Study of Complex Oscillation of Solutions of a Second Order Linear Differential Equation With Entire Coefficients of (α, β, γ)-Order

BENHARRAT BELAÏDI*
Department of Mathematics, Laboratory of Pure and Applied Mathematics
University of Mostaganem (UMAB)
B. P. 227 Mostaganem
ALGERIA

TANMAY BISWAS
Rajbari, Rabindrapally, R. N. Tagore Road, P.O.- Krishnagar,
P.S. Kotwali, Dist-Nadia, PIN-741101, West Bengal
INDIA

Abstract: In this paper, we deal with the complex oscillation of solutions of linear differential equation. We mainly study the interaction between the growth, zeros of solutions with the coefficients of second order linear differential equations in terms of (α, β, γ)-order and obtain some results in general form which considerably extend some results of [5], [18] and [21].

Key-Words: Linear differential equations, (α, β, γ)-order, (α, β, γ)-exponent of convergence of zero sequence.
AMS Subject Classification (2010): 30D35, 34M10.

Received: July 23, 2021. Revised: April 20, 2022. Accepted: May 25, 2022. Published: June 14, 2022.

1 Introduction, Definitions and Notations

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of entire and meromorphic functions which are available in [11, 19, 25] and therefore we do not explain those in details. The theory of complex linear equations has been developed since 1960s. Many authors have investigated the second order linear differential equation

$$f'' + A(z)f = 0, \qquad (1)$$

where $A(z)$ is an entire function or a meromorphic function of finite order or finite iterated order, and have obtained many results about the interaction between the solutions and the coefficient of (1) (see [1, 2, 3, 18]). Moreover some authors have investigated the exponent of convergence of zero sequence and pole-sequence of the solutions of second order differential equations and have obtained some interesting results (see [7, 8, 18, 24]). Mulyava et al. [20] have investigated the properties of solutions of a heterogeneous differential equation of the second order under some different conditions using the concept of generalized order. For details one may see [20].

We denote the linear measure and the logarithmic measure of a set $E \subset (1, +\infty)$ by $mE = \int_E dt$ and $m_l E = \int_E \frac{dx}{x}$. Now let L be a class of continuous non-negative on $(-\infty, +\infty)$ function α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$.

Recently Heittokangas et al. [14] have introduced a new concept of φ-order of entire and meromorphic function considering φ as subadditive function. For details one may see [14]. Now it is interesting to investigate the interaction between the growth, zeros of solutions with the coefficients of second order linear differential equations using the revised idea of Heittokangas et al. [14], which is the main aim of this paper. For this purpose, we introduce the definition of the (α, β, γ)-order of a meromorphic function in the following way:

Definitions 1. *Let $\alpha \in L$, $\beta \in L$ and $\gamma \in L$. The (α, β, γ)-order denoted by $\sigma_{(\alpha,\beta,\gamma)}[f]$ and (α, β, γ)-lower order denoted by $\mu_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f are, respectively, defined by*

$$\sigma_{(\alpha,\beta,\gamma)}[f] = \limsup_{r \to +\infty} \frac{\alpha(\log T(r, f))}{\beta(\log \gamma(r))}$$

and

$$\mu_{(\alpha,\beta,\gamma)}[f]=\liminf_{r\to+\infty}\frac{\alpha(\log T(r,f))}{\beta(\log\gamma(r))}.$$

Remark 1. *Let f be a meromorphic function. One can see that $\alpha(r)=\log^{[p]}r$, $(p\geq 0)$, $\beta(r)=\log^{[q]}r$, $(q\geq 0)$ and $\gamma(r)=r$ belong to the class L, where $\log^{[k]}x=\log(\log^{[k-1]}x)$ $(k\geq 1)$, with convention that $\log^{[0]}x=x$. So, when $p=0$ and $q=0$, i.e., $\alpha(r)=\beta(r)=r$, the Definition 1 coincides with the usual order and lower order, when $\alpha(r)=\log^{[p-1]}r$ $(p\geq 1)$ and $\beta(r)=r$, we obtain the iterated p-order and iterated lower p-order (see [18, 22]), moreover when $\alpha(r)=\log^{[p-1]}r$ and $\beta(r)=\log^{[q-1]}r$, $(p\geq q\geq 1)$, we get the (p,q)-order and lower (p,q)-order (see [15, 16]). Further, if $\alpha(r)=\varphi(e^r)$, where φ is an increasing unbounded function on $[1,+\infty)$ and $\beta(r)=r$, we obtain the φ-order and the lower φ-order (see [4, 9]). Finally if $\alpha(r)=\beta(r)=r$ and $\gamma(r)=\varphi(r)$, where $\varphi:(R_0,+\infty)\to(0,+\infty)$ is a non-decreasing unbounded function satisfying the condition $\varphi(a+b)\leq\varphi(a)+\varphi(b)$ for all $a,b\geq R_0$, we obtain the new definition of φ-order and the lower φ-order introduced by Heittokangas et al. [14].*

Similarly to Definition 1, we can also define the (α,β,γ)-exponent of convergence of the zero-sequence and (α,β,γ)-exponent of convergence of the distinct zero sequence of a meromorphic function f in the following way:

Definitions 2. *Let $\alpha\in L$, $\beta\in L$ and $\gamma\in L$. The (α,β,γ)-exponent of convergence of the zero-sequence denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f is defined by*

$$\lambda_{(\alpha,\beta,\gamma)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log n(r,1/f))}{\beta(\log\gamma(r))}.$$

Similarly, the (α,β,γ)-exponent of convergence of the distinct zero-sequence denoted by $\overline{\lambda}_{(\alpha,\beta,\gamma)}[f]$ of f is defined by

$$\overline{\lambda}_{(\alpha,\beta,\gamma)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log\overline{n}(r,1/f))}{\beta(\log\gamma(r))}.$$

We say that $\alpha\in L_1$, if $\alpha(a+b)\leq\alpha(a)+\alpha(b)+c$ for all $a,b\geq R_0$ and fixed $c\in(0,+\infty)$. Further we say that $\alpha\in L_2$, if $\alpha\in L$ and $\alpha(x+O(1))=(1+o(1))\alpha(x)$ as $x\to+\infty$. Finally, $\alpha\in L_3$, if $\alpha\in L$ and $\alpha(a+b)\leq\alpha(a)+\alpha(b)$ for all $a,b\geq R_0$, i.e., α is subadditive. Clearly $L_3\subset L_1$.

Particularly, when $\alpha\in L_3$, then one can easily verify that $\alpha(mr)\leq m\alpha(r)$, $m\geq 2$ is an integer. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0,+\infty)$ and satisfies $\alpha(0)\geq 0$, then for $t\in[0,1]$,

$$\alpha(tx)=\alpha(tx+(1-t)\cdot 0)$$

$$\geq t\alpha(x)+(1-t)\alpha(0)\geq t\alpha(x),$$

so that by choosing $t=\frac{a}{a+b}$ or $t=\frac{b}{a+b}$,

$$\begin{aligned}\alpha(a+b)&=\frac{a}{a+b}\alpha(a+b)+\frac{b}{a+b}\alpha(a+b)\\&\leq\alpha\Big(\frac{a}{a+b}(a+b)\Big)+\alpha\Big(\frac{b}{a+b}(a+b)\Big)\\&=\alpha(a)+\alpha(b),\quad a,b\geq 0.\end{aligned}$$

As a non-decreasing, subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r)\leq\alpha(r+R_0)\leq\alpha(r)+\alpha(R_0)$$

for any $R_0\geq 0$. This yields that $\alpha(r)\sim\alpha(r+R_0)$ as $r\to+\infty$.

Now we add two conditions on α, β and γ: (i) Always $\alpha\in L_1$, $\beta\in L_2$ and $\gamma\in L_3$; and (ii) $\alpha(\log^{[p]}x)=o(\beta(\log\gamma(x)))$, $p\geq 2$ is an integer as $x\to+\infty$.

Throughout this paper, we assume that α, β and γ always satisfy the above two conditions unless otherwise specifically stated.

Proposition 1. *Let f_1, f_2 be non-constant meromorphic functions with $\sigma_{(\alpha,\beta,\gamma)}[f_1]$ and $\sigma_{(\alpha,\beta,\gamma)}[f_2]$ as their (α,β,γ)-order. Then*
(i) $\sigma_{(\alpha,\beta,\gamma)}[f_1\pm f_2]\leq\max\{\sigma_{(\alpha,\beta,\gamma)}[f_1],\sigma_{(\alpha,\beta,\gamma)}[f_2]\}$;
(ii) $\sigma_{(\alpha,\beta,\gamma)}[f_1\cdot f_2]\leq\max\{\sigma_{(\alpha,\beta,\gamma)}[f_1],\sigma_{(\alpha,\beta,\gamma)}[f_2]\}$;
(iii) If $\sigma_{(\alpha,\beta,\gamma)}[f_1]\neq\sigma_{(\alpha,\beta,\gamma)}[f_2]$, then $\sigma_{(\alpha,\beta,\gamma)}[f_1\pm f_2]=\max\{\sigma_{(\alpha,\beta,\gamma)}[f_1],\sigma_{(\alpha,\beta,\gamma)}[f_2]\}$;
(iv) If $\sigma_{(\alpha,\beta,\gamma)}[f_1]\neq\sigma_{(\alpha,\beta,\gamma)}[f_2]$, then $\sigma_{(\alpha,\beta,\gamma)}[f_1\cdot f_2]=\max\{\sigma_{(\alpha,\beta,\gamma)}[f_1],\sigma_{(\alpha,\beta,\gamma)}[f_2]\}$.

Proof. (i) Without loss of generality, we assume that $\sigma_{(\alpha,\beta,\gamma)}[f_1]\leq\sigma_{(\alpha,\beta,\gamma)}[f_2]<+\infty$. From the definition of (α,β,γ)-order, for any $\varepsilon>0$, we obtain for all sufficiently large values of r that

$$T(r,f_1)<\exp(\alpha^{-1}((\sigma_{(\alpha,\beta,\gamma)}[f_1]+\varepsilon)\beta(\log\gamma(r))))\tag{2}$$

and

$$T(r,f_2)<\exp(\alpha^{-1}((\sigma_{(\alpha,\beta,\gamma)}[f_2]+\varepsilon)\beta(\log\gamma(r)))).\tag{3}$$

Since $T(r,f_1\pm f_2)\leq T(r,f_1)+T(r,f_2)+\log 2$ for all large r, we get from (2) and (3) for all sufficiently large values of r that

$$\begin{aligned} T(r,f_1 \pm f_2) &< 2\exp(\alpha^{-1}((\sigma_{(\alpha,\beta,\gamma)}[f_2] \\ &\quad + \varepsilon)\beta(\log\gamma(r)))) + \log 2 \\ i.e.,\ T(r,f_1 \pm f_2) &< 3\exp(\alpha^{-1}((\sigma_{(\alpha,\beta,\gamma)}[f_2] \\ &\quad + \varepsilon)\beta(\log\gamma(r)))) \\ i.e.,\ \log T(r,f_1 \pm f_2) &< \alpha^{-1}((\sigma_{(\alpha,\beta,\gamma)}[f_2] \\ &\quad + \varepsilon)\beta(\log\gamma(r))) + \log 3 \\ i.e.,\ \alpha(\log T(r,f_1 \pm f_2)) &< (\sigma_{(\alpha,\beta,\gamma)}[f_2] \\ &\quad + \varepsilon)\beta(\log\gamma(r))) \\ &\quad + \alpha(\log 3) + c,\ \ (c > 0)\,, \end{aligned}$$

which implies that

$$\limsup_{r\to+\infty} \frac{\alpha(\log T(r,f_1 \pm f_2))}{\beta(\log\gamma(r)))} \le \sigma_{(\alpha,\beta,\gamma)}[f_2] + \varepsilon$$

holds for any $\varepsilon > 0$. Hence

$$\sigma_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] \le \max\{\sigma_{(\alpha,\beta,\gamma)}[f_1], \sigma_{(\alpha,\beta,\gamma)}[f_2]\}. \tag{4}$$

(iii) Further without loss of any generality, let $\sigma_{(\alpha,\beta,\gamma)}[f_1] < \sigma_{(\alpha,\beta,\gamma)}[f_2] < +\infty$ and $f = f_1 \pm f_2$. Then in view of (4) we get that $\sigma_{(\alpha,\beta,\gamma)}[f] \le \sigma_{(\alpha,\beta,\gamma)}[f_2]$. As, $f_2 = \pm(f - f_1)$ and in this case we obtain that $\sigma_{(\alpha,\beta,\gamma)}[f_2] \le \max\ \{\sigma_{(\alpha,\beta,\gamma)}[f], \sigma_{(\alpha,\beta,\gamma)}[f_1]\}$. As we assume that $\sigma_{(\alpha,\beta,\gamma)}[f_1] < \sigma_{(\alpha,\beta,\gamma)}[f_2]$, therefore we have $\sigma_{(\alpha,\beta,\gamma)}[f_2] \le \sigma_{(\alpha,\beta,\gamma)}[f]$ and hence $\sigma_{(\alpha,\beta,\gamma)}[f] = \sigma_{(\alpha,\beta,\gamma)}[f_2] = \max\{\sigma_{(\alpha,\beta,\gamma)}[f_1], \sigma_{(\alpha,\beta,\gamma)}[f_2]\}$.

(ii) and (iv) Similarly, from $T(r, f_1 \cdot f_2) \le T(r, f_1) + T(r, f_2)$ for all large r, we can also get that

$$\sigma_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] \le \max\{\sigma_{(\alpha,\beta,\gamma)}[f_1],\ \sigma_{(\alpha,\beta,\gamma)}[f_2]\}$$

and if $\sigma_{(\alpha,\beta,\gamma)}[f_1] \ne \sigma_{(\alpha,\beta,\gamma)}[f_2]$, then

$$\sigma_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] = \max\{\sigma_{(\alpha,\beta,\gamma)}[f_1],\ \sigma_{(\alpha,\beta,\gamma)}[f_2]\},$$

which completes the proof of Proposition 1. □

Proposition 2. *Let f_1, f_2 be non-constant meromorphic functions with $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1]$ and $\sigma_{(\alpha(\log),\beta,\gamma)}[f_2]$ as their $(\alpha(\log), \beta, \gamma)$ -order. Then*
(i) $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1 \pm f_2] \le \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[f_1], \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]\}$;
(ii) $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1 \cdot f_2] \le \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[f_1], \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]\}$;
(iii) If $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1] \ne \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]$, then $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1 \pm f_2] = \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[f_1], \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]\}$;
(iv) If $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1] \ne \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]$, then $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1 \cdot f_2] = \max\ \{\sigma_{(\alpha(\log),\beta,\gamma)}[f_1], \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]\}$.

Since $\alpha(a+b) \le \alpha(a)+\alpha(b)+c$ for all $a, b \ge R_0$ and fixed $c \in (0, +\infty)$, the proof of Proposition 2 would run parallel to that of Proposition 1. We omit the details.

Proposition 3. *(i) If f is an entire function, then*

$$\begin{aligned} \sigma_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(\log T(r,f))}{\beta(\log\gamma(r))} \\ &= \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} M(r,f))}{\beta(\log\gamma(r))} \end{aligned}$$

and

$$\begin{aligned} \mu_{(\alpha,\beta,\gamma)}[f] &= \liminf_{r\to+\infty} \frac{\alpha(\log T(r,f))}{\beta(\log\gamma(r))} \\ &= \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]} M(r,f))}{\beta(\log\gamma(r))}. \end{aligned}$$

(ii) If f is a meromorphic function, then

$$\begin{aligned} \lambda_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(\log n(r,1/f))}{\beta(\log\gamma(r))} \\ &= \limsup_{r\to+\infty} \frac{\alpha(\log N(r,1/f))}{\beta(\log\gamma(r))} \end{aligned}$$

and

$$\begin{aligned} \overline{\lambda}_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(\log \overline{n}(r,1/f))}{\beta(\log\gamma(r))} \\ &= \limsup_{r\to+\infty} \frac{\alpha(\log \overline{N}(r,1/f))}{\beta(\log\gamma(r))}. \end{aligned}$$

Proof. (i) By the inequality $T(r, f) \le \log^+ M(r, f) \le \frac{R+r}{R-r} T(R, f)$ $(0 < r < R)$ (cf. [11]) for an entire function f, set $R = \eta r$ $(\eta > 1)$, we have

$$T(r,f) \le \log^+ M(r,f) \le \frac{\eta+1}{\eta-1} T(\eta r, f). \tag{5}$$

By (5), $\alpha(a + b) \le \alpha(a) + \alpha(b) + c$ for all $a, b \ge R_0$ and fixed $c \in (0, +\infty)$, $\beta((1 + o(1))x) = (1 + o(1))\beta(x)$ as $x \to +\infty$ and $\gamma(a + b) \le \gamma(a) + \gamma(b)$ for all $a, b \ge R_0$, it is easy to see that conclusion (i) holds.

(ii) Without loss of generality, assume that $f(0) \ne 0$, then $N(r, 1/f) = \int_0^r \frac{n(t,1/f)}{t} dt$. We get for $0 < r_0 < r$

$$\begin{aligned} N(r,1/f) - N(r_0,1/f) &= \int_{r_0}^{r} \frac{n(t,1/f)}{t} dt \\ &\le n(r,1/f) \log \frac{r}{r_0}, \end{aligned}$$

that is

$$N(r,1/f)\leq N(r_0,1/f)+n(r,1/f)\times\log\frac{r}{r_0}\ (0<r_0<r),$$

$$i.e.,\ N(r,1/f)\leq\Big(1+\frac{N(r_0,1/f)}{n(r,1/f)\log\frac{r}{r_0}}\Big)\times n(r,1/f)\log\frac{r}{r_0}\ (0<r_0<r),$$

which implies that

$$\log N(r,1/f)\leq\log n(r,1/f)+\log\log r+\log\Big(1-\frac{\log r_0}{\log r}\Big)+\log\Big(1+\frac{N(r_0,1/f)}{n(r,1/f)\log\frac{r}{r_0}}\Big)\ (0<r_0<r), \tag{6}$$

then by the condition on α and (6), we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha(\log N(r,1/f))}{\beta(\log\gamma(r))}$$

$$\leq\limsup_{r\to+\infty}\frac{\alpha(\log n(r,1/f))}{\beta(\log\gamma(r))}+\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}r)}{\beta(\log\gamma(r))}+\limsup_{r\to+\infty}\frac{\alpha\Big(\log\Big(1-\frac{\log r_0}{\log r}\Big)\Big)}{\beta(\log\gamma(r))}+\limsup_{r\to+\infty}\frac{\alpha\Big(\log\Big(1+\frac{N(r_0,1/f)}{n(r,1/f)\log\frac{r}{r_0}}\Big)\Big)}{\beta(\log\gamma(r))}+\limsup_{r\to+\infty}\frac{c}{\beta(\log\gamma(r))}$$

$$=\limsup_{r\to+\infty}\frac{\alpha(\log n(r,1/f))}{\beta(\log\gamma(r))},\ \ (c>0), \tag{7}$$

since $\alpha(\log^{[2]}x)=o(\beta(\log\gamma(x)))$ as $x\to+\infty$ we have $\frac{\alpha(\log^{[2]}r)}{\beta(\log\gamma(r))}\to 0$ as $r\to+\infty$.

On the other hand, we have

$$N(er,1/f)=\int_0^{er}\frac{n(t,1/f)}{t}dt\geq\int_r^{er}\frac{n(t,1/f)}{t}dt\geq n(r,1/f)\log e=n(r,1/f). \tag{8}$$

By (8), we have

$$\limsup_{r\to+\infty}\frac{\alpha(\log N(er,1/f))}{\beta(\log\gamma(r))}\geq\limsup_{r\to+\infty}\frac{\alpha(\log n(r,1/f))}{\beta(\log\gamma(r))}.$$

By the conditions $\beta((1+o(1))x)=(1+o(1))\beta(x)$ as $x\to+\infty$ and $\gamma(er)\leq\gamma(3r)\leq 3\gamma(r)$, we can write

$$\limsup_{r\to+\infty}\frac{\alpha(\log N(er,1/f))}{\beta(\log\gamma(r))}\leq\limsup_{r\to+\infty}\frac{\alpha(\log N(er,1/f))}{\beta(\log\frac{1}{3}\gamma(er))}=\limsup_{r\to+\infty}\frac{\alpha(\log N(er,1/f))}{\beta(\log\frac{1}{3}+\log\gamma(er))}=\limsup_{r\to+\infty}\frac{\alpha(\log N(er,1/f))}{\beta((1+o(1))\log\gamma(er))}=\limsup_{r\to+\infty}\frac{\alpha(\log N(er,1/f))}{(1+o(1))\beta(\log\gamma(er))}=\limsup_{r\to+\infty}\frac{\alpha(\log N(er,1/f))}{\beta(\log\gamma(er))}=\limsup_{r\to+\infty}\frac{\alpha(\log N(r,1/f))}{\beta(\log\gamma(r))},$$

it follows that

$$\limsup_{r\to+\infty}\frac{\alpha(\log N(r,1/f))}{\beta(\log\gamma(r))}\geq\limsup_{r\to+\infty}\frac{\alpha(\log n(r,1/f))}{\beta(\log\gamma(r))}. \tag{9}$$

By (7) and (9), it is easy to see that

$$\lambda_{(\alpha,\beta,\gamma)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log n(r,1/f))}{\beta(\log\gamma(r))}=\limsup_{r\to+\infty}\frac{\alpha(\log N(r,1/f))}{\beta(\log\gamma(r))}.$$

By the same proof above, we can obtain the conclusion

$$\overline{\lambda}_{(\alpha,\beta,\gamma)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log\overline{n}(r,1/f))}{\beta(\log\gamma(r))}=\limsup_{r\to+\infty}\frac{\alpha(\log\overline{N}(r,1/f))}{\beta(\log\gamma(r))}.$$

□

Proposition 4. *(i) If f is an entire function, then*

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f]=\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}T(r,f))}{\beta(\log\gamma(r))}=\limsup_{r\to+\infty}\frac{\alpha(\log^{[3]}M(r,f))}{\beta(\log\gamma(r))}$$

and

$$\mu_{(\alpha(\log),\beta,\gamma)}[f]=\liminf_{r\to+\infty}\frac{\alpha(\log^{[2]}T(r,f))}{\beta(\log\gamma(r))}=\liminf_{r\to+\infty}\frac{\alpha(\log^{[3]}M(r,f))}{\beta(\log\gamma(r))}.$$

(ii) If f is a meromorphic function, then

$$\lambda_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} n(r,1/f))}{\beta(\log\gamma(r))} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} N(r,1/f))}{\beta(\log\gamma(r))}$$

and

$$\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \overline{n}(r,1/f))}{\beta(\log\gamma(r))} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \overline{N}(r,1/f))}{\beta(\log\gamma(r))}.$$

Since $\alpha(a+b) \leq \alpha(a)+\alpha(b)+c$ for all $a, b \geq R_0$ and fixed $c \in (0, +\infty)$, $\beta((1 + o(1))x) = (1 + o(1))\beta(x)$ as $x \to +\infty$ and $\gamma(a + b) \leq \gamma(a) + \gamma(b)$ for all $a, b \geq R_0$, the proof of Proposition 4 would run parallel to that of Proposition 3. We omit the details.

2 Main Results

In this paper, our aim is to make use of the concept of (α, β, γ)-order of entire functions to investigate the growth, zeros of the solutions of equation (1) which considerably extend some results of [21].

Theorem 1. *Let $A(z)$ be an entire function satisfying $\sigma_{(\alpha,\beta,\gamma)}[A] > 0$. Then $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha,\beta,\gamma)}[A]$ holds for all non-trivial solutions of* (1).

Theorem 2. *Let $A(z)$ be an entire function satisfying $\sigma_{(\alpha,\beta,\gamma)}[A] > 0$, let f_1 and f_2 be two linearly independent solutions of* (1) *and denote $F = f_1 \cdot f_2$. Then* max $\{\lambda_{(\alpha(\log),\beta,\gamma)}[f_1], \lambda_{(\alpha(\log),\beta,\gamma)}[f_2]\} = \lambda_{(\alpha(\log),\beta,\gamma)}[F] = \sigma_{(\alpha(\log),\beta,\gamma)}[F] \leq \sigma_{(\alpha,\beta,\gamma)}[A]$. *If $\sigma_{(\alpha(\log),\beta,\gamma)}[F] < \sigma_{(\alpha,\beta,\gamma)}[A]$, then $\lambda_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha,\beta,\gamma)}[A]$ holds for all solutions of type $f = c_1 f_1 + c_2 f_2$, where $c_1 \cdot c_2 \neq 0$.*

Theorem 3. *Let $A(z)$ be an entire function satisfying $\overline{\lambda}_{(\alpha,\beta,\gamma)}[A] < \sigma_{(\alpha,\beta,\gamma)}[A]$. Then $\lambda_{(\alpha(\log),\beta,\gamma)}[f] \leq \sigma_{(\alpha,\beta,\gamma)}[A] \leq \lambda_{(\alpha,\beta,\gamma)}[f]$ holds for all non-trivial solutions of* (1).

Remark 2. *This article may be understood as an extension and an improvement of [5], [18] and [21].*

3 Some Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *([12, 13, 19]) Let f be a transcendental entire function, and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then, for all $|z|$ outside a set E_1 of r of finite logarithmic measure, we have*

$$\frac{f^{(j)}(z)}{f(z)} = \Big(\frac{\nu(r,f)}{z}\Big)^j (1 + o(1)) \quad (j \in \mathbb{N}), \quad (10)$$

where $\nu(r, f)$ is the central index of f.

Lemma 2. *([10, 19]) Let $g : [0, +\infty) \to \mathbb{R}$ and $h : [0, +\infty) \to \mathbb{R}$ be monotone nondecreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E_2 of finite linear measure or finite logarithmic measure. Then, for any $d > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(dr)$ for all $r > r_0$.*

Lemma 3. *([[13], Theorems 1.9 and 1.10, or [17], Satz 4.3 and 4.4]) Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ be any entire function, $\mu(r, f)$ be the maximum term, i.e., $\mu(r, f) = \max\{|a_n| r^n; n = 0, 1, ...\}$, and $\nu(r, f)$ be the central index of f.*
(i) If $|a_0| \neq 0$, then

$$\log\mu(r,f) = \log|a_0| + \int_0^r \frac{\nu(t,f)}{t} dt. \quad (11)$$

(ii) For $r < R$, we have

$$M(r,f) < \mu(r,f)\Big(\nu(R,f) + \frac{R}{R-r}\Big). \quad (12)$$

Lemma 4. *Let f be an entire function satisfying $\sigma_{(\alpha,\beta,\gamma)}[f] = \sigma_1$ and $\mu_{(\alpha,\beta,\gamma)}[f] = \mu_1$, and let $\nu(r, f)$ be the central index of f. Then*

$$\limsup_{r\to+\infty} \frac{\alpha(\log\nu(r,f))}{\beta(\log\gamma(r))} = \sigma_1$$

and

$$\liminf_{r\to+\infty} \frac{\alpha(\log\nu(r,f))}{\beta(\log\gamma(r))} = \mu_1.$$

Proof. In view of the first part of Lemma 3, one may obtain that

$$\log\mu(2r,f) = \log|a_0| + \int_0^{2r} \frac{\nu(t,f)}{t} dt$$

$$\geq \log|a_0| + \int_r^{2r} \frac{\nu(t,f)}{t} dt \geq \log|a_0| + \nu(r,f)\log 2. \quad (13)$$

Also by Cauchy's inequality, it is well known that (cf. [23])

$$\mu(r,f) \leq M(r,f). \quad (14)$$

Therefore, one may obtain from (13) and (14) that

$$\nu(r,f)\log 2 \le \log M(2r,f) - \log|a_0|.$$

Thus from above, we get that

$$\log\nu(r,f) + \log^{[2]} 2 \le \log^{[2]} M(2r,f) + \log\Big(1 - \frac{\log|a_0|}{\log M(2r,f)}\Big).$$

By using condition on α, we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{\beta(\log\gamma(r))} \le \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]} M(2r,f))}{\beta(\log\gamma(r))} + \limsup_{r\to+\infty}\frac{\alpha\Big(\log\Big(1-\frac{\log|a_0|}{\log M(2r,f)}\Big)\Big)}{\beta(\log\gamma(r))} + \limsup_{r\to+\infty}\frac{\alpha(-\log^{[2]} 2)}{\beta(\log\gamma(r))} + \limsup_{r\to+\infty}\frac{c}{\beta(\log\gamma(r))} = \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]} M(2r,f))}{\beta(\log\gamma(r))}.$$

By using $\gamma(2r) \le 2\gamma(r)$, it follows that

$$\begin{aligned}
&\limsup_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{\beta(\log\gamma(r))}\\
&\le \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]} M(2r,f))}{\beta(\log\frac{1}{2}\gamma(2r))}\\
&= \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]} M(2r,f))}{\beta((1+o(1))\log\gamma(2r))}\\
&= \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]} M(2r,f))}{(1+o(1))\,\beta(\log\gamma(2r))}\\
&= \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]} M(r,f))}{\beta(\log\gamma(r))} = \sigma_1,
\end{aligned}$$

$$i.e.,\ \sigma_1 \ge \limsup_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{\beta(\log\gamma(r))} \tag{15}$$

and consequently

$$\mu_1 \ge \liminf_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{\beta(\log\gamma(r))}. \tag{16}$$

Further for any constant K_1, one may get from the second part of Lemma 3, that (cf. [6])

$$\log M(r,f) < \nu(r,f)\log r + \log\nu(2r,f) + K_1.$$

Therefore from above we obtain that

$$\begin{aligned}
\log M(r,f) &< \nu(2r,f)\log r + \nu(2r,f) + K_1,\\
i.e.,\ \log M(r,f) &< \nu(2r,f)(1+\log r) + K_1,\\
i.e.,\ \log M(r,f) &< \nu(2r,f)\log(e\cdot r) + K_1,
\end{aligned}$$

$$i.e.,\ \log^{[2]} M(r,f) < \log\nu(2r,f) + \log^{[2]}(e\cdot r) + \log\Big(1 + \frac{K_1}{\nu(2r,f)\log(e\cdot r)}\Big),$$

$$\begin{aligned}
i.e.,\ &\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]} M(r,f))}{\beta(\log\gamma(r))}\\
&\le \limsup_{r\to+\infty}\frac{\alpha(\log\nu(2r,f))}{\beta(\log\gamma(r))} + \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}(e\cdot r))}{\beta(\log\gamma(r))}\\
&+ \limsup_{r\to+\infty}\frac{\alpha\Big(\log\Big(1+\frac{K_1}{\nu(2r,f)\log(e\cdot r)}\Big)\Big)}{\beta(\log\gamma(r))}\\
&+ \limsup_{r\to+\infty}\frac{c}{\beta(\log\gamma(r))} = \limsup_{r\to+\infty}\frac{\alpha(\log\nu(2r,f))}{\beta(\log\gamma(r))},
\end{aligned}$$

where $c > 0$. Since $\gamma(2r) \le 2\gamma(r)$, so from above we have

$$\begin{aligned}
\sigma_1 &= \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]} M(r,f))}{\beta(\log\gamma(r))}\\
&\le \limsup_{r\to+\infty}\frac{\alpha(\log\nu(2r,f))}{\beta(\log\gamma(r))}\\
&\le \limsup_{r\to+\infty}\frac{\alpha(\log\nu(2r,f))}{\beta(\log\frac{1}{2}\gamma(2r))}\\
&= \limsup_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{\beta(\log\gamma(r))},
\end{aligned}$$

$$i.e.,\ \sigma_1 \le \limsup_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{\beta(\log\gamma(r))} \tag{17}$$

and accordingly

$$\mu_1 \le \liminf_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{\beta(\log\gamma(r))}. \tag{18}$$

Combining (15), (17) and (16), (18) we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{\beta(\log\gamma(r))} = \sigma_1$$

and

$$\liminf_{r\to+\infty}\frac{\alpha(\log\nu(r,f))}{\beta(\log\gamma(r))} = \mu_1.$$

This proves the lemma. □

Lemma 5. *Let f be an entire function satisfying $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_2$ and $\mu_{(\alpha(\log),\beta,\gamma)}[f] = \mu_2$, and let $\nu(r,f)$ be the central index of f. Then*

$$\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\nu(r,f))}{\beta(\log\gamma(r))} = \sigma_2$$

and

$$\liminf_{r\to+\infty}\frac{\alpha(\log^{[2]}\nu(r,f))}{\beta(\log\gamma(r))} = \mu_2.$$

In the line of Lemma 4 one can easily deduce the conclusion of Lemma 5 and so its proof is omitted.

Lemma 6. *Let f_1 and f_2 be entire functions of (α, β, γ)-exponent of convergence of the zero-sequence and denote $F = f_1 \cdot f_2$. Then*

$$\lambda_{(\alpha,\beta,\gamma)}[F] = \max\{\lambda_{(\alpha,\beta,\gamma)}[f_1],\ \lambda_{(\alpha,\beta,\gamma)}[f_2]\}.$$

Proof. Let $n(r, 0, F)$, $n(r, 0, f_1)$ and $n(r, 0, f_2)$ be unintegrated counting functions for the number of zeros of F, f_1 and f_2. For any $r > 0$, it is easy to see that

$$n(r, 0, F) \geq \max\{n(r, 0, f_1),\ n(r, 0, f_2)\}. \quad (19)$$

By Definition 2 and (19), we have

$$\lambda_{(\alpha,\beta,\gamma)}[F] \geq \max\{\lambda_{(\alpha,\beta,\gamma)}[f_1],\ \lambda_{(\alpha,\beta,\gamma)}[f_2]\}. \quad (20)$$

On the other hand, since the zeros of F must be the zeros of f_1 and the zeros of f_2, for any $r > 0$, we have

$$\begin{aligned} n(r, 0, F) &= n(r, 0, f_1) + n(r, 0, f_2) \\ &\leq 2\max\{n(r, 0, f_1),\ n(r, 0, f_2)\}. \end{aligned} \quad (21)$$

By Definition 2 and (21), we get that

$$\lambda_{(\alpha,\beta,\gamma)}[F] \leq \max\{\lambda_{(\alpha,\beta,\gamma)}[f_1],\ \lambda_{(\alpha,\beta,\gamma)}[f_2]\}. \quad (22)$$

Therefore, by (20) and (22), we have

$$\lambda_{(\alpha,\beta,\gamma)}[F] = \max\{\lambda_{(\alpha,\beta,\gamma)}[f_1],\ \lambda_{(\alpha,\beta,\gamma)}[f_2]\}.$$

This complete the proof. □

Lemma 7. *Let f_1 and f_2 be entire functions of $(\alpha(\log), \beta, \gamma)$-exponent of convergence of the zero-sequence and denote $F = f_1 \cdot f_2$. Then*

$$\lambda_{(\alpha(\log),\beta,\gamma)}[F] = \max\{\lambda_{(\alpha(\log),\beta,\gamma)}[f_1],\ \lambda_{(\alpha(\log),\beta,\gamma)}[f_2]\}.$$

In the line of Lemma 6 one can easily deduce the conclusion of Lemma 7 and so its proof is omitted.

Lemma 8. *Let f be a transcendental meromorphic function satisfying $\sigma_{(\alpha,\beta,\gamma)}[f] = \sigma_3$ and let $k \geq 1$ be an integer. Then, for any $\varepsilon > 0$, there exists a set E_3 having finite linear measure such that for all $r \notin E_3$, we have*

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = O(\alpha^{-1}((\sigma_3 + \varepsilon)\beta(\log \gamma(r)))).$$

Proof. Set $k = 1$. Since $\sigma_{(\alpha,\beta,\gamma)}[f] = \sigma_3 < +\infty$, for sufficiently large r and for any given $\varepsilon > 0$, we have

$$T(r, f) < \exp(\alpha^{-1}((\sigma_3 + \varepsilon)\beta(\log \gamma(r)))). \quad (23)$$

By the lemma of logarithmic derivative, we have

$$m\Big(r, \frac{f'}{f}\Big) = O(\log r + \log T(r, f)) \quad (r \notin E_3), \quad (24)$$

where $E_3 \subset [0, +\infty)$ is a set of finite linear measure, not necessarily the same at each occurrence. By (23) and (24) and the condition $\alpha(\log^{[2]} x) = o(\beta(\log \gamma(x)))$ as $x \to +\infty$, we have

$$m\Big(r, \frac{f'}{f}\Big) = O(\alpha^{-1}((\sigma_3 + \varepsilon)\beta(\log \gamma(r)))) \quad (r \notin E_3).$$

We assume that

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = O(\alpha^{-1}((\sigma_3 + \varepsilon)\beta(\log \gamma(r)))) \ (r \notin E_3) \quad (25)$$

holds for a certain integer $k \geq 1$. By $N(r, f^{(k)}) \leq (k+1)N(r, f)$, for all $r \notin E_3$, we have

$$\begin{aligned} T(r, f^{(k)}) &= m(r, f^{(k)}) + N(r, f^{(k)}) \\ &\leq m\Big(r, \frac{f^{(k)}}{f}\Big) + m(r, f) \\ &+ (k+1)N(r, f) \\ &\leq (k+1)T(r, f) \\ &+ O(\alpha^{-1}((\sigma_3 + \varepsilon)\beta(\log \gamma(r)))). \end{aligned} \quad (26)$$

By (24) and (26), for $r \notin E_3$, we obtain that

$$\begin{aligned} m\Big(r, \frac{f^{(k+1)}}{f^{(k)}}\Big) &= m\Big(r, \frac{\big(f^{(k)}\big)'}{f^{(k)}}\Big) \\ &= O(\log r + \log T(r, f^{(k)})) \\ &= O(\alpha^{-1}((\sigma_3 + \varepsilon)\beta(\log \gamma(r)))). \end{aligned} \quad (27)$$

Therefore, by (25) and (27), for $r \notin E_3$, we have that

$$\begin{aligned} m\Big(r, \frac{f^{(k+1)}}{f}\Big) &\leq m\Big(r, \frac{f^{(k+1)}}{f^{(k)}}\Big) + m\Big(r, \frac{f^{(k)}}{f}\Big) \\ &= O(\alpha^{-1}((\sigma_3 + \varepsilon)\beta(\log \gamma(r)))). \end{aligned}$$

Hence the lemma follows. □

4 Proof of the Main Results

Proof of Theorem 1. Set $\sigma_{(\alpha,\beta,\gamma)}[A] = \sigma_4 > 0$. First, we prove that every solution of (1) satisfies $\sigma_{(\alpha(\log),\beta,\gamma)}[f] \leq \sigma_4$. If f is a polynomial solution of (1), it is easy to show that $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = 0 \leq \sigma_4$ holds. Let f be a transcendental solution of (1). By (1), we can write that

$$\left|\frac{f''(z)}{f(z)}\right| = |A(z)|,$$

so, by Lemma 1, there exists a set $E_1 \subset (1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$ and $|f(z)| = M(r,f)$, we have

$$\Big(\frac{\nu(r,f)}{r}\Big)^2(1+o(1)) \leq \exp^{[2]}\Big(\alpha^{-1}\Big(\Big(\sigma_4+\frac{\varepsilon}{2}\Big)\beta(\log\gamma(r))\Big)\Big),$$

and hence, we obtain for $r \notin E_1$ that

$$\nu(r,f) \leq r\exp^{[2]}(\alpha^{-1}((\sigma_4+\varepsilon)\beta(\log\gamma(r)))). \quad (28)$$

Therefore by (28) and Lemma 2, there exists some $\eta_1 > 1$ such that for all $r > r_1$, we have

$$\nu(r,f) \leq \eta_1 r\exp^{[2]}(\alpha^{-1}((\sigma_4+\varepsilon)\beta(\log\gamma(\eta_1 r)))). \quad (29)$$

By (29), Lemma 5, and the conditions on α, β and γ, we obtain that

$$\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to+\infty}\frac{\alpha(\log^{[2]}\nu(r,f))}{\beta(\log\gamma(r))} \leq \sigma_4. \quad (30)$$

On the other hand, since f is a transcendental, so by (1), we get that

$$\begin{aligned} m(r,A) &= m\Big(r,-\frac{f''}{f}\Big) = O(\log rT(r,f)) \\ &= O(\log r + \log T(r,f)), \quad (r \notin E_3), \end{aligned}$$

where $E_3 \subset [0,+\infty)$ is a set of finite linear measure. By using Lemma 2, for any $\eta_2 > 1$ such that for all $r > r_2$, we get that

$$m(r,A) = m\Big(r,-\frac{f''}{f}\Big) \leq K_2(\log\eta_2 r + \log T(\eta_2 r,f)), \quad (31)$$

where $K_2 > 0$ is some constant. Since $A(z)$ is an entire function, so by (31) and using the inequality $\log(x+y) \leq \log x + \log y + \log 2$ $(x,y \geq 1)$, we have

$$\begin{aligned} \sigma_{(\alpha,\beta,\gamma)}[A] &= \limsup_{r\to+\infty}\frac{\alpha(\log m(r,A))}{\beta(\log\gamma(r))} \\ &\leq \limsup_{r\to+\infty}\frac{\alpha(\log 2K_2)}{\beta(\log\gamma(r))} \\ &\quad +\limsup_{r\to+\infty}\frac{\alpha(\log\log\eta_2 r)}{\beta(\log\gamma(r))} \\ &\quad +\limsup_{r\to+\infty}\frac{\alpha(\log\log T(\eta_2 r,f))}{\beta(\log\gamma(r))} \\ &\quad +\limsup_{r\to+\infty}\frac{c}{\beta(\log\gamma(r))} \\ &= \limsup_{r\to+\infty}\frac{\alpha(\log\log\eta_2 r)}{\beta(\log\gamma(r))} \\ &\quad +\limsup_{r\to+\infty}\frac{\alpha(\log\log T(\eta_2 r,f))}{\beta(\log\gamma(r))} \ (c>0). \end{aligned}$$

Since $\gamma(\eta_2 r) \leq \gamma(([\eta_2]+1)r) \leq ([\eta_2]+1)\gamma(r)$, where $[\eta_2]$ is the integer part of the number η_2, so from the inequality above and (30), we get that $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha,\beta,\gamma)}[A]$ holds for all non-trivial solutions of (1).
Thus Theorem 1 follows.

Proof of Theorem 2. Set $\sigma_{(\alpha,\beta,\gamma)}[A] = \sigma_5 > 0$, by Theorem 1, we have $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1] = \sigma_{(\alpha(\log),\beta,\gamma)}[f_2] = \sigma_{(\alpha,\beta,\gamma)}[A] = \sigma_5$. Hence, we get

$$\begin{aligned} \lambda_{(\alpha(\log),\beta,\gamma)}[F] &\leq \sigma_{(\alpha(\log),\beta,\gamma)}[F] \\ &\leq \max\{\sigma_{(\alpha(\log),\beta,\gamma)}[f_1], \sigma_{(\alpha(\log),\beta,\gamma)}[f_2]\} \\ &= \sigma_{(\alpha,\beta,\gamma)}[A]. \end{aligned} \quad (32)$$

By (32) and Lemma 7, we have

$$\begin{aligned} \max\{\lambda_{(\alpha(\log),\beta,\gamma)}[f_1], \lambda_{(\alpha(\log),\beta,\gamma)}[f_2]\} &= \lambda_{(\alpha(\log),\beta,\gamma)}[F] \\ &\leq \sigma_{(\alpha(\log),\beta,\gamma)}[F] \\ &\leq \sigma_{(\alpha,\beta,\gamma)}[A]. \end{aligned} \quad (33)$$

It remains to show that $\lambda_{(\alpha(\log),\beta,\gamma)}[F] = \sigma_{(\alpha(\log),\beta,\gamma)}[F]$. By (1), we have (see [18, pp. 76-77]) that all zeros of F are simple and that

$$F^2 = C^2\Big(\Big(\frac{F'}{F}\Big)^2 - 2\Big(\frac{F''}{F}\Big) - 4A\Big)^{-1}, \quad (34)$$

where $C \neq 0$ is a constant. Hence,

$$2T(r,F) = T\Big(r,\Big(\frac{F'}{F}\Big)^2 - 2\Big(\frac{F''}{F}\Big) - 4A\Big) + O(1)$$

$$\leq O\Big(\overline{N}\Big(r,\frac{1}{F}\Big) + m\Big(r,\frac{F'}{F}\Big) + m\Big(r,\frac{F''}{F}\Big) + m(r,A)\Big). \quad (35)$$

By $\sigma_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha,\beta,\gamma)}[A] = \sigma_5 < +\infty$ and Lemma 8, for all $r \notin E_3$, we have $m(r,A) = m\Big(r,\frac{f''}{f}\Big) = O(\exp(\alpha^{-1}((\sigma_5+\varepsilon)\beta(\log\gamma(r)))))$, $m\Big(r,\frac{F'}{F}\Big) = O(\exp(\alpha^{-1}((\sigma_5+\varepsilon)\beta(\log\gamma(r)))))$ and $m\Big(r,\frac{F''}{F}\Big) = O(\exp(\alpha^{-1}((\sigma_5+\varepsilon)\beta(\log\gamma(r)))))$. Therefore, by (35), for all $r \notin E_3$, we obtain

$$T(r,F) = O\Big(\overline{N}\Big(r,\frac{1}{F}\Big) + \exp(\alpha^{-1}((\sigma_5+\varepsilon)\beta(\log\gamma(r))))\Big). \quad (36)$$

Now let us assume that $\lambda_{(\alpha(\log),\beta,\gamma)}[F] < \kappa < \sigma_{(\alpha(\log),\beta,\gamma)}[F]$. Since all zeros of F are simple, we have

$$\begin{aligned} \overline{N}\Big(r,\frac{1}{F}\Big) &= N\Big(r,\frac{1}{F}\Big) \\ &= O(\exp^{[2]}(\alpha^{-1}(\kappa\beta(\log\gamma(r))))). \end{aligned} \quad (37)$$

Hence by (36) and (37), for all $r \notin E_3$, we get that

$$T(r,F) = O(\exp^{[2]}(\alpha^{-1}(\kappa\beta(\log\gamma(r))))).$$

By Definition 1 and Lemma 2, we have $\sigma_{(\alpha(\log),\beta,\gamma)}[F] \leq \kappa < \sigma_{(\alpha(\log),\beta,\gamma)}[F]$, this is a contradiction. Therefore, the first assertion is proved. If $\sigma_{(\alpha(\log),\beta,\gamma)}[F] < \sigma_{(\alpha,\beta,\gamma)}[A]$, let us assume that $\lambda_{(\alpha(\log),\beta,\gamma)}[f] < \sigma_{(\alpha,\beta,\gamma)}[A]$ holds for any solution of type $f = c_1f_1 + c_2f_2$ ($c_1c_2 \neq 0$). We denote $F = f_1 \cdot f_2$ and $F_1 = f \cdot f_1$, then we have $\lambda_{(\alpha(\log),\beta,\gamma)}[F] < \sigma_{(\alpha,\beta,\gamma)}[A]$ and $\lambda_{(\alpha(\log),\beta,\gamma)}[F_1] < \sigma_{(\alpha,\beta,\gamma)}[A]$. Since (36) holds for F and F_1, where $F_1 = f \cdot f_1 = (c_1f_1 + c_2f_2)f_1 = c_1f_1^2 + c_2F$, then we get that

$$\begin{aligned} T(r,f_1) &= O(T(r,F_1) + T(r,F)) \\ &= O\Big(\overline{N}\Big(r,\frac{1}{F_1}\Big) + \overline{N}\Big(r,\frac{1}{F}\Big) \\ &\quad + \exp(\alpha^{-1}((\sigma_5+\varepsilon)\beta(\log\gamma(r))))\Big). \end{aligned} \tag{38}$$

By $\lambda_{(\alpha(\log),\beta,\gamma)}[F] < \sigma_{(\alpha,\beta,\gamma)}[A]$, $\lambda_{(\alpha(\log),\beta,\gamma)}[F_1] < \sigma_{(\alpha,\beta,\gamma)}[A]$ and (37), for some $\kappa < \sigma_{(\alpha,\beta,\gamma)}[A]$, we obtain

$$T(r,f_1) = O(\exp^{[2]}(\alpha^{-1}(\kappa\beta(\log\gamma(r))))). \tag{39}$$

By Definition 1 and (39), we have $\sigma_{(\alpha(\log),\beta,\gamma)}[f_1] \leq \kappa < \sigma_{(\alpha,\beta,\gamma)}[A]$, this is a contradiction with Theorem 1. Therefore, we have that $\lambda_{(\alpha(\log),\beta,\gamma)}[f] = \sigma_{(\alpha,\beta,\gamma)}[A]$ holds for all solutions of type $f = c_1f_1 + c_2f_2$, where $c_1c_2 \neq 0$. Hence the theorem follows.

Proof of Theorem 3. By Theorem 1 and $\lambda_{(\alpha(\log),\beta,\gamma)}[f] \leq \sigma_{(\alpha(\log),\beta,\gamma)}[f]$, it is easy to show that $\lambda_{(\alpha(\log),\beta,\gamma)}[f] \leq \sigma_{(\alpha,\beta,\gamma)}[A]$ holds. It remains to show that $\sigma_{(\alpha,\beta,\gamma)}[A] \leq \lambda_{(\alpha,\beta,\gamma)}[f]$. Let us assume that $\sigma_{(\alpha,\beta,\gamma)}[A] > \lambda_{(\alpha,\beta,\gamma)}[f]$. By (1) and a similar proof of Theorem 5.6 in [18, pp. 82], we obtain

$$T\Big(r,\frac{f}{f'}\Big) = O\Big(\overline{N}\Big(r,\frac{1}{f}\Big) + \overline{N}\Big(r,\frac{1}{A}\Big)\Big) \quad (r \notin E_3). \tag{40}$$

By (40), the assumption $\sigma_{(\alpha,\beta,\gamma)}[A] > \lambda_{(\alpha,\beta,\gamma)}[f]$ and $\overline{\lambda}_{(\alpha,\beta,\gamma)}[A] < \sigma_{(\alpha,\beta,\gamma)}[A]$, for some $\kappa < \sigma_{(\alpha,\beta,\gamma)}[A]$, we obtain that

$$T\Big(r,\frac{f}{f'}\Big) = O(\exp(\alpha^{-1}(\kappa\beta(\log\gamma(r))))). \tag{41}$$

Further by Definition 1 and (41), we have $\sigma_{(\alpha,\beta,\gamma)}\left[\frac{f}{f'}\right] = \sigma_{(\alpha,\beta,\gamma)}\left[\frac{f'}{f}\right] \leq \kappa < \sigma_{(\alpha,\beta,\gamma)}[A]$. Therefore by

$$-A(z) = \Big(\frac{f'}{f}\Big)' + \Big(\frac{f'}{f}\Big)^2,$$

we get that $\sigma_{(\alpha,\beta,\gamma)}[A] \leq \sigma_{(\alpha,\beta,\gamma)}\left[\frac{f'}{f}\right] < \sigma_{(\alpha,\beta,\gamma)}[A]$, this is a contradiction. Hence, we have that $\lambda_{(\alpha(\log),\beta,\gamma)}[f] \leq \sigma_{(\alpha,\beta,\gamma)}[A] \leq \lambda_{(\alpha,\beta,\gamma)}[f]$ holds for all non-trivial solutions of (1).
The proof is complete.

5 Conclusion

Throughout this article, we have generalized some previous results to general (α,β,γ)-order. Defining new order of growth in the complex plane is discussed and is applied to complex differential equations with entire coefficients to solve some problems related to growth of solutions. It is interesting now to study the growth of solutions of complex differential equations with meromorphic coefficients.

6 Acknowledgements

The authors are grateful to the referees for their many valuable remarks and suggestions which lead to the improvement of the original version of this paper.

References:

[1] Bank, S., Laine, I.: *On the oscillation theory of $f'' + Af = 0$ where A is entire*. Trans. Amer. Math. Soc. 273 (1982), no. 1, 351–363.

[2] Bank, S., Laine, I.: *On the zeros of meromorphic solutions of second-order linear differential equations*. Comment. Math. Helv. 58 (1983), no. 4, 656–677.

[3] Bank, S., Laine, I., Langley, J.: *Oscillation results for solutions of linear differential equations in the complex domain*. Results Math. 16 (1989), no. 1-2, 3–15.

[4] Belaïdi, B.: *Growth of ρ_φ -order solutions of linear differential equations with entire coefficients*. PanAmer. Math. J. 27 (2017), no. 4, 26–42.

[5] Belaïdi, B., Biswas, T.: *Complex oscillation of a second order linear differential equation with entire coefficients of (α,β)-order*, Silesian J. Pure Appl. Math. vol. 11 (2021), 1–21.

[6] Chen, Z. X., Yang, C. C.: *Some further results on the zeros and growths of entire solutions of second order linear differential equations*. Kodai Math. J. 22 (1999), no. 2, 273–285.

[7] Cao, T. B , Li, L. M.: *Oscillation results on meromorphic solutions of second order differential equations in the complex plane*. Electron. J. Qual. Theory Differ. Equ. 2010, No. 68, 13 pp.

[8] Chyzhykov, I., Heittokangas, J., Rättyä, J.: *Finiteness of φ-order of solutions of linear differential equations in the unit disc*. J. Anal. Math. 109 (2009), 163–198.

[9] Chyzhykov, I., Semochko, N.: *Fast growing entire solutions of linear differential equations*. Math. Bull. Shevchenko Sci. Soc. 13 (2016), 1–16.

[10] Gundersen, G. G.: *Finite order solutions of second order linear differential equations*. Trans. Amer. Math. Soc. 305 (1988), no. 1, 415–429.

[11] Hayman, W. K.: *Meromorphic functions*. Oxford Mathematical Monographs, Clarendon Press, Oxford 1964.

[12] Hayman, W. K.: *The local growth of power series: a survey of the Wiman-Valiron method*. Canad. Math. Bull. 17 (1974), no. 3, 317–358.

[13] He, Y. Z., Xiao, X. Z.: *Algebroid Functions and Ordinary Differential Equations*. Science Press, Beijing (1988) (in Chinese).

[14] Heittokangas, J., Wang, J., Wen, Z. T., Yu, H. : *Meromorphic functions of finite φ-order and linear q-difference equations*. J. Difference Equ. Appl. 27 (2021), no. 9, 1280–1309. DOI: 10.1080/10236198.2021.1982919.

[15] Juneja, O. P., Kapoor, G. P., Bajpai, S. K.: *On the (p, q)-order and lower (p, q)-order of an entire function*. J. Reine Angew. Math. 282 (1976), 53–67.

[16] Juneja, O. P., Kapoor, G. P., Bajpai, S. K.: *On the (p, q)-type and lower (p, q)-type of an entire function*. J. Reine Angew. Math. 290 (1977), 180–190.

[17] Jank, G., Volkmann, L.: *Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen*. Birkhäuser Verlag, Basel, 1985.

[18] Kinnunen, L.: *Linear differential equations with solutions of finite iterated order*. Southeast Asian Bull. Math. 22 (1998), no. 4, 385–405.

[19] Laine, I.: *Nevanlinna theory and complex differential equations*. De Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin, 1993.

[20] Mulyava, O. M., Sheremeta, M. M., Trukhan, Yu. S.: *Properties of solutions of a heterogeneous differential equation of the second order*. Carpathian Math. Publ. 11 (2019), no. 2, 379–398.

[21] Shen, X., Tu, J., Xu, H. Y.: *Complex oscillation of a second-order linear differential equation with entire coefficients of $[p, q]$-φ order*. Adv. Difference Equ. 2014, 2014:200, 14 pp.

[22] Sato, D.: *On the rate of growth of entire functions of fast growth*. Bull. Amer. Math. Soc. 69 (1963), 411–414.

[23] Singh, A. P., Baloria, M. S.: *On the maximum modulus and maximum term of composition of entire functions*. Indian J. pure appl. Math. 22 (1991), 1019–1026.

[24] Xu, H. Y., Tu, J.: *Oscillation of meromorphic solutions to linear differential equations with coefficients of $[p, q]$-order*. Electron. J. Differential Equations 2014, No. 73, 14 pp.

[25] Yang, C. C., Yi, H. X.: *Uniqueness theory of meromorphic functions*. Mathematics and its Applications, 557. Kluwer Academic Publishers Group, Dordrecht, 2003.