Note on the transcendental equation with three unknowns $\sqrt{2f(z)-4}=\sqrt{x-P'(t)+\sqrt{P(t)}(y+2)}\pm\sqrt{x-P'(t)-\sqrt{P(t)}(y+2)}$

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Abstract: Let $P := P(t)$ be a non square polynomial and $f := f(z)$ be a bijective application over Z. Using the method of continuous fractions, we consider, in this paper, the number of integer solutions of transcendental equation

$$\sqrt{2f(z)-4}=\sqrt{x-P'(t)+\sqrt{P(t)}(y+2)}\pm\sqrt{x-P'(t)-\sqrt{P(t)}(y+2)}$$

under the condition that

$$x^2-P(t)y^2-2P'(t)x+4P(t)y+(P'(t))^2-4P(t)-1=0.$$

We extend a previous result given by A. S. Sriram and P. Veeramallan.

Key-Words: Transcendental equation, Integer solutions, Diophantine Equation, Pell equation, Polynomial, Bijection, Continued fraction, Recurrence relation.

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1 Introduction

An algebraic equation is an equation of the form $f(x) = 0$, where $f(x)$ is entirely a polynomial in x, such as $x^5 - x^3 + x^2 - 1 = 0$. However, if $f(x)$ contains trigonometrical, arithmetic, or exponential terms, it is referred to as a transcendental equation, such as $xe^x - 2 = 0$ and $xlog10x - 1.2 = 0$.

Transcendental equations are widely used in science and engineering because they enable the modeling and simulation of physical phenomena. The importance of Transcendental equations is demonstrated by the fact that they can be found practically everywhere in mathematical analysis, which has aided in the development of theoretical sciences and the development of new technologies. Transcendental equations enable the analysis of mechanical vibration in the field of physics [1, 2], the analysis of alternating current in electrical circuits [1, 2], electro magnetics theory [3, 4], quantum mechanics [5, 6], digital signal processing [7, 8], and the modeling of wave heat conduction [9, 10].

In the case of logarithms, another application for transcendental functions is the creation of phase and magnitude plots in Bode analysis [2], [9], [10].

Transcendental functions enabled the development of mathematical tools for analysis such as Fourier [11], [12], and Laplace transform [1], [13]. Furthermore, hyperbolic functions are important in mathematical analysis for science and engineering; for example, in civil engineering, they have applications in the study and design of catenary forms in chains and cables for suspended bridges [14], in electrical engineering for the design of free hanging electric power cables [15], [16], [17], and in naval and civil engineering for the modeling of sea wave behavior [18].

The determination of transcendental function roots is a problem that appears in a wide range of engineering applications (For more details, one can see [3, 4, 5]). There are numerous numerical approaches available for approximating the solution to any desired level of accuracy. In terms of practicality, such root discovery algorithms are often simple to implement and provide an adequate method for obtaining root values. However, having an exact mathematical solution to the problem under investigation is sometimes advantageous. For example, analytical derivatives for uncertainty analyses and sensitivity studies can be developed using an explicit expression for the root. Analytical derivatives, in many circumstances, provide far more insight into the problem than numerical derivatives.

Explicit expressions can also be used to ensure that approximation root seeking techniques are convergent. Haji-Sheikh and Beck presented, in 2000, a closed formulas for many analytical heat transfer problems and detailed their applications [8]. Using Cauchy's integral theorem from complex analysis, Luck Rogelio, and James W. Stevens presented, in [9], a straightforward way for formulating accurate explicit solutions for the roots of analytic Transcendental equations. Their method was presented along with various examples.

In [6], Rogelio Luck, Gregory J. Zdaniuk, and Heejin Cho described a method for finding a polynomial equation with the same roots as a transcendental equation and solving it for all of its roots within a bounded region. Using Cauchy's integral theorem for complex variables, the proposed method transforms the problem of finding the roots of a transcendental equation into an equivalent problem of finding the roots of a polynomial equation with exactly the same roots.

The coefficients of the polynomial form an interesting vector that lies in the null space of a Hankel matrix made up of the Fourier series coefficients of the inverse of the original transcendental equation.

The explicit solution can then be easily obtained by employing the complex fast Fourier transform.

In this paper, we aim to discover integer solutions for the given transcendental equation using the continued fraction method. For a non-square polynomial $P(t)$, we consider the Diophantine equation $H(x,y,P(t)) = x^2 - P(t)y^2 - 2P'(t)x + 4P(t)y + (P'(t))^2 - 4P(t) = 1$ and then the Pell equation $x2 - P(t)y2 = 1$. We also go over some of the most important characteristics of the simple continued fraction.

In the detailed, I referred to [1, 2] in the extensive, but I considered [7] as the major source of inspiration that we will give a great generalization of its results.

In the following, definitions and results needed in our paper.

2 Method

Consider the transcendental equations

$$E^{+} := \sqrt{2f(z)-4} = \sqrt{x - P'(t) + \sqrt{P(t)}(y+2)} + \sqrt{x - P'(t) - \sqrt{P(t)}(y+2)}$$

and

$$E^{-} := \sqrt{2f(z)-4} = \sqrt{x - P'(t) + \sqrt{P(t)}(y+2)} - \sqrt{x - P'(t) - \sqrt{P(t)}(y+2)}$$

verifying $H(x,y,P(t)) = x^2 - P(t)y^2 - 2P'(t)x + 4P(t)y + (P'(t))^2 - 4P(t) = 1$, such that $P := P(t)$ be a non square polynomial and $f := f(z)$ be a bijective application over Z.

We get by squaring both sides of (E^+ and E^-) respectively

$$f(z) - 2 = x - P'(t) \pm \sqrt{(x - P'(t))^2 - P(t)(y+2)^2} \quad (1)$$

Then, we have

$$f(z) - 2 = x - P'(t) \pm \sqrt{H(x,y,P(t))} \quad (2)$$

and thus x, y are given from the Diophantine equation

$$H(x,y,P(t)) = 1 \quad (3)$$

and

$$z = f^{-1}(x - P'(t) + 2 \pm 1)$$

That is

$$z = f^{-1}(x - P'(t) + 3) \text{ or } z = f^{-1}(x - P'(t) + 1).$$

Note that the resolution of (3), in its current form, was described in [chandoul], by transforming it into a Pell equation which can be easily solved. To get it, it had set

$$T : \begin{cases} u = x - P'(t) \\ v = y + 2 \end{cases} \quad (4)$$

we get,

$$T(3) := \widetilde{(3)} : \quad u^2 - P(t)v^2 = 1$$

This Pell equation is always known to be solved. Its solutions are related to the $\sqrt{P(t)}$'s continued fraction expansion.

We will be looking at continued fraction expansions of $\sqrt{P(t)}$, where $P(t)$ is a non-square. In fact, the following theorem summarizes a very interesting form of continued fractions.

Theorem 1 *Let $P(t)$ be a non square polynomial. Then*

$$\sqrt{P(t)} = [a_0; \overline{a_1, a_2, \cdots, a_l, 2a_0}],$$

Proof. Let A_1 the polynomial part of $\sqrt{P(t)}$. Then $deg(\sqrt{P(t)} + A_1) > 1$ and $deg(-\sqrt{P(t)} + A_1) < 1$. Thus $\sqrt{P(t)}+A_1$, is a reduced quadratic irrational with degree of polynomial part is less or equal to $2deg(A_1)$. We can confirm that

$$\sqrt{P(t)} + A_1 = [\overline{2A_1, A_2, \cdots, A_n}]$$

for some n. which is equivalent to

$$\sqrt{P(t)} + A_1 = [2A_1, \overline{A_2, \cdots, A_n, 2A_1}]$$

consequently

$$\sqrt{P(t)} = [A_1, \overline{A_2, \cdots, A_n, 2A_1}]$$

3 Main result

The following theorem express our main finding. we consider the result without giving their proof since it can be proved by induction as similat to that of Theorems in [1, 2, 7] were proved.

Theorem 2 *Let $\sqrt{P(t)} = [a_0; \overline{a_1, a_2, \cdots, a_l, 2a_0}]$ be the continuied fraction expansion of period length l, of $\sqrt{P(t)}$, where $P(t)$ is a non square polynomial and let $\frac{p_n}{q_n}$ its n^{th} convergent. Then the following assertions holds*

(1) The fundamental solutions of (E^+) is (x_1, y_1, z_1), such as

$$\begin{cases} x_1 = p_{l-1} + P'(t), \\ y_1 = q_{l-1} + 2, & if\ l\ is\ even \\ z_1 = f^{-1}(p_{l-1} + 3) \end{cases}$$

and

$$\begin{cases} x_1 = p_{2l-1} + P'(t), \\ y_1 = q_{2l-1} + 2, & if\ l\ is\ odd \\ z_1 = f^{-1}(p_{2l-1} + 3) \end{cases}$$

(2) The fundamental solutions of (E^-) is (x_1, y_1, z_1), such as

$$\begin{cases} x_1 = p_{l-1} + P'(t), \\ y_1 = q_{l-1} + 2, & if\ l\ is\ even \\ z_1 = f^{-1}(p_{l-1} + 1) \end{cases}$$

and

$$\begin{cases} x_1 = p_{2l-1} + P'(t), \\ y_1 = q_{2l-1} + 2, & if\ l\ is\ odd \\ z_1 = f^{-1}(p_{2l-1} + 1) \end{cases}$$

(3) Define the sequence

$$\{(x_n, y_n, z_n)\}_{n\geq 1} = \{(u_n+P'(t), v_n+2, f^{-1}(u_n+2\pm 1))\},$$

where $\{(u_n, v_n)\}$ defined by

$$\begin{cases} u_n = u_1u_{n-1} + (a_0u_1 + \alpha)v_{n-1} & \\ & \forall n \geq 2, \\ v_n = v_1u_{n-1} + (a_0v_1 + \beta)v_{n-1} & \\ with\ \alpha = x_{l-2}\ and\ \beta = x_{l-2}, if\ l\ is\ even. \end{cases}$$

and

$$\begin{cases} u_n = u_1u_{n-1} + (a_0u_1 + \alpha)v_{n-1} & \\ & \forall n \geq 2, \\ v_n = v_1u_{n-1} + (a_0v_1 + \beta)v_{n-1} & \\ with\ \eta = x_{2l-2}\ and\ \delta = x_{2l-2}, , if\ l\ is\ odd. \end{cases}$$

Then (x_n, y_n, z_n) is a solution of (E^+,), respectively of (E^-). So (E^+,) respectively (E^-) have infinitely many integer solutions $(x_n, y_n, z_n) \in Z^3$.

(4) The solutions (x_n, y_n, z_n) of (E^+,) satisfy the recurrence relations

$$\begin{cases} x_k & = u_1x_{k-1} + (a_0u_1 + \alpha)y_{n-1} - u_1(2a_0 + P'(t)) \\ & -2\alpha + P'(t),\ if\ l\ iseven \\ y_k & = v_1x_{k-1} + (a_0v_1 + \beta)y_{n-1} - v_1(2a_0 + P'(t)) \\ & -2\beta + 2 \\ z_k & = f^{-1}(u_1x_{k-1} + (a_0u_1 + \alpha)y_{n-1} - u_1(2a_0 \\ & +P'(t)) - 2\alpha) + 3), \end{cases}$$

$\forall k \geq 2$, if l is even,
and

$$\begin{cases} x_k & = u_1x_{k-1} + (a_0u_1 + \alpha)y_{n-1} - u_1(2a_0 + P'(t)) \\ & -2\alpha + P'(t),\ if\ l\ iseven \\ y_k & = v_1x_{k-1} + (a_0v_1 + \beta)y_{n-1} - v_1(2a_0 + P'(t)) \\ & -2\beta + 2 \\ z_k & = f^{-1}(u_1x_{k-1} + (a_0u_1 + \alpha)y_{n-1} - u_1(2a_0 \\ & +P'(t)) - 2\alpha) + 3), \end{cases}$$

$\forall k \geq 2$, if l is odd.

(5) The solutions (x_n, y_n, z_n) of (E^-), satisfy the recurrence relations

$$\begin{cases} x_k & = u_1x_{k-1} + (a_0u_1 + \alpha)y_{n-1} - u_1(2a_0 + P'(t)) \\ & -2\alpha + P'(t),\ if\ l\ iseven \\ y_k & = v_1x_{k-1} + (a_0v_1 + \beta)y_{n-1} - v_1(2a_0 + P'(t)) \\ & -2\beta + 2 \\ z_k & = f^{-1}(u_1x_{k-1} + (a_0u_1 + \alpha)y_{n-1} - u_1(2a_0 \\ & +P'(t)) - 2\alpha) + 1), \end{cases}$$

$\forall k \geq 2$, if l is even,
and

$$\begin{cases} x_k & = u_1x_{k-1} + (a_0u_1 + \alpha)y_{n-1} - u_1(2a_0 + P'(t)) \\ & -2\alpha + P'(t),\ if\ l\ iseven \\ y_k & = v_1x_{k-1} + (a_0v_1 + \beta)y_{n-1} - v_1(2a_0 + P'(t)) \\ & -2\beta + 2 \\ z_k & = f^{-1}(u_1x_{k-1} + (a_0u_1 + \alpha)y_{n-1} - u_1(2a_0 \\ & +P'(t)) - 2\alpha) + 1), \end{cases}$$

$\forall k \geq 2$, *if l is odd.*

4 Examples

Remark 1 *to deduce all the results given in [7], just take $f := f(z) = z$, $P(t) := C$ and set $v = y + 2$. Extract first (x, y, v) and then deduce (x, y, z).*

Example 1: Let $f(z) = z + 3$ and $P(t) = t^4 + 4t^3 + 6t^2 + 4t + 2$, It is easy to verify that $f^{-1}(z) = z - 3$ and

$$\sqrt{P(t)} = \left[t^2 + 2t + 1; \overline{2t^2 + 4t + 2}\right].$$

It can be seen that $P(t)$ become $D(n) = n^2 + 1$ with $n = t^2 + 2t + 1$. Then the transcendental equations become

$$E^+ := \sqrt{2z+2} = \sqrt{x - 2n + \sqrt{n^2+1}(y+2)} + \sqrt{x - 2n - \sqrt{n^2+1}(y+2)}$$

and

$$E^- := \sqrt{2z+2} = \sqrt{x - 2n + \sqrt{n^2+1}(y+2)} - \sqrt{x - 2n - \sqrt{n^2+1}(y+2)}.$$

We have

$$\sqrt{D(n)} = [n; \overline{2n}].$$

So, the fundamental solution of E^+ is $(x_1, y_1, z_1) = (2n^2 + 2n + 1, 2n + 2, 2n^2 + 1)$ and the other solutions are given, for $k \geq 2$, by

$$\begin{cases} x_k = (2n^2+1)x_{k-1} + (2n^3+2n)y_{k-1} \\ \quad -8n^3 - 4n \\ y_k = 2nx_{k-1} + (2n^2+1)y_{k-1} - 8n^2 \\ \quad +2n - 2 \\ z_k = (2n^2+1)x_{k-1} + (2n^3+2n)y_{k-1} - 8n^3 \\ \quad -6n \end{cases}$$

Similarly, the fundamental solution of E^- is

$$(x_1, y_1, z_1) = (2n^2 + 2n + 1, 2n + 2, 2n^2 - 1)$$

and the other solutions are given, for $k \geq 2$, by

$$\begin{cases} x_k = (2n^2+1)x_{k-1} + (2n^3+2n)y_{k-1} \\ \quad -8n^3 - 4n \\ y_k = 2nx_{k-1} + (2n^2+1)y_{k-1} - 8n^2 \\ \quad +2n - 2 \\ z_k = (2n^2+1)x_{k-1} + (2n^3+2n)y_{k-1} - 8n^3 \\ \quad -6n - 2 \end{cases}$$

Further, for $t = 1$, $P(t) = 17$ and $\sqrt{P(t)} = [4; \overline{8}]$. So, $(x_1, y_1, z_1) = (65, 10, 62)$ is then the fundamental solution of E^+. and the other solutions are given, for $k \geq 2$, by

$$\begin{cases} x_k = 33x_{k-1} + 136y_{k-1} - 1296 \\ y_k = 8x_{k-1} + 33y_{k-1} - 320 \\ z_k = 33x_{k-1} + 136y_{k-1} - 1296 \end{cases}$$

Similarly, $(x_1, y_1, z_1) = (65, 10, 62)$ is then the fundamental solution of E^+. and the other solutions are given, for $k \geq 2$, by

$$\begin{cases} x_k = 33x_{k-1} + 136y_{k-1} - 1296 \\ y_k = 8x_{k-1} + 33y_{k-1} - 320 \\ z_k = 33x_{k-1} + 136y_{k-1} - 1294 \end{cases}$$

Example 2: Let $f(z) = z + 3$ and $P(t) = t^6 + t + 1$, a nonsquare polynomial in F_3. It is easy to verify that

$$\sqrt{P(t)} = [t^3, \overline{2t^2+t+2, t+1, t+1, 2t, 2t +2, 2t, t+1, t+1, 2t^2+t+2, 2t^3}]$$

and then we deduce the solutions.

5 Conclusion

Using the continued fraction method, we found all feasible non-negative integer solutions to the given transcendental equation in this study. It's also noteworthy to note that all of the results in [7], can be derived easily from our theorem, it is in fact a case of our general result.

This extension enable us to solve a wide range of equations. On the integer solutions of Diophantine and Pell equations, we also deduce various recurrence relations.

Another advantage of our findings is that the procedure may be run on a computer, allowing us to retrieve all of the answers after inserting the coecients and verifying the method's criteria.

Furthermore, it is important to note that the proposed method can be extended to solve many different types of transcendental equations, with the goal of expanding the set of resolvable transcendental equations and making them available for use in various fields of science and engineering. Finally, it will be interesting to include these new functions in commercial or open source mathematical software such as Maple, Mathematica, Matlab, and GNU Octave.

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