A Variable Step Reduction Block Solver for Stiff ODEs

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Abstract: - This research study is aimed at developing variable step reduction block solver (VSRBS) for stiff ODEs. This step reduction block solver will embrace the technic of variable step-variable order to determine suited variable step size. The trigonometrically fitted method will represent the basis function approximation to be utilized together with the method of interpolation and collocation to derive (VSRBS). VSRBS comes with advantages to overcome the barrier of stability requirement pose by definition 4. Some selected modelled examples of stiff ODEs will solved and compared with existing methods to establish the efficiency and accuracy.

Key-words- Trigonometrically Fitted Method; Stiff Odes; Variable Step Reduction; Block Solver; Tolerance Level.

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1 Introduction

Every technique for estimating the analytical solution to initial-value problems possesses error terms that require a higher differential of the analytical solution of the equation. Suppose the differential will be fairly bounded, then this technique will possess a predictable error bound that will be utilized to estimate the accuracy of the approximation. Still, assume the differential increases as the steps increase; the error will be maintained in relative control and provided that the solution also increase in magnitude. Problems often spring up, still, if the magnitude of the differential increases but the analytical solution does not. In this position, the error will increase so high that it controls the computations. Initial-value problems for which this is probably to appear are referred to as stiff equations and are often common, especially in the areas of oscillations, chemical responses and electric circuits [6].

Stiff differential equations will be qualified as those whose analytic solution holds a term of the class e^{-cx}, where c is a high positive constant coefficient. Thus, it forms part of the analytic solution called the transient result. Mostly, the essential part of the analytic solution is called the stiff-state result. The transient part of a stiff equation will quickly decay to nothing as x increases, because the nth differential of this term holds magnitude $c^n e^{-cx}$, the differential will not decay as rapidly. As a matter of fact, because the differential in the error term is calculated not at x, simply at a number between nothing and x, the differential terms will increase as x increases and very quickly surely. By good fortune, stiff equations in general can be anticipated from the real life problem from which the equation is derived and, with caution, the error can be maintained below control. See [6] for more info

Definition 1: Consider the form of stiff equations of

$y' = Ay + \emptyset(x), \qquad y(\alpha) = y_0 \qquad (1)$

where $y, \emptyset \in R$ and A is an $m \times m$ rectangular array of rows and columns with eigenvalues $\lambda_x \in \mathbb{C}, x = 1,2,\dots,n$ (presume distinct) and matching eigenvectors $c_x \in \mathbb{C}, x = 1,2,\dots,n$. In general, the solution to (1) assumes the class

$y(x) = \sum_{x=1}^{n} v_x \exp(\lambda_x x)\, c_x + \emptyset(x) \qquad (2)$

where the v_x is any constant coefficients and $\emptyset(x)$ is a particular integral. See [12-13] for more

The above introduction will be supported with the following definitions.

Definition 2: The initial-value problem (1) is said to be stiff whenever every of its eigenvalues possesses negative real form and the stiffness ratio is large. The stiffness ratio $S > 1$ and $re\lambda_i < 0, i = 1(1)n$. See [8, 12-13] for more info.

Definition 3: The initial-value problem (1) is said to be stiff vibrating whenever the eigenvalues $(\lambda_i = a_i + jb_i, i = 1(1)n)$ of the Jacobian $J = \left(\frac{\partial f}{\partial y}\right)$ satisfies the following conditions:

$$a_i < 0, i = 1(1)n,$$
$$\max_{1 \le i \le n} |a_i| > \min_{1 \le i \le n} |a_i|,$$

or whenever the stiffness ratio meets

$$S = \max_{i,j} \left|\frac{a_i}{a_j}\right| > 1$$

and $\qquad |a_i| < |a_j| \qquad (3)$

for at least a single pair of $i\ in\ 1 \le i \le n$. See [8] for more info.

Definition 4: Stiffness appears whenever stability demands, rather than those of accuracy restraint the step length. See [8, 12-13] for more info.

Theorem 1: Suppose $z: R \to R$ is continuously 2π −periodic. Then for whatever $\varepsilon > 0$ there will be a trigonometric polynomial $P(u) = \sum_{j=-n}^{k} c_j\, e^{iju}$ such that for entirely $u, |z(u) - P(u)| < \varepsilon$. Equally essential, whenever any such f there must subsist a sequentially polynomial of $P_n \to z$ in a sequential order on R. See [4] for more items.

Authors have suggested that computing stiff and extremely vibrating initial-value problems generally demands the acceptance of several computational methods. Among them includes; [1] integrated the stiff ODEs using block backward differentiation formulas of order six. The method is derived via the expansion of linear multistep method and executed with fixed step size. MATLAB solver ode15s is used to achieve the computational result with better accuracy. Suitable step and convergence of the result is not established. [[3] formulated the new numerical method for solving stiff initial value problems. The derivation of the method is done with linear operator and implemented with fixed step size. Stiff problems solved have vibrating and oscillating solutions. Solving the method is done with fixed step size to determine the maximum errors. [9] proposed the diagonally implicit block backward differentiation formula $(\rho - DIBBDF)$ which relies on the best choice of the parameter ρ that has optimum stability attributes resulting to more precise results. Although, $(\rho - DIBBDF)$ is self-initiating but utilize the uniform step size to carry out the implementation process. Convergence of the result is done with uniform step size. [10] suggested the derivation of diagonally implicit block backward differentiation formulas for solving stiff Initial value problems. The derivation and implementation are carried out using Lagrange polynomial and fixed step size. [10] consider the linear and nonlinear stiff problems whose analytical solutions are exponential in nature with fixed step size. [11] implemented the BBDF$-\alpha$ for solving stiff ordinary differential equations with oscillating solutions. This method is derived using Lagrange polynomial and solves stiff problems with oscillating solutions and uses fixed step size to implement the method. [14] developed an accurate block solver for stiff initial value problems. Linear difference operator is used to derive the method. Variable step size and tolerance contributes immensely to the successful implementation of the method. Problems solved have oscillating and vibrating solutions. [24] proposed the numerical algorithm for solving stiff ordinary differential equations. An efficient scheme in selecting the step size and order has been introduced and implemented throughout the numerical calculation. Test problem considered has analytical solution with exponentially and trigonometrically fitted in nature. Lagrange polynomial is used as basis function approximation. Variable step size and tolerance level were both utilized to establish the convergence of the result. [25] implemented fully implicit block method with four-point for computing ODEs. The derivation of the method is done with Lagrange polynomial and problems solved have oscillating/vibrating solutions. Variable step size implementation and tolerance level were used as well. The BBDF has strong nature of region of absolute stability that is considered as a better method to proffer solutions to stiff problems. This study will suggest variable step reduction block solver for stiff ODEs for the purpose of introducing variable step-variable order and finding suitable variable step size to provide a better solution to trigonometrically exact solution. Trigonometrically fitted methods used as basis function approximation agrees with the oscillatory or vibration solutions to ensure stability of the results achieved. VSRBS have the capacity to determine for every loop a suited variable step to overcome the stability demand of definition 4. Again, VSRBS is introduced to ensure better efficiency and accuracy. See [12-13].

The motivation of this study originates from [12-13] to yield better efficiency and precision via the introduction of variable step-variable order-variable step size. Secondly, VSRBS is proposed to bypass the obstacle pose by backward differentiation formula to adopt region of absolute stability as the criteria for better result. Thus, this study will implement variable step-variable order and finding a suitable variable step size. Again, trigonometrically fitted method will be utilized as the basis function approximation to suit the trigonometrically exact solution or oscillating and vibrating solutions whose solutions are trigonometrically in nature. This idea of using trigonometrically fitted method supersedes the use of Lagrange polynomial and other basis function utilized by other researchers. See [1, 11, 14, 25] for details.

2 Formulation of Variable Step Reduction Block Solver

The variable step reduction block solver formulation will employ the concept of variable step and variable order strategy. This involves the combination of block predictor method and block corrector method. The block predictor corrector method takes y_{n-1} as point of interpolation and $f_{n-1}, f_{n-2}, f_{n-3}, f_{n-4}$ as points of collocation while the block corrector method use y_{n-2} and $f_{n+1}, f_{n+2}, f_{n+3}$ as points of collocation. The block predictor method has $3-step\ of\ order\ 4$ and $2-step\ of\ order\ 3$ for the block corrector method. The step reduction block solver utilizes the trigonometrically fitted method as the basis function approximation in accordance with the oscillating and vibrating solutions. The trigonometrically fitted method for the 3-step block predictor method of order 4 is defined as

$$y(x) = \sum_{i=0}^{2} a_i \left(\frac{x-x_n}{h}\right)^i + a_3 \sum_{i=1,3}^{2} w^i \left(\frac{x-x_n}{h}\right)^i + a_4 \sum_{i=0,2,4}^{3} w^i \left(\frac{x-x_n}{h}\right)^i \qquad (4)$$

Similarly, the trigonometrically fitted method for the 2-step block corrector method of order 3 is defined as

$$y(x) = \sum_{i=0}^{1} a_i \left(\frac{x-x_n}{h}\right)^i + a_2 \sum_{i=1,3}^{2} w^i \left(\frac{x-x_n}{h}\right)^i + a_4 \sum_{i=0,2}^{3} w^i \left(\frac{x-x_n}{h}\right)^i. \qquad (5)$$

Interpolating and collocating (4) and (5) using the selected points of block predictor method and block corrected method will yield the expression as

$$matrixa = \left\{\left\{1, -1, 1, -w + \frac{w^3}{6}, 1 - \frac{w^2}{2} + \frac{w^4}{24}\right\}, \left\{0, 1, -2, w - \frac{w^3}{2}, w^2 - \frac{w^4}{6}\right\}, \left\{0, 1, -4, w - 2w^3, 2w^2 - \frac{4w^3}{3}\right\}, \left\{0, 1, -6, w - \frac{9w^3}{2}, 3w^2 - \frac{9w^4}{2}\right\}, \left\{0, 1, -8, w - 8w^3, 4w^2 - \frac{32w^4}{3}\right\}\right\};$$

$$b\{y[n-1], f[n-1], f[n-2], f[n-3], f[n-4]\}; \qquad (6)$$

$$\{a[0], a[1], a[2], a[3], a[4]\} = Inverse[matrixz].b$$

$matrixa = \left\{\left\{1, -2, -2w + \frac{4w^3}{3}, 1 - 2w^2\right\}, \left\{0,1, w - \frac{w^3}{2}, w^2\right\}, \{0,1, w - 2w^3, -2w^2\}, \left\{0,1, w - \frac{9w^3}{2}, -3w^2\right\}\right\};$
$b\{y[n-2], f[n+1], f[n+2], f[n+3]\}$; 7)
$\{a[0], a[1], a[2], a[3]\} = Inverse[matrixz].b$

Equations (6) and (7) represent the Mathematica matrix format written under Mathematica Kernel 9. Solving (6) and (7) using Mathematica Kernel 9 and substituting into (4) and (5) will bring forth the continuous scheme of both the block predictor method and block corrector method as

$$y(x) = y_{n-1} + \left[\left(\frac{-\left(2w^3 - \frac{55w^7}{12}\right)}{2w^7} - \frac{(-6w^5 - 8w^7)}{2w^7}\left(\frac{x-x_n}{h}\right) - \frac{\left(-w^5 - \frac{13w^7}{3}\right)}{2w^7}\left(\frac{x-x_n}{h}\right)^2 - \frac{3}{w^3}\left(\frac{x-x_n}{h}\right)^3 + \frac{1}{w^4}\left(\frac{x-x_n}{h}\right)^4\right)hf_{n-1} + \left(\frac{-\left(6w^3 + \frac{59w^7}{12}\right)}{2w^7} - \frac{(16w^5 + 12w^7)}{2w^7}\left(\frac{x-x_n}{h}\right) - \frac{\left(3w^5 + \frac{19w^7}{2}\right)}{2w^7}\left(\frac{x-x_n}{h}\right)^2 + \frac{8}{w^3}\left(\frac{x-x_n}{h}\right)^3 - \frac{3}{w^4}\left(\frac{x-x_n}{h}\right)^4\right)hf_{n-2} + \left(\frac{-\left(6w^3 - \frac{37w^7}{12}\right)}{2w^7} - \frac{(-14w^5 - 8w^7)}{2w^7}\left(\frac{x-x_n}{h}\right) - \frac{(-3w^5 - 7w^7)}{2w^7}\left(\frac{x-x_n}{h}\right)^2 - \frac{7}{w^3}\left(\frac{x-x_n}{h}\right)^3 + \frac{3}{w^4}\left(\frac{x-x_n}{h}\right)^4\right)hf_{n-3} + \left(\frac{-\left(-2w^3 + \frac{3w^7}{4}\right)}{2w^7} - \frac{(4w^5 + 2w^7)}{2w^7}\left(\frac{x-x_n}{h}\right) - \frac{\left(w^5 + \frac{11w^7}{6}\right)}{2w^7}\left(\frac{x-x_n}{h}\right)^2 + \frac{2}{w^3}\left(\frac{x-x_n}{h}\right)^3 - \frac{1}{w^4}\left(\frac{x-x_n}{h}\right)^4\right)hf_{n-4}\right]. \qquad (8)$$

$$y(x) = y_{n-2} + \left[\left(\frac{-\left(\frac{5w^3}{2} - \frac{375}{3}\right)}{w^5} - \frac{(-w^3 - 3w^5)}{w^5}\left(\frac{x-x_n}{h}\right) - \frac{1}{w^3}\left(\frac{x-x_n}{h}\right)^2 + \frac{5}{2w^2}\left(\frac{x-x_n}{h}\right)^3\right)hf_{n+1} + \left(\frac{-\left(4w^3 + \frac{50w^5}{3}\right)}{w^5} - \frac{(2w^3 + 3w^5)}{w^5}\left(\frac{x-x_n}{h}\right) + \frac{2}{w^3}\left(\frac{x-x_n}{h}\right)^2 - \frac{4}{w^2}\left(\frac{x-x_n}{h}\right)^3\right)hf_{n+2} + \left(\frac{-\left(\frac{3w^3}{2} - \frac{19w^5}{3}\right)}{w^5} - \frac{(-w^3 - w^5)}{w^5}\left(\frac{x-x_n}{h}\right) - \frac{1}{w^3}\left(\frac{x-x_n}{h}\right)^2 + \frac{3}{w^2}\left(\frac{x-x_n}{h}\right)^3\right)hf_{n+3} + \right]. \; (9)$$

Evaluating equations (9) and (10) at $x = x_n + h, x_n + 2h \; and \; x_n + 3h$ will yield block predictor method and block corrector method of:

$$y(x) = y_{n-1} + h[\beta_0(w,x)f_{n-1} + \beta_1(w,x)f_{n-2} + \beta_2(x,x)f_{n-3} + \beta_4(w,x)f_{n-4})], \qquad (10)$$

$$y(x) = y_{n-2} + h[\beta_0(w,x)f_{n+1} + \beta_1(w,x)f_{n+2} + \beta_2(x,x)f_{n+3}], \qquad (11)$$

w is the recognized frequency, $\beta_0(w,x), \beta_1,(w,x), \beta_2(w,x) \; and \; \beta_4(w,x)$ is called fixed constant coefficients. Equations (10) and (11) are called the Variable step reduction block solver (VSRBS) implemented via variable step and variable order strategy. See [12-13, 15-23] for info.

2.1 Deriving the Tolerance Level of Variable Step Reduction Block Solver

To obtain the derivation of the tolerance lever of variable step reduction block solver, $3 - step$ block predictor method of order 4 and $2 - step$ of block corrector method of order 3 with different order and step is been considered. A collection of [2, 5-7, 12-13, 19-23] suggest the possibility of finding estimate of the principal local truncation error of the block predictor method-block corrector pair without solving higher differential constant coefficients, $y(x)$. Making the presumption that whenever $\tilde{k} = \bar{k}$, $\bar{k}$ and $\tilde{k}$ represents the order of block predictor method and block corrector method. Right way, inquiry of $3 - step$ block predictor method of fourth order to yield the principal local truncation errors as stated by

$$\begin{aligned} &\tilde{C}^{[1]}_{\tilde{k}+4} h^{\tilde{k}+4} y^{(\tilde{k}+4)}(\tilde{x}_n) = y(x_{n+1}) - y^{[l_1]}_{n+1} + \left(\frac{77}{720} - \frac{11}{12w^3} + \frac{5}{3w^2}\right) + O\left(h^{\tilde{k}+5}\right), \\ &\tilde{C}^{[2]}_{\tilde{k}+4} h^{\tilde{k}+4} y^{(\tilde{k}+4)}(\tilde{x}_n) = y(x_{n+2}) - y^{[l_2]}_{n+2} + \left(\frac{533}{720} + \frac{45}{2w^4} - \frac{22}{3w^3} + \frac{29}{6w^2}\right) + O\left(h^{\tilde{k}+5}\right), \qquad (15) \\ &\tilde{C}^{[3]}_{\tilde{k}+4} h^{\tilde{k}+4} y^{(\tilde{k}+4)}(\tilde{x}_n) = y(x_{n+3}) - y^{[l_3]}_{n+3} + \left(\frac{2249}{720} + \frac{120}{w^4} - \frac{99}{4w^3} + \frac{19}{2w^2}\right) + O\left(h^{\tilde{k}+5}\right). \end{aligned}$$

In likewise manner, investigating the breakdown of $2 - step$ block corrector method will give rise to principal local truncation errors as defined by

$\bar{C}^{[1]}_{\bar{k}+3}h^{\bar{k}+3}y^{(\bar{k}+3)}(\bar{x}_n) = y(x_{n+1}) - y^{[q_1]}_{n+1} + \left(\frac{210-210w-239w^3}{24w^3}\right) + O(h^{\bar{p}+4})$,

$\bar{C}^{[2]}_{\bar{k}+3}h^{\bar{k}+3}y^{(\bar{k}+3)}(\bar{x}_n) = y(x_{n+2}) - y^{[q_2]}_{n+2} + \left(\frac{177}{12w^3} - \frac{283}{12w^2} - \frac{41}{3}\right) + O(h^{\bar{p}+4})$ (16)

$\bar{C}^{[3]}_{\bar{k}+3}h^{\bar{k}+3}y^{(\bar{k}+3)}(\bar{x}_n) = y(x_{n+3}) - y^{[q_3]}_{n+3} + \left(\frac{-183w^4+432w-1288w^2}{24w^4}\right) + O(h^{\bar{p}+4})$,

$\tilde{C}^{[1]}_{\tilde{k}+4}, \tilde{C}^{[2]}_{\tilde{k}+4}, \tilde{C}^{[1]}_{\tilde{k}+4}, \bar{C}^{[1]}_{\bar{k}+3}, \bar{C}^{[2]}_{\bar{k}+3}$ and $\bar{C}^{[3]}_{\bar{k}+3}$ exists as a separate device with step size h. $y(x)$ represents the analytical result of the differential constant coefficient which agrees with the assumptions $y(x_n) \approx y_n$. See [2, 5-7, 12-13, 19-23].

Moving ahead, the assumptions is fixed for lesser valuates of h to get

$y^{(4)}(\tilde{u}_n) \approx y^{(3)}(\bar{u}_n)$. (17)

Step reduction block solver banks on the assumption (17) to be implemented.

Furthermore, solving (15) and (16) and truncating terms of degree $O(h^{\bar{p}+5})$ and $O(h^{\bar{p}+4})$ will achieve the step reduction block solver estimates

$\bar{C}^{[1]}_{\bar{k}+3}h^{\bar{k}+3}y^{(\bar{k}+3)}(\bar{x}_n) \approx \frac{7170}{8897}\left[y^{[l_1]}_{n+1} - y^{[q_1]}_{n+1}\right] < \sigma_1$,

$\bar{C}^{[2]}_{\bar{k}+3}h^{\bar{k}+3}y^{(\bar{k}+3)}(\bar{x}_n) \approx \frac{7440}{12323}\left[y^{[l_2]}_{n+2} - y^{[q_2]}_{n+2}\right] < \sigma_2$, (18)

$\bar{C}^{[3]}_{\bar{k}+3}h^{\bar{k}+3}y^{(\bar{k}+3)}(\bar{x}_n) \approx \frac{2745}{44798}\left[y^{[l_3]}_{n+3} - y^{[q_3]}_{n+3}\right] < \sigma_3$.

Stating the assertions that $y^{[l_1]}_{n+1} \neq y^{[q_1]}_{n+1}$, $y^{[l_2]}_{n+2} \neq y^{[q_2]}_{n+2}$ and $y^{[l_3]}_{n+3} \neq y^{[q_3]}_{n+3}$ are known as the block predicting and block correcting estimate achieved by the block solver kth order. $\bar{C}^{[1]}_{\bar{k}+4}h^{\bar{k}+3}y^{(\bar{k}+3)}(\bar{x}_n)$, $\bar{C}^{[2]}_{\bar{k}+4}h^{\bar{k}+3}y^{(\bar{k}+3)}(\bar{x}_n)$ and $\bar{C}^{[3]}_{\bar{k}+4}h^{\bar{k}+3}g^{(\bar{k}+3)}(\bar{x}_n)$ will distinctly be called principal local truncation errors. σ_1, σ_2 and σ_3 will be tolerance level of the block solver.

Again, the results of (18) is employ to take decision to accept or reject the computed results of the loop or redo the loop with a lesser suitable variable step size. Accepting the computed results is certainly based on successful loop as specified by (18). See [2, 5-7, 12-13, 19-23] for more details.

2.2 Variable Step Size Variation Strategy for VSRBS

This study uses the principal local truncation errors of order four (4) of the 3-step block predictor method and order three (3) of the 2-step block corrector method to introduce the idea. The block predictor method of order four (4) and the block corrector method of order three (3) will be employed to find the suitable vary step of the step reduction block solver. The principal local truncation errors of the block predictor method-block corrector method will be computed by this mathematical expression

$\left|y(x_{n+1}) - \bar{y}^{[\bar{l}_1]}_{n+1}\right| \approx \frac{\left|z(x_{n+1}) - \bar{y}^{[\bar{q}_1]}_{n+1}\right|}{h} \approx \frac{7170}{8897h}\left|y^{[\bar{l}_1]}_{n+1} - \bar{y}^{[\bar{q}_1]}_{n+1}\right|$

$\left|y(x_{n+2}) - \bar{y}^{[\bar{l}_2]}_{n+2}\right| \approx \frac{\left|z(x_{n+2}) - \bar{y}^{[\bar{q}_2]}_{n+2}\right|}{h} \approx \frac{7440}{12323h}\left|y^{[\bar{l}_2]}_{n+2} - \bar{y}^{[\bar{q}_2]}_{n+2}\right|$ (19)

$\left|y(x_{n+3}) - \bar{y}^{[\bar{l}_3]}_{n+3}\right| \approx \frac{\left|z(x_{n+3}) - \bar{y}^{[\bar{q}_3]}_{n+3}\right|}{h} \approx \frac{2745}{44798h}\left|y^{[\bar{l}_3]}_{n+3} - \bar{y}^{[\bar{q}_3]}_{n+3}\right|$,

where z define the exact result to the first order stiff problem satisfying the initial condition $z(x_n) = y(x_n)$.

Presume rebuilding the process of utilizing anew suitable variable step size qh to generate a anew estimates of $y^{[\bar{l}_1]}_{n+1}, \bar{y}^{[\bar{q}_1]}_{n+1}$, $y^{[\bar{l}_2]}_{n+2}, \bar{y}^{[\bar{q}_2]}_{n+2}$ and $y^{[\bar{l}_3]}_{n+3}$ and $\bar{y}^{[\bar{q}_3]}_{n+3}$. To ascertain and ensure the principal local truncation errors in ε, choosing q such that

$\frac{\left|z(x_n+qh) - \bar{y}^{[\bar{q}_1]}_{n+1}(using\ the\ step\ size\ qh)\right|}{qh} < \varepsilon$

$\frac{\left|z(x_n+qh) - \bar{y}^{[\bar{q}_2]}_{n+2}(using\ the\ step\ size\ qh)\right|}{qh} < \varepsilon$ (20)

$\frac{\left|z(x_n+qh) - \bar{y}^{[\bar{q}_3]}_{n+3}(using\ the\ step\ size\ qh)\right|}{qh} < \varepsilon$

Employing the principal local truncation errors of the block predictor method and block corrector method together with (20) will achieve the result as

$\frac{\left|z(x_n+qh) - \bar{y}^{[\bar{q}_1]}_{n+1}(utilizing\ qh)\right|}{qh} = \frac{239}{24} = \left|z^{(4)}(\bar{x}_n)\right|q^3h^3 \approx \frac{239}{24}\left[\frac{720}{8897}\left|y^{[\bar{l}_1]}_{n+1} - \bar{y}^{[\bar{q}_1]}_{n+1}\right|\right]q^4h^4$,

$\frac{\left|z(x_n+qh) - \bar{y}^{[\bar{q}_2]}_{n+2}(utilizing\ qh)\right|}{qh} = \frac{31}{4} = \left|z^{(4)}(\bar{x}_n)\right|q^3h^3 \approx \frac{41}{3}\left[\frac{720}{12323}\left|y^{[\bar{l}_2]}_{n+2} - \bar{y}^{[\bar{q}_2]}_{n+2}\right|\right]q^4h^4$,

(21)

$\frac{\left|z(x_n+qh)-\bar{y}_{n+3}^{[\bar{q}_3]}(utilizing\ qh)\right|}{qh}=\frac{61}{8}=\left|z^{(4)}(\bar{x}_n)\right|q^3h^3\approx \frac{61}{8}\left[\frac{180}{22399}\left|y_{n+3}^{[\bar{l}_3]}-\bar{y}_{n+3}^{[\bar{q}_3]}\right|\right]q^4h^4.$

Thus, requesting for the new selection of q by resolving (21) will further gives

$$\frac{239}{24}\left[\frac{720}{8897}\left|y_{n+1}^{[\bar{l}_1]}-\bar{y}_{n+1}^{[\bar{q}_1]}\right|\right]q^3h^3=\frac{7170}{8897}\frac{\left|y_{n+1}^{[\bar{l}_1]}-\bar{y}_{n+1}^{[\bar{q}_1]}\right|}{h}<\varepsilon,$$
$$\frac{41}{3}\left[\frac{720}{12323}\left|y_{n+2}^{[\bar{l}_2]}-\bar{y}_{n+2}^{[\bar{q}_2]}\right|\right]q^3h^3=\frac{7440}{12323}\frac{\left|y_{n+2}^{[\bar{l}_2]}-\bar{y}_{n+2}^{[\bar{q}_2]}\right|}{h}<\varepsilon, \quad (22)$$
$$\frac{61}{8}\left[\frac{180}{22399}\left|y_{n+3}^{[\bar{l}_3]}-\bar{y}_{n+3}^{[\bar{q}_3]}\right|\right]q^3h^3=\frac{2745}{44798}\frac{\left|y_{n+3}^{[\bar{l}_3]}-\bar{y}_{n+3}^{[\bar{q}_3]}\right|}{h}<\varepsilon.$$

Hence, requesting for change in step size from h to qh, where q will be adjusted as

$$q<\left(\left(\frac{7170}{8897}\right)\frac{h\varepsilon}{\left|y_{n+1}^{[\bar{l}_1]}-\bar{y}_{n+1}^{[\bar{q}_1]}\right|}\right)^{\frac{1}{4}}\approx 0.80589\left(\frac{h\varepsilon}{\left|y_{n+1}^{[\bar{l}_1]}-\bar{y}_{n+1}^{[\bar{q}_1]}\right|}\right),$$
$$q<\left(\left(\frac{7440}{12323}\right)\frac{h\varepsilon}{\left|y_{n+1}^{[\bar{l}_1]}-\bar{y}_{n+1}^{[\bar{q}_1]}\right|}\right)^{\frac{1}{4}}\approx 0.603749\left(\frac{h\varepsilon}{\left|y_{n+2}^{[\bar{l}_2]}-\bar{y}_{n+2}^{[\bar{q}_2]}\right|}\right), \quad (23)$$
$$q<\left(\left(\frac{2745}{44798}\right)\frac{h\varepsilon}{\left|y_{n+1}^{[\bar{l}_1]}-\bar{y}_{n+1}^{[\bar{q}_1]}\right|}\right)^{\frac{1}{4}}\approx 0.0612751\left(\frac{h\varepsilon}{\left|y_{n+3}^{[\bar{l}_3]}-\bar{y}_{n+3}^{[\bar{q}_3]}\right|}\right).$$

In addition, the successful execution of VSRBS relies on (22) and (23). This demand of employing the 3-step block predictor method of order four and 2-step block corrector method of order three combines with (22) or (23) is iteratively solved to agree with the tolerance level. Again, this iteration process is implemented repeatedly until the newly selected suitable step size satisfies the tolerance level. Whenever the newly selected variable step size is successful achieved then it eventually becomes the suitable vary step size to get the desired results with better accuracy and efficiency. Vary step size strategies involve varying the step size during the iteration process until the tolerance levels are achieved. A step size changes for SRBS is very costly to implement with high demand to achieve and as such, mathematical software package is used to easy the execution. See [2, 5-7, 12-13, 19-23] for details.

3 Examples of Stiff Problems

Stiff problems solved involve trigonometrically solution with oscillating and vibrating behaviour. Three stiff problems were considered and solved using the VSRBS. These stiff problems were extracted from [9, 11, 25]

Stiff Problem 1

A two-torso orbit mildly stiff problem

$y_1'=y_3,\ y_2'=y_4,\ y_3'=\frac{y_1}{r^3},\ y_4'=\frac{y_2}{r^3},\ r=(y_1^2+y_2^2)^{\frac{1}{2}},\ 0\le x\le 20,$

$y_1(0)=1,\ y_2(0)=0,\ y_3(0)=0,\ y_4(0)=1.$

Exact result: $y_1(x)=cosx, y_2(x)=sinx, y_3(x)=-sinx,\ y_4(x)=cosx.$

Source. See [25] more info.

Stiff Problem 2

$y'(x)=-2\pi sin(2\pi x)-\frac{1}{10^{-3}}\left(y-cos(2\pi x)\right),$

$y(0)=1,\ [0,1].$

Exact result: $y(x)=\cos(2\pi x),$

Source: see [9] for details.

Stiff Problem 3

$y_1'=y_3, y_2'=y_4, y_3'=-y_1+\frac{1}{10}(y_1^2+y_2^2+y_3^2+y_4^2),\ y_3'=-1000y_2+\frac{1}{10}(y_1^2+y_2^2+y_3^2+y_4^2-1),$

$[0,3], y_1(0)=1, y_2(0)=0, y_3(0)=0, y_4(0)=0.$

Exact result: $y_1(x)=Cosx,\ y_2(x)=0,\ y_3(x)=-sinx,\ y_4(x)=0.$

Source. See [11] for info.

4 Results and Discussion

This study considers specifically stiff problems whose exact result is trigonometrically in nature with oscillating and vibrating solutions. Again, this study will solve stiff problems with tolerance level and without tolerance level. The following tolerance level of $10^{-1}, 10^{-2}, 10^{-4}, 10^{-6}, 10^{-7}, 10^{-8}, 10^{-10}$ and 10^{-11} were used during the implementation process. The results of the three stiff problems were displayed in Table 1, Table 2 and Table 3. Headings in the tables are defined in the nomenclature. The VSRBS is coded under Mathematica Kernel 9 and implemented as well. The process of implementation can be viewed from sections 2.1 and 2.2. VSRBS is very tedious computational procedure that involves the combination of equations (11) and (12) together with equation (18) written in Mathematica language. See [9, 11, 25] for more info.

Table 1. Result of Stiff Problem I

$MTHD_{EMPLOYED}$	MAX_{ERROR}	TOL
4PIFI VSRBS	$7.4085e-5$ $3.37258e-4$	10^{-2}
4PIFI VSRBS	$7.9610e-7$ $3.37498e-6$	10^{-4}
4PIFI VSRBS	$6.4643e-9$ $3.375e-8$	10^{-6}
4PIFI VSRBS	$4.0474e-11$ $3.375e-10$	10^{-8}
4PIFI VSRBS	$3.3125e-13$ $3.37508e-12$	10^{-10}

Table 2. Result of Stiff Problem 2

$MTHD_{EMPLOYED}$	MAX_{ERROR}	TOL
ρ-DIBBDF(0.50) ρ-DIBBDF(0.95) VSRBS	$1.04695e-1$ $1.70999e-1$ $3.74965e-3$	10^{-1}
ρ-DIBBDF(-0.75) ρ-DIBBDF(-60) VSRBS	$3.61318e-2$ $3.83043e-2$ $4.47403e-4$	10^{-2}
ρ-DIBBDF(0.95) VSRBS	$1.18569e-6$ $4.47833e-8$	10^{-6}
ρ-DIBBDF(-0.75) ρ-DIBBDF(-0.60) ρ-DIBBDF(0.50) VSRBS	$5.14905e-7$ $5.25483e-7$ $6.58550e-7$ $3.78993e-9$	10^{-7}
ρ-DIBBDF(0.95) VSRBS	$4.17385e-10$ $4.47842e-12$	10^{-10}
ρ-DIBBDF(-0.75) ρ-DIBBDF(-0.60) ρ-DIBBDF(0.50) VSRBS	$6.28992e-11$ $6.44415e-11$ $9.41198e-11$ $3.79141e-13$	10^{-11}

Table 3. Result of Stiff Problem 3

$MTHD_{EMPLOYED}$	MAX_{ERROR}	TOL
BBDF$-\alpha = 0.3$ VSRBS	$5.159812e-4$ $4.43756e-6$	10^{-4}
BBDF$-\alpha = 0.3$ VSRBS	$5.235607e-6$ $4.4376e-8$	10^{-6}
BBDF$-\alpha = 0.3$ VSRBS	$5.243138e-8$ $4.4376e-10$	10^{-8}
BBDF$-\alpha = 0.3$ VSRBS	$5.261320e-10$ $4.43778e-12$	10^{-10}

4.1 Nomenclature

The following nomenclature will be used to show the results in Tables 1, 2 and 3.

VSRBS: variable step reduction block solver

TOL : the tolerance level employed

Max_{error} : the magnitude of the maximum errors of VSRBS.

$Mthd_{used}$: method employed.

4PIFI: implementation of the four-point one-block fully implicit method using variable step size. See [25] for more info.

ρ-DIBBDF (ρ_i): ρ –Diagonally implicit block backward differentiation formula (ρ value). See [9] for more info.

BBDF-α: block backward differentiation α-formulas. See [11] for more details.

5 Conclusion

A variable step reduction block solver for stiff ODEs has been suggested. This VSRBS emanates from the 3-step block predictor method of order four and 2-step block corrector method of order three. The VSRBS has the capacity to vary the step-vary the order and implement a suitable vary step size with the support of the tolerance level. The derivation of the VSRBS is done via a special trigonometrically fitted method used as the basis function approximation for the purpose of approximating the trigonometrically exact solution. The (VSRBS) evaluated three stiff problems and compare the results with existing methods. The performance of VSRBS competes favourably with [25] in terms of the maximum errors as a result of finding a suitable variable step size for VSRBS to satisfy the tolerance level. [9, 11, 25] belongs to the backward differentiation formula (family) which has been strictly designed to solve stiff problems with strong region of absolute stability compare to VSRBS of Adams family which is projected for non-stiff problems. VSRBS involves tedious computation of using a specially designed block predictor and block corrector method to find a suitable variable step size to satisfy the tolerance criteria. The VSRBS performs better than [9, 11] due to the execution of (4) and (5) as the basis function approximation compare to others using Lagrange polynomial and Newton iteration as basis function approximation. Also, the successful implementation is attributed to implementing variable step-variable order-finding a suitable variable step size at every loop process. Thus, the VSRBS is efficient and accurate for stiff ordinary differential equations. Further studies will be to design a block solver with the capacity to handle exponentially exact solution.

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References:

[1] A. N. Amiratul, B. I. Zarina, R. Haliza, On the integration of stiff ODEs using block backward differentiation formulas of order six, *Symmetry,* Vol. 12, No. 6, 2020, pp. 1-13.

[2] U. M. Ascher, L. R. Petzoid, *Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations,* SIAM, USA, 1998.

[3] B. Babangida, H. Musa, L. K. Ibrahim, A new numerical method for solving stiff initial value problems, *Fluid Mechanics,* Vol. 3, No. 2, 2016, pp. 1-5.

[4] M. Bond, *Convolutions and the Weierstrass Approximation Theorem, Department of Mathematics,* Michigan State University, USA, 2009.

[5] J. R. Dormand, *Numerical Methods for Differential Equations,* CRC Press, New York, 1996.

[6] J. D. Faires, R. L. Burden, *Initial-Value Problems for ODEs,* Brooks Cole, Dublin City University, 2002.

[7] J. D. Faires, R. L. Burden, *Initial-Value Problems for ODEs,* Brooks Cole, Dublin City University, 2012.

[8] S. O. Fatunla, *Numerical Methods for Initial Value Problems in Ordinary Differential Equations,* ACADEMIC Press, Inc., New York, 1988.

[9] M. I Hazizah, B. I. Zarina, Diagonally implicit block backward differentiation formula with optimal stability properties for stiff ordinary differential equations, *Symmetry*, Vol. 11, No. 11, 2019, pp. 1-18.

[10] S. M. Z. Iskandar, B. I. Zarina, Derivation of diagonally implicit block backward differentiation formulas for solving stiff initial value problems, *Mathematical Problems in Engineering,* Vol. 2015, 2015, pp. 1-13.

[11] S. M. Z. Iskandar, B. I. Zarina, BBDF-α for solving stiff ordinary differential equations with oscillating solutions, *Tamkang Journal of Mathematics,* Vol. 51, No. 2, 2020, pp. 123-136.

[12] J. D. Lambert, *Computational Methods in Ordinary Differential Equations,* John Wiley & Sons, New York, 1973.

[13] J. D. Lambert, *Numerical Methods for Ordinary Differential Systems,* John Wiley & Sons, New York, 1991.

[14] H. Musa, M. B. Suleiman, F. Ismail, N. Senu, B.I. Zarina, An accurate block solver for stiff

initial value problems, *Applied Mathematics,* Vol. 2013, 2013, pp. 1-10.

[15] F. F. Ngwane, S. N. Jator, Block hybrid method using trigonometric basis for initial value problems with oscillating solutions, *Numerical Algorithm,* Vol. 63, 2013, pp. 713-725.

[16] F. F. Ngwane, S. N. Jator, Solving oscillatory problems using a block hybrid trigonometrically fitted method with two off-step points, Electronic Journal of Differential Equations, *Conference on Differential Equations and Computational Simulations,* Vol. 20, 2013, pp. 119-132.

[17] F. F. Ngwane, S. N. Jator, Trigonometrically-fitted second derivative method for oscillatory problems, *SpringerPlus*, Vol. 3, No. 304, 2014, pp. 1-11.

[18] F. F. Ngwane, S. N. Jator, Solving the telegraph and oscillatory differential equations by a block hybrid trigonometrically fitted algorithm, *Hindawi Publishing Corporation,* Vol. 2015, 2015, pp. 1-15.

[19] J. G. Oghonyon, S. A. Okunuga, N. A. Omoregbe, O. O. Agboola, A computational approach in estimating the amount of pond and determining the long time behavioural representation of pond pollution, *Global Journal of Pure and Applied Mathematics,* Vol. 11, No. 5, 2015, pp. 2773-2785.

[20] J. G. Oghonyon, J. Ehigie, S. K. Eke, Investigating the convergence of some selected properties on block predictor-corrector methods and it's applications, *Journal of Engineering and Applied Sciences,* Vol. 11, No. 11, 2017, pp. 2402-2408.

[21] J. G. Oghonyon, O. A. Adesanya, H. Akewe, H. I. Okagbue, Softcode of multi-processing Milne's device for estimating first-order ordinary differential equations, *Asian Journal of Scientific Research,* Vol. 11, No. 4, 2018, pp. 553-559.

[22] J. G. Oghonyon, O. F. Imaga, P. O. Ogunniyi, The reversed estimation of variable step size implementation for solving nonstiff ordinary differential equations, *International Journal of Civil Engineering and Technology,* Vol. 9, No. 8, 2018, pp. 332-340.

[23] J. G. Oghonyon, S. A. Okunuga, H. I. Okagbue, Expanded trigonometrically matched block variable-step-size technics for computing oscillating vibrations, *Lecture Notes in Engineering and Computer Science,* Vol. 2239, 2019, pp. 552-557.

[24] S. A. M. Yatim, B. I. Zarina, K. I. Othman, M. B. Suleiman, A numerical algorithm for solving stiff ordinary differential equations, *Mathematical Problems in Engineering,* Vol. 2013, 2013, pp. .1-12.

[25] A. M. Zarina, M. B. Suleiman, Implementation of four-point fully implicit block method for solving ordinary differential equations, *Applied Mathematics and Computation,* Vol. 184, No. 2, 2007, pp. 514-522.

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-Jimevwo Godwin Oghonyon found the idea, method and implemented the code using Mathematica.
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