6** *It is not known whether these problems are DP-complete. In [14, p. 415], it is said that “U-SAT is not believed to be DP-complete”.*

**Proposition 7** [8] *The decision problems U-SAT and U-VC have equivalent complexity, up to polynomials. In particular, there exists a polynomial reduction from U-1-3-SAT to U-VC:  $U-1-3-SAT \leq_p U-VC$ .*  $\diamond$

After the following remark is made, we shall be ready to investigate the problems of uniqueness of Hamiltonian cycle or path.

**Remark 8** *In [13], it is shown that the following problem is  $P^{NP}$ -complete (or  $\Delta_2$ -complete).*

**Problem U-OTS (Unique Optimal Travelling Salesman):**

**Instance:** A set of  $n$  vertices, a  $n \times n$  symmetric matrix  $[c_{ij}]$  of (nonnegative) integers giving the distance between any two vertices  $i$  and  $j$ .

**Question:** Is there a unique optimal tour, that is, a unique way of visiting every vertex exactly once and coming back, with the smallest distance sum?

At best, a polynomial reduction from any instance  $G = (V, E)$  of U-HAMC[U] to U-OTS would show that U-HAMC[U] belongs to  $P^{NP}$ , but we have a better result in Theorem 15(b), with U-HAMC[U] belonging to DP; no useful information for our Hamiltonian problems can be induced from this result on U-OTS.

## 2 Locating the Problems of Uniqueness

We prove that our six Hamiltonian problems have the same complexity as any of the three problems U-SAT, U- $k$ -SAT ( $k \geq 3$ ) and U-1-3-SAT by proving the chain of polynomial reductions given by Figure 2.

**Theorem 9** *There exists a polynomial reduction from U-1-3-SAT to U-HAMP[O]:  $U-1-3-SAT \leq_p U-HAMP[O]$ .*

**Proof.** A polynomial reduction from the problem U-1-3-SAT to U-HAMP[O], via U-VC, can be found in the Appendix; it is an elaborate variation on the polynomial reduction from 3-SAT to VC in [12], [6, pp. 54–56] and the polynomial reduction from VC to HAMC[U] (see [6, pp. 56–60]).  $\diamond$

**Proposition 10** *There exists a polynomial reduction from U-HAMP[O] to U-HAMC[O]:  $U-HAMP[O] \leq_p U-HAMC[O]$ .*

**Proof.** We start from an oriented graph  $H = (X, A)$  which is an instance of U-HAMP[O] and build a graph which is an instance of U-HAMC[O] by adding two extra vertices  $y, z$ , together with the arc  $(y, z)$  and all the arcs  $(x, y)$  and  $(z, x)$ ,  $x \in X$ . This transformation is polynomial and clearly preserves the number of solutions, in particular the uniqueness.  $\diamond$

**Proposition 11** *There is a polynomial reduction from U-HAMP[O] to U-HAMP[D] and from U-HAMC[O] to U-HAMC[D]:  $U-HAMP[O] \leq_p U-HAMP[D]$  and  $U-HAMC[O] \leq_p U-HAMC[D]$ .*

**Proof.** It suffices to consider the identity as the polynomial reduction.  $\diamond$

**Theorem 12** *There is a polynomial reduction from U-HAMP[D] to U-HAMP[U] and from U-HAMC[D] to U-HAMC[U]:  $U-HAMP[D] \leq_p U-HAMP[U]$  and  $U-HAMC[D] \leq_p U-HAMC[U]$ .*

**Proof.** The method is borrowed from [12]. Consider any instance of U-HAMP[D] or U-HAMC[D], i.e., a directed graph  $H = (X, A)$  on  $n$  vertices. We build the undirected graph  $G = (V, E)$ , the instance of U-HAMP[U] or U-HAMC[U], as follows: every vertex  $x \in X$  is triplicated into three

vertices  $x^- \in V$  (a *minus-type vertex*),  $x^* \in V$  (a *star-type vertex*) and  $x^+ \in V$ , linked by the edges  $x^-x^* \in E$  and  $x^*x^+ \in E$ ; for every arc  $(x, y) \in A$ , we create the edge  $x^+y^-$  in  $E$ . The graph  $G$  thus constructed has order  $3n$ .

We claim that there is a unique Hamiltonian cycle (respectively, path) in  $G$  if and only if there is a unique directed Hamiltonian cycle (respectively, path) in  $H$ .

(1) Assume first that  $H$  admits a directed Hamiltonian cycle  $\langle x_1x_2 \dots x_n(x_1) \rangle$ . Then

$$\langle x_1^-x_1^*x_1^+x_2^-x_2^*x_2^+ \dots x_{n-1}^+x_n^-x_n^*x_n^+(x_1^-) \rangle$$

is a Hamiltonian cycle in  $G$ . Moreover, two different directed Hamiltonian cycles in  $H$  provide two different Hamiltonian cycles in  $G$ .

Conversely, assume that  $G$  admits a Hamiltonian cycle  $\mathcal{HC}$ . This cycle must go through all the star-type vertices  $x^*$ , so it necessarily goes through all the edges  $x^-x^*$  and  $x^*x^+$ . Without loss of generality,  $\mathcal{HC}$  reads:

$$\mathcal{HC} = \langle x_1^-x_1^*x_1^+x_2^-x_2^*x_2^+ \dots x_{n-1}^+x_n^-x_n^*x_n^+(x_1^-) \rangle; \quad (1)$$

indeed, we may assume that we “start” with the edge  $x_1^-x_1^*$ , then  $x_1^*x_1^+$ ; now, because the edges which have no star-type vertex as one of their extremities are necessarily of the type  $x^+y^-$ , the other neighbour of  $x_1^+$  is a minus-type vertex, say  $x_2^-$ ; step by step, we see that  $\mathcal{HC}$  has necessarily the previous form (1). Now we claim that  $\langle x_1x_2 \dots x_{n-1}x_n(x_1) \rangle$  is a directed Hamiltonian cycle in  $H$ .

Indeed, for every  $i \in \{1, \dots, n-1\}$ , the edge  $x_i^+x_{i+1}^-$  in  $G$  implies the existence of the arc  $(x_i, x_{i+1})$  in  $H$ ; the same is true for the arc  $(x_n, x_1)$  in  $H$ , thanks to the edge  $x_n^+x_1^-$  in  $G$ . Furthermore, observe that two different Hamiltonian cycles in  $G$  provide two different directed Hamiltonian cycles in  $H$ .

So,  $G$  admits a unique Hamiltonian cycle if and only if  $H$  admits a unique directed Hamiltonian cycle.

(2) Exactly the same argument works with paths, apart from the fact that we need not consider the arc  $(x_n, x_1)$  in  $H$ , nor the edge  $x_n^+x_1^-$  in  $G$ .  $\diamond$

**Proposition 13** *There exists a polynomial reduction from U-HAMP[U] to U-HAMC[U]:  $U-HAMP[U] \leq_p U-HAMC[U]$ .*

**Proof.** We start from an undirected graph  $G = (V, E)$  which is an instance of U-HAMP[U] and build a graph which is an instance of U-HAMC[U] by adding the extra vertex  $y$ , together with all the edges  $xy$ ,  $x \in V$ . This transformation is polynomial and clearly preserves the number of solutions, in particular the uniqueness.  $\diamond$

**Theorem 14** *There exists a polynomial reduction from U-HAMC[U] to U-SAT:  $U\text{-HAMC}[U] \leq_p U\text{-SAT}$ .*

**Proof.** We start from an instance of U-HAMC[U], an undirected graph  $G = (V, E)$  with  $V = \{x^1, \dots, x^{|V|}\}$ ; we assume that  $|V| \geq 3$ . We create the set of variables  $\mathcal{X} = \{x_j^i : 1 \leq j \leq |V|, 1 \leq i \leq |V|\}$  and the following clauses:

- (a<sub>1</sub>) for  $1 \leq i \leq |V|$ , clauses of size  $|V|$ :  $\{x_1^i, x_2^i, \dots, x_{|V|}^i\}$ ;
- (a<sub>2</sub>) for  $1 \leq i \leq |V|, 1 \leq j < j' \leq |V|$ , clauses of size two:  $\{\bar{x}_j^i, \bar{x}_{j'}^i\}$ ;
- (b<sub>1</sub>) for  $1 \leq j \leq |V|$ , clauses of size  $|V|$ :  $\{x_j^1, x_j^2, \dots, x_j^{|V|}\}$ ;
- (b<sub>2</sub>) for  $1 \leq i < i' \leq |V|, 1 \leq j \leq |V|$ , clauses of size two:  $\{\bar{x}_j^i, \bar{x}_j^{i'}\}$ ;
- (c) for  $1 \leq i < i' \leq |V|$  such that  $x^i x^{i'} \notin E$ , for  $1 \leq j \leq |V|$ , clauses of size two:  $\{\bar{x}_j^i, \bar{x}_{j+1}^{i'}\}$  and  $\{\bar{x}_j^i, \bar{x}_{j-1}^{i'}\}$ , with computations performed modulo  $|V|$ ;
- (d<sub>1</sub>)  $\{x_1^1\}$ ;
- (d<sub>2</sub>) for  $2 \leq j < j' \leq |V|$ , clauses of size two:  $\{\bar{x}_j^2, \bar{x}_{j'}^3\}$ .

Assume that we have a unique Hamiltonian cycle in  $G$ ,  $\mathcal{HC}_1 = \langle x^{p_1} x^{p_2} x^{p_3} \dots x^{p_{|V|-1}} x^{p_{|V|}} (x^{p_1}) \rangle$ . Note that for the time being, we could also write  $\mathcal{HC}_1 = \langle x^{p_1} x^{p_{|V|}} x^{p_{|V|-1}} \dots x^{p_3} x^{p_2} (x^{p_1}) \rangle$ , or “start” on a vertex other than  $x^{p_1}$ , cf. Introduction. This is why, without loss of generality, we set  $p_1 = 1$ , i.e., we “fix” the first vertex, and we also choose the “direction” of the cycle, by deciding, e.g., that  $x^2$  appears “before”  $x^3$  in the cycle —cf. (d<sub>1</sub>)-(d<sub>2</sub>). Define the assignment  $\mathcal{A}_1$  by  $\mathcal{A}_1(x_q^{p_q}) = \text{T}$  for  $1 \leq q \leq |V|$ , and all the other variables are set FALSE by  $\mathcal{A}_1$ . We claim that  $\mathcal{A}_1$  satisfies all the clauses.

(a<sub>1</sub>) for  $1 \leq i \leq |V|$ , if the vertex  $x^i$  has position  $j$  in the cycle, then the variable  $x_j^i$  satisfies the clause; (a<sub>2</sub>) if  $\{\bar{x}_j^i, \bar{x}_{j'}^i\}$  is not satisfied by  $\mathcal{A}_1$  for some  $i, j, j'$ , then  $\mathcal{A}_1(x_j^i) = \mathcal{A}_1(x_{j'}^i) = \text{T}$ , which means that the vertex  $x^i$  appears at least twice in the cycle;

(b<sub>1</sub>) for  $1 \leq j \leq |V|$ , if the position  $j$  is occupied by the vertex  $x^i$ , then the variable  $x_j^i$  satisfies the clause; (b<sub>2</sub>) if  $\{\bar{x}_j^i, \bar{x}_j^{i'}\}$  is not satisfied by  $\mathcal{A}_1$  for some  $i, i', j$ , then two different vertices are the  $j$ -th vertex in the cycle.

(c) If one of the two clauses is not satisfied, say the first one, then the positions  $j$  and  $j + 1$  in the cycle are occupied by two vertices not linked by any edge in  $G$ .

(d<sub>1</sub>)  $\{x_1^1\}$  is satisfied by  $\mathcal{A}_1$  thanks to the assumption on the first vertex of the cycle; (d<sub>2</sub>) if for some

$j < j'$ , the clause  $\{\bar{x}_j^2, \bar{x}_{j'}^3\}$  is not satisfied, then the vertex  $x^3$  occupies a position  $j$  smaller than the position  $j'$  of  $x^2$ , which contradicts our assumption on  $x^2$  and  $x^3$ .

Is  $\mathcal{A}_1$  unique? Assume on the contrary that another assignment,  $\mathcal{A}_2$ , also satisfies the constructed instance of U-SAT. Then by (a<sub>1</sub>) and (a<sub>2</sub>), for every  $i \in \{1, \dots, |V|\}$ , there is at least, then at most, one  $j = j(i)$  such that  $\mathcal{A}_2(x_j^i) = \text{T}$ ; by (b<sub>1</sub>) and (b<sub>2</sub>), for every  $j \in \{1, \dots, |V|\}$ , there is at least, then at most, one  $i = i(j)$  such that  $\mathcal{A}_2(x_j^i) = \text{T}$ ; so we have “a place for everything and everything in its place”, with exactly  $|V|$  variables which are TRUE by  $\mathcal{A}_2$  and an ordering of the vertices according to the one-to-one correspondence given by  $\mathcal{A}_2$ : the vertex  $x^i$  is in position  $j$  if and only if  $\mathcal{A}_2(x_j^i) = \text{T}$ . Next, thanks to the clauses (c), two vertices following each other in this ordering, including the last and first ones, are necessarily neighbours, so that this ordering is a Hamiltonian cycle,  $\mathcal{HC}_2$ . Since we have assumed the uniqueness of the Hamiltonian cycle  $\mathcal{HC}_1$  in  $G$ , the two cycles can differ only by their starting points or their “directions”. However these differences are ruled out by the clauses (d<sub>1</sub>) and (d<sub>2</sub>), so that the two cycles coincide vertex to vertex, and  $\mathcal{A}_1 = \mathcal{A}_2$ . So a YES answer for U-HAMC[U] leads to a YES answer for U-SAT.

Assume now that the answer to U-HAMC[U] is negative. If it is negative because there are at least two Hamiltonian cycles, then we have at least two assignments satisfying the instance of U-SAT: we have seen above how to construct a suitable assignment from a cycle, and different cycles obviously lead to different assignments. If there is no Hamiltonian cycle, then there is no assignment satisfying U-SAT, because such an assignment would give a cycle, as we have seen above with  $\mathcal{A}_2$ . So in both cases, a NO answer to U-HAMC[U] implies a NO answer to U-SAT.  $\diamond$

Gathering all our previous results, we obtain the following theorem.

**Theorem 15** *For every integer  $k \geq 3$ , the decision problems U-SAT, U-k-SAT and U-1-3-SAT have the same complexity as U-HAMP[U], U-HAMC[U], U-HAMP[O], U-HAMC[O], U-HAMP[D], and U-HAMC[D], up to polynomials. Therefore,*

(a) *the decision problems U-HAMP[U], U-HAMC[U], U-HAMP[O], U-HAMC[O], U-HAMP[D], and U-HAMC[D] are co-NP-hard and thus NP-hard by Remark 1;*

(b) *the decision problems U-HAMP[U], U-HAMC[U], U-HAMP[O], U-HAMC[O], U-HAMP[D], and U-HAMC[D] belong to the class DP.*  $\diamond$

Note that the membership to  $DP$  could have been proved directly.

### 3 Conclusion

By Theorem 15, for every integer  $k \geq 3$ , the three decision problems U-SAT, U- $k$ -SAT, U-1-3-SAT have the same complexity, up to polynomials, as the problem of the uniqueness of a path or of a cycle in a graph, undirected, directed, or oriented; all are co- $NP$ -hard (and  $NP$ -hard by Remark 1) and belong to the class  $DP$ , and it is thought that they are not  $DP$ -complete. Anyway, they can be found somewhere in the shaded area of Figure 3.

**Open problem.** Find a better location for any of these problems inside the hierarchy of complexity classes.

In [2], the authors wonder whether

- (A) U-SAT is  $NP$ -hard, but here we believe that what they mean is: does there exist a *polynomial* reduction from an  $NP$ -complete problem to U-SAT? i.e., they use the *second* definition of  $NP$ -hardness; finally, they show that (A) is true if and only if
- (B) U-SAT is  $DP$ -complete.

So, if one is careless and considers that U-SAT is  $NP$ -hard without checking according to which definition, one might easily jump too hastily to the conclusion that U-SAT is  $DP$ -complete, which, to our knowledge, is not known to be true or not. As for U-3-SAT, we do not know where to locate it more precisely either; in [3] the problems U- $k$ -SAT and more particularly U-3-SAT are studied, but it appears that they are versions where the given set of clauses has zero or one solution, which makes quite a difference with our problem.

### Appendix: the Proof of Theorem 9

#### A) From U-1-3-SAT to U-VC

From an arbitrary instance of U-1-3-SAT with  $m$  clauses and  $n$  variables, we mimick the reduction from 3-SAT to VC in [12], [6, pp. 54–56], and we construct the instance  $G_{VC} = (V_{VC}, E_{VC})$  of U-VC as follows (see Figure 4 for an example): we construct for each clause  $c_j$  a triangle  $T_j = \{a_j, b_j, d_j\}$ , and for each variable  $x_i$  a component  $G_i = (V_i = \{x_i, \bar{x}_i\}, E_i = \{x_i \bar{x}_i\})$ . Then we link the components  $G_i$  on the one hand, and the triangles  $T_j$  on the other hand, according to which literals appear in which clauses (“membership edges”). For each clause  $c_j = \{\ell_1, \ell_2, \ell_3\}$ , we also add the triangular set of edges  $E'_j = \{\bar{\ell}_1 \bar{\ell}_2, \bar{\ell}_1 \bar{\ell}_3, \bar{\ell}_2 \bar{\ell}_3\}$ . Finally, we set  $k = n + 2m$ .

The order of  $G_{VC}$  is  $3m + 2n$  and its number of edges is at most  $n + 9m$  (the edge sets  $E'_j$  are not necessarily disjoint).

Note already that if  $V^*$  is a vertex cover, then each triangle  $T_j$  contains at least two vertices, each component  $G_i$  at least one vertex, and  $|V^*| \geq 2m + n = k$ ; if  $|V^*| = 2m + n$ , then each triangle contains exactly two vertices, and each component  $G_i$  exactly one vertex. We can also observe that, because of the edge sets  $E'_j$ , at least two vertices among  $\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3$  belong to any vertex cover.

(a) Let us first assume that the answer to U-1-3-SAT is YES: there is a unique truth assignment 1-3-satisfying the clauses of  $\mathcal{C}$ . Then, by taking, in each  $G_i$ , the vertex corresponding to the literal which is TRUE, and in every triangle  $T_j$ , the two vertices which are linked to the two false literals of  $c_j$ , we obtain a vertex cover  $V^*$  whose size is equal to  $k$ . Moreover, once we have put the  $n$  vertices corresponding to the true literals in the vertex cover  $V^*$  in construction, we have *no choice* for the completion of  $V^*$  with  $k - n = 2m$  vertices: when we take two vertices in  $T_j$ , we *must* take the two vertices which cover the membership edges linked to the two false literals (in the example of Figure 4, the vertices  $b_1, d_1$  and  $b_2, d_2$ ). So, if another vertex cover  $V^+$  of size  $k$  exists, it must have a different distribution of its vertices over the components  $G_i$ , still with exactly one vertex in each  $G_i$ ; this in turn defines a valid truth assignment, by setting  $x_i = T$  if  $x_i \in V^+$ ,  $x_i = F$  if  $\bar{x}_i \in V^+$ . Now this assignment 1-3-satisfies  $\mathcal{C}$ , thanks in particular to our observation on the covering of the edges in  $E'_j$ . So we have two truth assignments 1-3-satisfying  $\mathcal{C}$ , contradicting the YES answer to U-1-3-SAT; therefore,  $V^*$  is the only vertex cover of size  $k$ .

(b) Assume next that the answer to U-1-3-SAT is NO: this may be either because no truth assignment 1-3-satisfies the instance, or because at least two assignments do; in the latter case, this would lead, using the same argument as in the previous paragraph, to at least two vertex covers of size  $k$ , and a NO answer to U-VC. So we are left with the case when the set of clauses  $\mathcal{C}$  cannot be 1-3-satisfied. But again, we have already seen that this would imply that no vertex cover of size (at most)  $k$  exists, since such a hypothetical vertex cover  $V^+$  would imply the existence of a suitable assignment.

We are now ready to construct an instance of U-HAMP[O]. In the sequel, we shall say “path” for “directed Hamiltonian path”.

#### B) Construction of the Instance of U-HAMP[O]

We look deeper into the proof of the  $NP$ -completeness of the problem Hamiltonian Cycle (see [6, pp. 56–60]), which uses a polynomial reduction from VC to HAMC[U] that, due to the so-called “selector vertices”, cannot cope with the problem of uniqueness; step by step, we construct an oriented graph  $H = (X, A)$  for which we will prove that:

(i) if there is a YES answer for the instance of U-1-3-SAT (which implies that there is a unique vertex cover  $V^*$  in  $G_{VC}$ , with cardinality at most  $k$ ), then there is a unique path in  $H$ ;

(ii) if there are at least two assignments 1-3-satisfying all the clauses (i.e., there are at least two vertex covers in  $G_{VC}$ , with cardinality at most  $k$ ), then there are at least two paths in  $H$ ;

(iii) if there is no assignment 1-3-satisfying the clauses (and no vertex cover in  $G_{VC}$  with cardinality at most  $k$ ), then there is no path in  $H$ .

**Step 1.** For each edge  $e = uv \in E_{VC}$ , we build one component  $H_e = (X_e, A_e)$  with 12 vertices and 14 arcs:  $X_e = \{(u, e, i), (v, e, i) : 1 \leq i \leq 6\}$ ,  $A_e = \{((u, e, i), (u, e, i + 1)), ((v, e, i), (v, e, i + 1)) : 1 \leq i \leq 5\} \cup \{((v, e, 3), (u, e, 1)), ((u, e, 3), (v, e, 1))\} \cup \{((v, e, 6), (u, e, 4)), ((u, e, 6), (v, e, 4))\}$ ; see Figure 5, which is the oriented copy of Figure 3.4 in [6, p. 57].

In the completed construction, the only vertices from this component that will be involved in any additional arcs are the vertices  $(u, e, 1)$ ,  $(u, e, 6)$ ,  $(v, e, 1)$ , and  $(v, e, 6)$ . This, together with the fact that there will be two particular vertices,  $\alpha_1$  and  $\delta$ , which will necessarily be the ends of any path, will imply that any path in the final graph  $H$  will have to meet the vertices in  $X_e$  in exactly one of the three configurations shown in Figure 6, which is the oriented copy of Figure 3.5, in [6, p. 58]. Thus, when the path meets the component  $H_e$  at  $(u, e, 1)$ , it will have to leave at  $(u, e, 6)$  and go through either (a) all 12 vertices in the component, in which case we shall say that the component is *completely visited from the u-side*, or (b) only the 6 vertices  $(u, e, i)$ ,  $1 \leq i \leq 6$ , in which case we shall say that the component is visited *in parallel* and needs two visits, i.e., another section of the path will re-visit the component, meeting the 6 vertices  $(v, e, i)$ ,  $1 \leq i \leq 6$ .

**Step 2.** We create  $n$  vertices  $\alpha_i$ ,  $1 \leq i \leq n$ , and  $2n$  arcs  $(\alpha_i, (x_i, x_i\bar{x}_i, 1))$ ,  $(\alpha_i, (\bar{x}_i, x_i\bar{x}_i, 1))$ , that is, we link  $\alpha_i$  to the “first” vertices of the component  $H_e$  whenever  $e = x_i\bar{x}_i$ . The vertices  $\alpha_i$  can be seen as literal selectors that will choose between  $x_i$  and  $\bar{x}_i$ . The vertex  $\alpha_1$  will have no other neighbours; this means in particular that it will have no in-neighbours, thus it will necessarily be the starting vertex of any Hamiltonian path, if such a path exists.

We choose an arbitrary order on the  $3m$  vertices of the triangles  $T_j$  in the graph  $G_{VC}$ , say  $\mathcal{O}_T = \langle a_1, b_1, d_1, a_2, \dots, d_m \rangle$  and an arbitrary order on the literals  $x_i, \bar{x}_i$ , say  $\mathcal{O}_\ell = \langle x_1, x_2, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n \rangle$ . For each literal  $\ell_i$  equal to  $x_i$  or  $\bar{x}_i$ , we do the following (see Figure 7 for an example):

Rule (a): If  $\ell_i$  appears  $q = q(\ell_i) \geq 0$

times in the clauses and is linked in  $G_{VC}$  to  $t_1, \dots, t_q$  where the  $t$ 's belong to the triangles  $T_j$  and follow the order  $\mathcal{O}_T$ , then we create the arcs  $((\ell_i, \ell_i\bar{\ell}_i, 6), (\ell_i, \ell_i t_1, 1))$ ,  $((\ell_i, \ell_i t_1, 6), (\ell_i, \ell_i t_2, 1))$ ,  $\dots$ ,  $((\ell_i, \ell_i t_{q-1}, 6), (\ell_i, \ell_i t_q, 1))$ .

Rule (b): We consider the triangular sets of edges  $E'_j$  described in the construction of  $G_{VC}$ .

- If  $\ell_i$  does not belong to any such edge, we create the arc  $((\ell_i, \ell_i t_q, 6), \alpha_{i+1})$  – or  $((\ell_i, \ell_i \bar{\ell}_i, 6), \alpha_{i+1})$  if  $\ell_i$  does not appear in any clause – unless  $i = n$ , in which case we create  $((\ell_i, \ell_i t_q, 6), \beta_1)$  or  $((\ell_i, \ell_i \bar{\ell}_i, 6), \beta_1)$ , where  $\beta_1$  is a new vertex that will be spoken of at the beginning of Step 3.

- If  $\ell_i$  belongs to  $s = s(\ell_i) > 0$  edges from  $E'_j$ , which link  $\ell_i$  to  $s$  literals  $\ell_{i_1}, \dots, \ell_{i_s}$  that follow the order  $\mathcal{O}_\ell$ , then we build the arc  $((\ell_i, \ell_i t_q, 6), (\ell_i, \ell_i \ell_{i_1}, 1))$  – or the arc  $((\ell_i, \ell_i \bar{\ell}_i, 6), (\ell_i, \ell_i \ell_{i_1}, 1))$  if  $q = 0$ ; next, the arcs  $((\ell_i, \ell_i \ell_{i_2}, 6), (\ell_i, \ell_i \ell_{i_2}, 1))$ ,  $\dots$ ,  $((\ell_i, \ell_i \ell_{i_{s-1}}, 6), (\ell_i, \ell_i \ell_{i_{s-1}}, 1))$  and  $((\ell_i, \ell_i \ell_{i_s}, 6), \alpha_{i+1})$ , unless  $i = n$ , in which case we create  $((\ell_i, \ell_i \ell_{i_s}, 6), \beta_1)$ .

**Remark 16** In the example of Figure 7, one can see that if a path takes, e.g., the arc  $(\alpha_1, (x_1, x_1\bar{x}_1, 1))$ , then it visits the vertices  $(x_1, x_1\bar{x}_1, 6)$ ,  $(x_1, x_1 a_1, 1)$ ,  $(x_1, x_1 a_1, 6)$ , and  $\alpha_2$ . If on the other hand, we use the arc  $(\alpha_1, (\bar{x}_1, x_1\bar{x}_1, 1))$ , we also go to  $\alpha_2$ . The same is true between  $\alpha_2$  and  $\alpha_3, \dots$ , between  $\alpha_{n-1}$  and  $\alpha_n$ , between  $\alpha_n$  and  $\beta_1$ .

We can see that so far,  $\alpha_1$  has (out-)degree 2,  $\alpha_2, \dots, \alpha_n$  have degree 4 (in- and out-degrees equal to 2), and  $\beta_1$  has (in-)degree 2.

**Step 3.** We consider the  $m$  clauses and the  $m$  corresponding triangles  $T_j$ .

We create  $2m$  vertices  $\beta_j, \beta'_j$ ,  $1 \leq j \leq m$ . As we have seen in the previous step,  $\beta_1$  has already two in-neighbours, which can be  $(\ell_n, \ell_n t_q, 6)$ , or  $(\ell_n, \ell_n \bar{\ell}_n, 6)$ , or  $(\ell_n, \ell_n \ell_{n_s}, 6)$ . We also create one more vertex  $\delta$ , which will have only in-neighbours, so that  $\alpha_1$  and  $\delta$  will necessarily be the ends of any directed Hamiltonian path, if such a path exists.

Now for the triangle  $T_j = \{a_j, b_j, d_j\}$ ,  $1 \leq j \leq m$ , associated to the clause  $c_j = \{\ell_{j_1}, \ell_{j_2}, \ell_{j_3}\}$  in the graph  $G_{VC}$ , we consider the six corresponding components  $H_{a_j b_j}$ ,  $H_{a_j d_j}$ ,  $H_{b_j d_j}$ ,  $H_{a_j \ell_{j_1}}$ ,  $H_{b_j \ell_{j_2}}$  and  $H_{d_j \ell_{j_3}}$ . The vertices  $\beta_j$  and  $\beta'_j$  can be seen as triangle selectors, intended to choose two vertices among three. With this in mind, we create the following arcs (see Figure 8), for  $j \in \{1, 2, \dots, m\}$ :

Rule (a):  $(\beta_j, (a_j, a_j b_j, 1))$ ,  $((a_j, a_j b_j, 6), (a_j, a_j d_j, 1))$ ,  $((a_j, a_j d_j, 6), (a_j, a_j \ell_{j_1}, 1))$ ,  $((a_j, a_j \ell_{j_1}, 6), \beta'_j)$ .

Rule (b):  $(\beta_j, (b_j, a_j b_j, 1)), (\beta'_j, (b_j, a_j b_j, 1)), ((b_j, a_j b_j, 6), (b_j, b_j d_j, 1)), ((b_j, b_j d_j, 6), (b_j, b_j \ell_{j_2}, 1)), ((b_j, b_j \ell_{j_2}, 6), \beta'_j)$ , plus the arc  $((b_j, b_j \ell_{j_2}, 6), \beta_{j+1})$ , unless  $j = m$ , in which case it is  $((b_j, b_j \ell_{j_2}, 6), \delta)$ .

Rule (c):  $(\beta'_j, (d_j, a_j d_j, 1)), ((d_j, a_j d_j, 6), (d_j, b_j d_j, 1)), ((d_j, b_j d_j, 6), (d_j, d_j \ell_{j_3}, 1)), ((d_j, d_j \ell_{j_3}, 6), \beta_{j+1})$ , unless  $j = m$ , in which case it is  $((d_j, d_j \ell_{j_3}, 6), \delta)$ .

**Remark 17** *In the example of Figure 8, there are three ways for going from  $\beta_1$  to  $\beta_2$  through the components  $H_{a_1 b_1}$ ,  $H_{a_1 d_1}$  and  $H_{d_1 b_1}$ .*

• *If a path starts by taking the arc  $(\beta_1, (a_1, a_1 b_1, 1))$ , then there are two possibilities, according to how we visit  $H_{a_1 b_1}$ :*

◦ *The first possibility corresponds to taking  $a_1$  and  $d_1$ , not  $b_1$ , in a vertex cover: the path completely visits the component  $H_{a_1 b_1}$  from the  $a_1$ -side, then the component  $H_{a_1 d_1}$  in parallel, then the component  $H_{a_1 x_1}$  in a so far unspecified way, then  $\beta'_1$ .*

*Next, it takes the arc  $(\beta'_1, (d_1, d_1 a_1, 1))$ , re-visits  $H_{d_1 a_1}$  in parallel, completely visits  $H_{d_1 b_1}$  from the  $d_1$ -side, then  $H_{d_1 x_3}$ , and ends this path section at  $\beta_2$ . One can see that the three components corresponding to edges incident to  $b_1$  must all be completely visited from the side opposite  $b_1$ , including the  $x_2$ -side.*

◦ *The second possibility corresponds to taking  $a_1$  and  $b_1$ , not  $d_1$ , in a vertex cover: the path follows the arc  $(\beta_1, (a_1, a_1 b_1, 1))$ , visits  $H_{a_1 b_1}$  in parallel, visits completely  $H_{a_1 d_1}$  from the  $a_1$ -side, then  $H_{a_1 x_1}$ , and  $\beta'_1$ .*

*Next, it takes the arc  $(\beta'_1, (b_1, b_1 a_1, 1))$ , goes through  $H_{b_1 a_1}$  in parallel, goes completely through  $H_{b_1 d_1}$  from the  $b_1$ -side, then  $H_{b_1 x_2}$ , and ends this path section at  $\beta_2$ . The component  $H_{d_1 x_3}$  is not yet visited.*

• *Alternatively, a path can start by taking the arc  $(\beta_1, (b_1, a_1 b_1, 1))$ ; this corresponds to taking  $b_1$  and  $d_1$ , not  $a_1$ , in a vertex cover and constitutes the third way for going from  $\beta_1$  to  $\beta_2$ . The path then completely visits  $H_{b_1 a_1}$  from the  $b_1$ -side,  $H_{b_1 d_1}$  in parallel,  $H_{b_1 x_2}$ , and  $\beta'_1$ .*

*Next, it completely visits  $H_{d_1 a_1}$  from the  $d_1$ -side,  $H_{d_1 b_1}$  in parallel,  $H_{d_1 x_3}$ , and this path section ends at  $\beta_2$ . The component  $H_{a_1 x_1}$  is not yet visited.*

*It is easy to see that these are the only three ways for going from  $\beta_1$  to  $\beta_2$  through the components  $H_{a_1 b_1}$ ,  $H_{a_1 d_1}$  and  $H_{d_1 b_1}$ , not taking into account the ways of going through the components  $H_{a_1 x_1}$ ,  $H_{b_1 x_2}$  and  $H_{d_1 x_3}$  (this issue will be treated later on, in the general case): indeed, the only possibility left would be to follow the arc  $(\beta_1, (b_1, a_1 b_1, 1))$  and visit  $H_{b_1 a_1}$  in parallel, but then the  $a_1$ -side of  $H_{b_1 a_1}$  cannot be reached.*

*The same will be true for the components  $H_{a_j b_j}$ ,  $H_{a_j d_j}$  and  $H_{d_j b_j}$  and the corresponding triangles  $T_j$ ,  $1 \leq j \leq m$ , between  $\beta_j$  and  $\beta_{j+1}$  (or between  $\beta_m$  and  $\delta$ ).*

The description of the oriented graph  $H$  is complete. Now  $\beta_1$  has increased its degree to 4, and  $\beta_2, \dots, \beta_m$  and  $\beta'_1, \dots, \beta'_m$  have degree 4. Actually, all the selectors but  $\alpha_1$  have in-degree 2 and out-degree 2 in  $H$ . These  $n$  selectors  $\alpha_i$ ,  $1 \leq i \leq n$ , and  $2m$  selectors  $\beta_j, \beta'_j$ ,  $1 \leq j \leq m$ , translate the choices we have to make when constructing a vertex cover with size  $2m + n$ : we have one choice among the  $n$  variables (take  $x_i$  or  $\bar{x}_i$ ); as for the  $m$  triangles  $T_j$  associated to the clauses, Remark 17 has shown how the selectors  $\beta_j, \beta'_j$ ,  $1 \leq j \leq m$ , can be used to choose two vertices among three. The number of selectors is one reason why there is no directed Hamiltonian path in  $H$  when the vertex covers in  $G_{VC}$  have size at least  $2m + n + 1$ .

The order of  $H$  is  $12|E_{VC}| + n + 2m + 1$ , which is at most  $12(n + 9m) + n + 2m + 1$ , so that the transformation is polynomial indeed.

We are now going to prove our claims about the existence or non-existence, uniqueness or non-uniqueness, of a directed Hamiltonian path in  $H$ .

### C) How it Works

Assume first that there is an assignment satisfying the instance of U-1-3-SAT, and therefore that there is a vertex cover  $V^*$  in  $G_{VC}$  with size  $2m + n$ . We construct a path in  $H$  in a straightforward way: every component  $H_{uv}$  ( $uv \in E_{VC}$ ) with  $\{u, v\} \subset V^*$  is visited in parallel, whereas  $H_{uv}$  is completely visited from the  $u$ -side whenever  $u \in V^*$ ,  $v \notin V^*$ . Let us have a closer look at how this works:

We start at  $\alpha_1$ , and visit completely the component  $H_{x_1 \bar{x}_1}$  from the  $x_1$ -side if  $x_1 = T$ , from the  $\bar{x}_1$ -side if  $x_1 = F$  (or, equivalently, if  $x_1 \in V^*$  or  $\bar{x}_1 \in V^*$ , respectively). If, say,  $x_1 = F$ , we then completely go through all the components corresponding to triangles  $T_j$  and involving  $\bar{x}_1$ , all from the  $\bar{x}_1$ -side; note that all the components just completely visited involve  $\bar{x}_1$  and a vertex not in  $V^*$ , by the very construction of the vertex cover  $V^*$ , which is possible because it stems from an assignment 1-3-satisfying all the clauses. Then we go through the components constructed from the edge sets  $E'_j$  and involving  $\bar{x}_1$ ; those involving a second vertex in  $V^*$  (i.e., a true literal) are visited in parallel, whereas those involving a vertex not in  $V^*$  are completely visited from the  $\bar{x}_1$ -side; then the path arrives at  $\alpha_2$ . The components involving  $x_1$ , apart from  $H_{x_1 \bar{x}_1}$ , remain completely unvisited for the time being, and the components that have been visited in parallel will have to be re-visited.



We act similarly between  $\alpha_2$  and  $\alpha_3, \dots, \alpha_n$  and  $\beta_1$ ; cf. Remark 16. When doing this, we revisit all the components that had been visited only in parallel, and completely visit the components involving a literal not in  $V^*$  and corresponding to edges in  $E'_j$ . The only components not visited yet between  $\alpha_1$  and  $\beta_1$  are those corresponding to edges between a false literal (not in  $V^*$ ) and its neighbours in the triangles  $T_j$ .

Next, starting from  $\beta_1$ , we use Remark 17 according to the three possible cases:

- (a)  $\{a_1, d_1\} \subset V^*, b_1 \notin V^*$ ,
- (b)  $\{a_1, b_1\} \subset V^*, d_1 \notin V^*$ ,
- (c)  $\{b_1, d_1\} \subset V^*, a_1 \notin V^*$ .

We give in detail only the third case, for the clause  $c_1 = \{\ell_1, \ell_2, \ell_3\}$ : we use the arc  $(\beta_1, (b_1, a_1 b_1, 1))$  and completely visit the component  $H_{a_1 b_1}$  from the  $b_1$ -side, then the component  $H_{b_1 d_1}$  in parallel, then the complete component  $H_{b_1 \ell_2}$  from the  $b_1$ -side (because if  $b_1 \in V^*$ , then  $\ell_2 \notin V^*$  and this component had not yet been visited) and end at  $\beta'_1$ . Next, we take the arc  $(\beta'_1, (d_1, d_1 a_1, 1))$ , we completely visit  $H_{d_1 a_1}$  from the  $d_1$ -side, re-visit  $H_{d_1 b_1}$  in parallel, completely visit  $H_{d_1 \ell_3}$  from the  $d_1$ -side, and this path section ends at  $\beta_2$ . Note that (a) the three components involving  $a_1$  between  $\beta_1$  and  $\beta_2$  have been completely visited, from the  $b_1$ -,  $d_1$ - or  $\ell_1$ -sides (because  $a_1 \notin V^*$  implies that  $\ell_1 \in V^*$ ); (b) any so far unvisited component involving a false literal (here, these are  $\ell_2$  and  $\ell_3$ ) and one of the vertices of the triangle  $T_1$  (here  $b_1$  and  $d_1$ ) has now been completely visited from the triangle sides (here from the  $b_1$ - and  $d_1$ -sides).

We act similarly between  $\beta_2$  and  $\beta_3, \dots, \beta_m$  and  $\delta$ ; cf. the end of Remark 17. The ultimate section takes us between  $\beta_m$  and  $\delta$ , the final vertex, and we have indeed built a directed Hamiltonian path, from  $\alpha_1$  to  $\delta$ , in the oriented graph  $H$ .

Obviously, two different assignments 1-3-satisfying all the clauses lead, following the above process, to two different paths in  $H$ . We still want to prove that 1) if no assignment 1-3-satisfying all the clauses exists, then no path exists, and 2) a unique assignment 1-3-satisfying all the clauses leads to a unique path.

1) We assume that there is a directed Hamiltonian path  $\mathcal{HP}$  in  $H$ , and exhibit an assignment 1-3-satisfying all the clauses.

Let us consider the vertex  $\alpha_1$ ; its two out-neighbours in  $H$  are  $(x_1, x_1 \bar{x}_1, 1)$  and  $(\bar{x}_1, x_1 \bar{x}_1, 1)$ . So exactly one of the arcs  $(\alpha_1, (x_1, x_1 \bar{x}_1, 1))$ ,  $(\alpha_1, (\bar{x}_1, x_1 \bar{x}_1, 1))$  is part of  $\mathcal{HP}$ . The same is true for  $\alpha_i$ ,  $1 < i \leq n$ . As a consequence, we can define a valid assignment of the variables  $x_i$ ,  $1 \leq i \leq n$ , by setting  $x_i = T$  if and only if the arc  $(\alpha_i, (x_i, x_i \bar{x}_i, 1))$  belongs to  $\mathcal{HP}$ .

Next, we address the vertices  $\beta_j$ ,  $1 \leq j \leq m$ . The construction in Steps 2 and 3 is such that each vertex  $\beta_j$ ,  $1 \leq j \leq m$ , has two out-neighbours,  $(a_j, a_j b_j, 1)$  and  $(b_j, a_j b_j, 1)$ .

This implies that the assignment defined above is such that there is at least one true literal in each clause. Indeed, if we assume that the clause  $c_j = \{\ell_{j_1}, \ell_{j_2}, \ell_{j_3}\}$  does not contain any true literal, then the component  $H_{a_j \ell_{j_1}}$  is completely visited by  $\mathcal{HP}$  from the  $a_j$ -side, because  $\ell_{j_1} = F$  implies that the arc  $(\alpha_j, (\ell_{j_1}, \ell_{j_1} \bar{\ell}_{j_1}, 1))$  is not part of  $\mathcal{HP}$  and does not give access to the  $\ell_{j_1}$ -side. Similarly, the components  $H_{b_j \ell_{j_2}}$  and  $H_{d_j \ell_{j_3}}$  are completely visited by  $\mathcal{HP}$  from the  $b_j$ - and  $d_j$ -sides, respectively. This in turn implies that in  $\mathcal{HP}$  we have the arcs  $((a_j, a_j \ell_{j_1}, 6), \beta'_j)$ ,  $(\beta_j, (a_j, a_j b_j, 1))$ ,  $((d_j, d_j \ell_{j_3}, 6), \beta_{j+1})$  and  $(\beta'_j, (d_j, d_j a_j, 1))$  — replace  $\beta_{j+1}$  by  $\delta$  if  $j = m$ . Now how does  $\mathcal{HP}$  go through  $(b_j, b_j \ell_{j_2}, 6)$ ? It cannot be with the help of the  $\ell_{j_2}$ -side of  $H_{b_j \ell_{j_2}}$ , so there are only two possibilities left: but if it is with the arc  $((b_j, b_j \ell_{j_2}, 6), \beta'_j)$ , then  $\beta'_j$  has three neighbours in  $\mathcal{HP}$ , which is impossible; and if it is with the arc  $((b_j, b_j \ell_{j_2}, 6), \beta_{j+1})$ , then in  $\mathcal{HP}$ , the vertex  $\beta_{j+1}$  has two in-neighbours, which is impossible — including when  $j = m$  and  $\beta_{j+1}$  is replaced by  $\delta$ . From this we can conclude that the clause  $c_j = \{\ell_{j_1}, \ell_{j_2}, \ell_{j_3}\}$  contains at least one true literal.

Assume next that one clause has at least two true literals: without loss of generality,  $c_j = \{\ell_{j_1}, \ell_{j_2}, \ell_{j_3}\}$  is such that  $\ell_{j_1} = \ell_{j_2} = T$ . Then  $\mathcal{HP}$  has no access to the  $\bar{\ell}_{j_1}$ - and  $\bar{\ell}_{j_2}$ - sides of the components involving  $\bar{\ell}_{j_1}$  or  $\bar{\ell}_{j_2}$ , but, since there is the edge  $\bar{\ell}_{j_1} \bar{\ell}_{j_2}$  in  $G_{VC}$ , this means that  $\mathcal{HP}$  has no way of visiting the component  $H_{\bar{\ell}_{j_1} \bar{\ell}_{j_2}}$ . Therefore, we have just established that the assignment derived from the path  $\mathcal{HP}$  1-3-satisfies all the clauses. This, together with the fact that two assignments 1-3-satisfying the clauses lead to two paths, shows that a NO answer to the instance of U-1-3-SAT implies a NO answer for the constructed instance  $H$  of U-HAMP[O].

2) We want to show that a unique assignment  $\mathcal{A}$  1-3-satisfying all the clauses leads to a unique path in  $H$ . This assignment leads to a unique vertex cover  $V^*$ , of size  $n + 2m$ , in  $G_{VC}$ , and to a path in  $H$ , as already seen. Now assume that we have a second path, so that these two paths, which we call  $\mathcal{HP}_1$  and  $\mathcal{HP}_2$ , both lead, with the above description in 1), to the same  $\mathcal{A}$  and the same  $V^*$ .

The two paths must behave in the same way over the components  $H_{x_i \bar{x}_i}$ ,  $1 \leq i \leq n$ : otherwise, from them we could define two different valid assignments, which would both, as seen previously, 1-3-satisfy the clauses.

Next, consider the clause  $c_j = \{\ell_{j_1}, \ell_{j_2}, \ell_{j_3}\}$  and assume without loss of generality that  $\mathcal{A}(\ell_{j_1}) = \text{T}$ ,  $\mathcal{A}(\ell_{j_2}) = \mathcal{A}(\ell_{j_3}) = \text{F}$ ; this implies, for both  $\mathcal{HP}_1$  and  $\mathcal{HP}_2$ , that the components  $H_{\ell_{j_1}\bar{\ell}_{j_1}}$ ,  $H_{\ell_{j_2}\bar{\ell}_{j_2}}$  and  $H_{\ell_{j_3}\bar{\ell}_{j_3}}$  are completely visited from the  $\ell_{j_1}$ -,  $\ell_{j_2}$ - and  $\bar{\ell}_{j_3}$ -sides, respectively, so that both paths have no access to the  $\ell_{j_2}$ - nor  $\ell_{j_3}$ -sides. As a consequence, between  $\beta_j$  and  $\beta_{j+1}$  (or  $\beta_m$  and  $\delta$ ), the components  $H_{b_j\ell_{j_2}}$  and  $H_{d_j\ell_{j_3}}$  are completely visited from the  $b_j$ - and  $d_j$ -sides, respectively. Then necessarily the following arcs belong to  $\mathcal{HP}_1$  and  $\mathcal{HP}_2$ , going along the  $d_j$ -side:

$((d_j, d_j\ell_{j_3}, 6), \beta_{j+1})$  – or  $((d_j, d_j\ell_{j_3}, 6), \delta)$  –,  
 $((d_j, d_jb_j, 6), (d_j, d_j\ell_{j_3}, 1))$ ,  
 $((d_j, d_ja_j, 6), (d_j, d_jb_j, 1))$ ,  
 $(\beta'_j, (d_j, d_ja_j, 1))$ ;

and going along the  $b_j$ -side:

$((b_j, b_j\ell_{j_2}, 6), \beta'_j)$  (because  $\beta_{j+1}$  – or  $\delta$  – cannot have two in-neighbours), and  $((b_j, b_jd_j, 6), (b_j, b_j\ell_{j_2}, 1))$ ,  
 $((b_j, b_ja_j, 6), (b_j, b_jd_j, 1))$ .

The component  $H_{b_jd_j}$  must be visited in parallel, and it is  $(\beta_j, (b_j, b_ja_j, 1))$  that belongs to the two paths.

We can see that all the components containing  $a_j$ , in particular  $H_{a_j\ell_{j_1}}$ , must be completely visited from the sides opposite  $a_j$ . So far, we have proved that the two paths  $\mathcal{HP}_1$  and  $\mathcal{HP}_2$  behave identically between  $\beta_j$  and  $\beta_{j+1}$  (or  $\beta_m$  and  $\delta$ ), including on the components corresponding to membership edges (between literals and triangles).

Consider now what happens between  $\alpha_i$  and  $\alpha_{i+1}$  (or  $\alpha_n$  and  $\beta_1$ ). Assume without loss of generality that, say,  $\mathcal{A}(x_i) = \text{T}$ , so that  $(\alpha_i, (x_i, x_i\bar{x}_i, 1))$  is part of the two paths. Consider the components involving  $x_i$  in  $H$ : there are first those involving vertices of type  $a$ ,  $b$  or  $d$ , which translate the membership of  $x_i$  to a certain number of clauses, and which we called  $t_1, \dots, t_q$  in Step 2(a); we have already seen in the previous paragraph that these components must be completely visited from the  $x_i$ -side.

Then we consider the components created from the edges in  $E'_j$ , cf. Step 2(b); here, some edges in  $G_{VC}$  can have both ends in  $V^*$ , but, using similar arguments as before, we can see that the two paths will visit all these components in the same way: consider the clause  $c_j = \{\ell_{j_1}, \ell_{j_2}, \ell_{j_3}\}$  and the corresponding set  $E'_j$ , and assume without loss of generality that  $x_i = \ell_{j_1}$ , so that  $\mathcal{A}(\ell_{j_1}) = \text{T}$ , which implies that  $\mathcal{A}(\ell_{j_2}) = \mathcal{A}(\ell_{j_3}) = \text{F}$ ; then  $H_{\bar{\ell}_{j_1}\bar{\ell}_{j_2}}$  and  $H_{\bar{\ell}_{j_1}\bar{\ell}_{j_3}}$  must be completely visited from the  $\bar{\ell}_{j_2}$ - and  $\bar{\ell}_{j_3}$ -sides, respectively, and  $H_{\bar{\ell}_{j_2}\bar{\ell}_{j_3}}$  in parallel, i.e., the two paths have no choice but to behave identically on all three components. As for the components with  $\bar{x}_i$ , they must be completely visited from the side which is not

the side of  $\bar{x}_i$ .

So we have just proved that the two paths are identical between  $\alpha_1$  and  $\alpha_2, \dots, \alpha_n$  and  $\beta_1$ .

Therefore, the two paths (between  $\alpha_1$  and  $\delta$ ) are one and the same.  $\diamond$

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### **Contribution of individual authors to the creation of a scientific article (ghostwriting policy)**

The two authors contribute equally to all the aspects of this work.

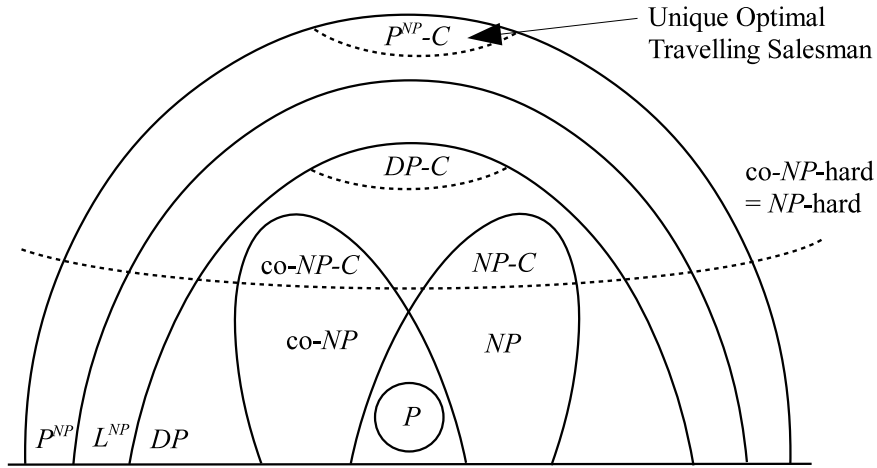


Figure 1: Some classes of complexity.

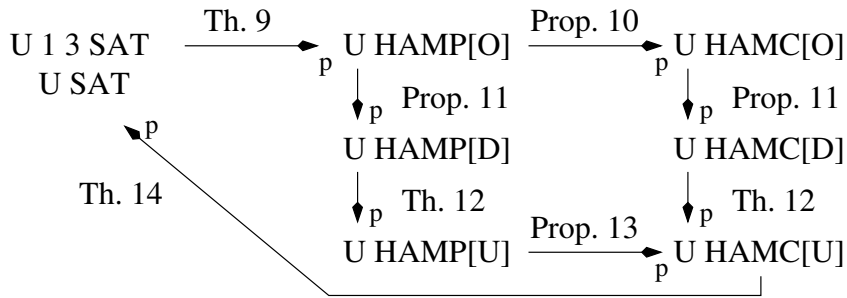


Figure 2: The chain of polynomial reductions, where an arrow from  $\pi_1$  to  $\pi_2$  stands for the relation  $\pi_1 \leq_p \pi_2$ .

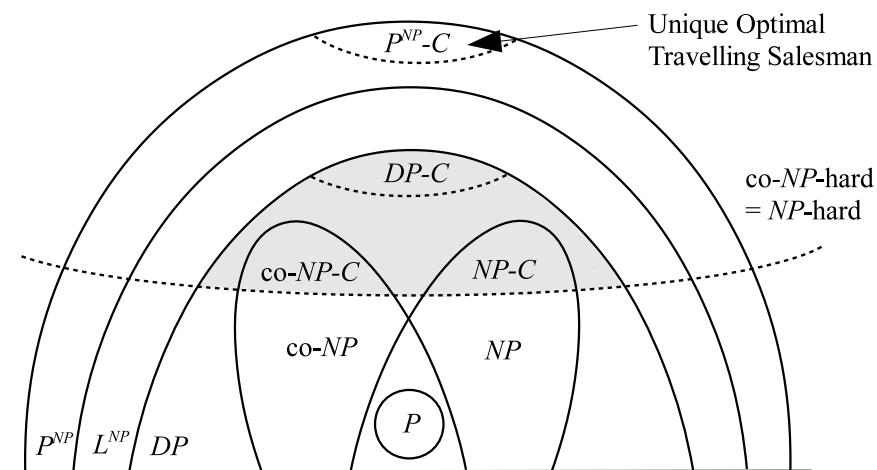


Figure 3: Some classes of complexity: Figure 1 re-visited.

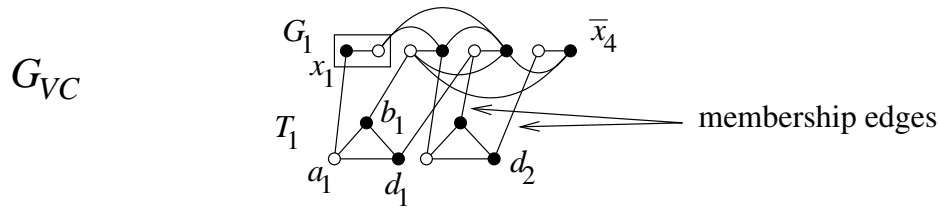


Figure 4: Illustration of the undirected graph constructed for the reduction from U-1-3-SAT to U-VC, with four variables and two clauses,  $c_1 = \{x_1, x_2, x_3\}$ ,  $c_2 = \{\bar{x}_2, x_3, x_4\}$ . Here,  $k = 8$ , and the black vertices form the (not unique) vertex cover  $V^*$  of size eight corresponding to the (not unique) truth assignment  $x_1 = T$ ,  $x_2 = F$ ,  $x_3 = F$ ,  $x_4 = F$  1-3-satisfying the clauses. As soon as we set  $V^* \cap (V_1 \cup V_2 \cup V_3 \cup V_4) = \{x_1, \bar{x}_2, \bar{x}_3, \bar{x}_4\}$ , the other vertices in  $V^*$  are forced.

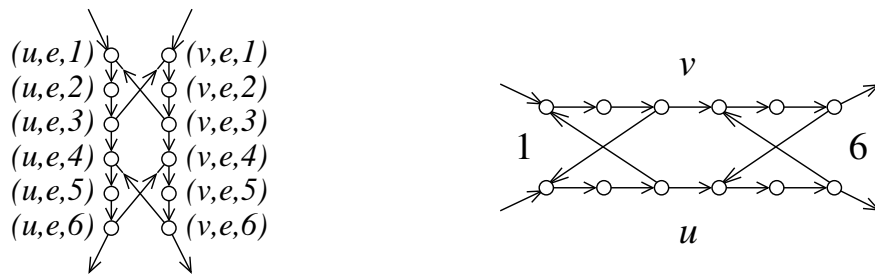


Figure 5: Two possible representations of the same component  $H_e$  for the edge  $e = uv \in E_{VC}$  (Step 1).

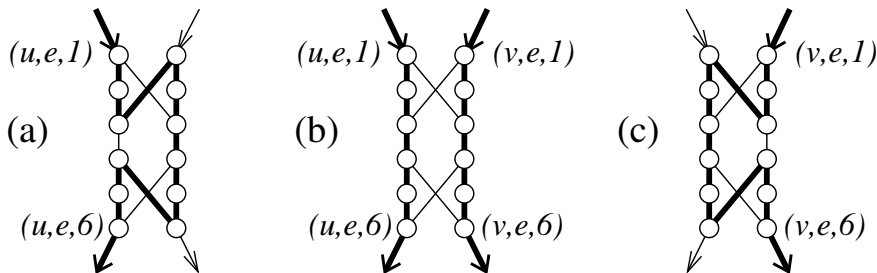


Figure 6: The three ways of going through the component  $H_e$  (Step 1). The arrows inside  $H_e$  are not represented.

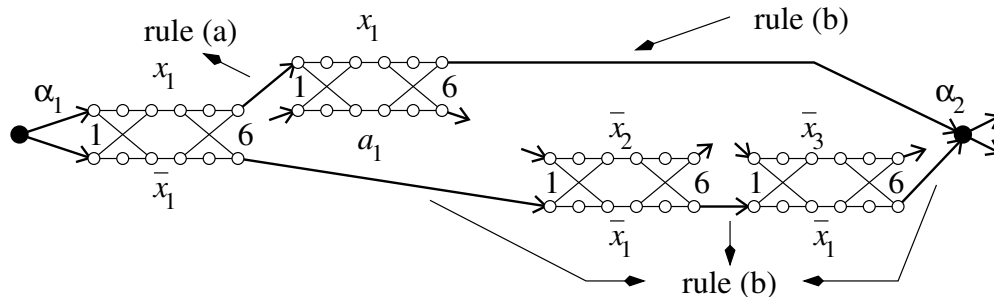


Figure 7: The example of the literals  $x_1$  and  $\bar{x}_1$  from Figure 4 (Step 2); here, Rule (a) applies with  $q(x_1) = 1$ ,  $q(\bar{x}_1) = 0$ , Rule (b) with  $s(\bar{x}_1) = 2$ . The arrows inside  $H_e$  are not represented.

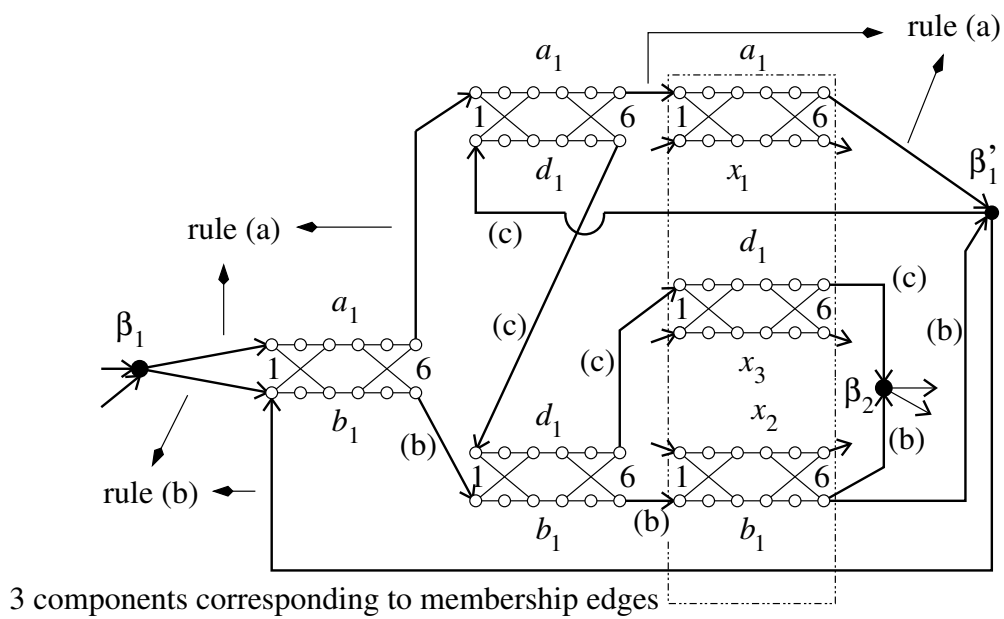


Figure 8: The treatment of the triangle  $T_1$  from Figure 4 (Step 3). The arrows inside  $H_e$  are not represented.