On Pairwise C-closed Bitopological Spaces

EMAN ALMUHUR
Department of Basic Science and Humanities, Faculty of Arts and Science,
Applied Science Private University,
Shafa Badran, Amman,
JORDAN

Abstract: - Ismail first proposed the concept of C-closed topological spaces in 1980, assuming that spaces which are countably compact are closed. In 2019, Omar and Hdeib introduced the notion of pairwise C-closed bitopological spaces. In this article, several results concerning these notions are proposed and discussed.
Many findings are summarized in relating pairwise strongly Lindelöf bitopological space and pairwise strongly C-lindelöf.

Key-Words: - Bitopological Space, Pairwise C-closed Space, Pairwise C-Lindelöf.

Received: July 16, 2021. Revised: April 16, 2022. Accepted: May 19, 2022. Published: June 8, 2022.

1 Introduction

In 1963, Kelly [1] initiated "Bitopological Spaces" in an article in the London Mathematical society, thereafter a large number of articles generalized topological concepts to bitoplogical ones. In 1980, Ismail [2] introduced C-closed topological spaces in which he assumed that every countably compact is closed. Omar and Hdeib [3] introduced the concept of pairwise C-closed bitopological spaces in 2019. The notion of strongly Lindelöf spaces was introduced in an article named "Strongly compact spaces" by Mashour in 1984 where he required that each preopen cover of the space to have a countable subcover.

Omar and Hdeib [3] called a bitopological space (X, τ_1, τ_2) a pairwise countably compact space if the countably open cover of X has a finite subcover. Also, they introduced the notion of pairwise C-closed spaces where every τ_1 −countably compact subset of a bitopological space (X, τ_1, τ_2) is τ_2 −closed and every τ_2 −countably compact subset of X is τ_1 −closed [4].

2 Preliminaries

The bitopological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (simply X and Y) are bitopological spaces on which no separation axioms are required unless clearly indicated throughout this work.

For the bitopological space (X, τ_1, τ_2) :

i. A cover $\widetilde{U}$ of the bitoplogical space (X, τ_1, τ_2) is pairwise open cover if $\widetilde{U}$ covers (X, τ_i) $\forall i = 1,2$.

ii. X is pairwise countably compact if the countably pairwise open cover of X has finite subcover.

iii. X is called p-Hausdorff [1] if $\forall x \neq y$ in X, $\exists W_1, W_2 : x \in W_1, y \in W_2$ and $W_1 \cap W_2 = \emptyset$.

vi. X is regular with respect to τ_2 if $\forall x \in X$ and τ_2 −closed subset K not containing x, $\exists W$ a τ_1 −open subset of X and V which is a τ_2 −open subset of X and disjoint from W such that $x \in W$ and $K \subseteq V$.

v. X is p-regular [1] if τ_1 is regular with respect to τ_2 and τ_2 is regular with respect to τ_1.

3 Pairwise C-closed Bitopological Spaces

Definition 1: If (X, τ_1, τ_2) is a bitopological space, then:

i. A subset W of X an (i,j) *−preopen* (resp. (i,j) *−preclosed)* [5] if $W \subseteq i - int(j - cl(W))$ (resp. $i - cl(j - int(W))$ $\forall i,j = 1,2$ $i \neq j$. If W is $(1,2)$ −preopen and $(2,1)$ −preopen, so W is *pairwise preopen*. A pairwise preopen complement subset is *pairwise preclosed.*

ii. A bitopological space (X, τ_1, τ_2) is said to be *pairwise strongly C-closed* [6] if a τ_1 −countably compact subset of X is $(2,1)$ −preclosed and τ_2 −countably compact subset of X is $(1,2)$ −preclosed.

iii. A bitopological space (X, τ_1, τ_2) is a *pairwise* C- *Lindelöf* [7] if every $(1,2)$ −preclosed subset is τ_1 −Lindelöf and every $(2,1)$ −preclosed subset is τ_2 −Lindelöf.

iv. A bitopological space (X, τ_1, τ_2) is *pairwise strongly C- Lindelöf* [4] if for every (i,j) −preclosed set A, there exists an (i,j) −preopen cover $\{u_\alpha: \alpha \in \Lambda\}$ of A contains a countable subfamily $\Lambda_1 = \{\alpha_1, \alpha_2, ...\}$ such that A is covered by $\{(j,i) - precl(u_{\alpha_k}): u_{\alpha_k} \in \Lambda_1\}$ where the preclosure of a subset is the smallest preclosed set containing it

Lemma1: A pairwise strongly Lindelöf bitopological space is pairwise strongly C- Lindelöf.
Proof: Assume that (X, τ_1, τ_2) is pairwise strongly Lindelöf space and let $\{u_\alpha: \alpha \in \Lambda\}$ be a cover of pairwise preopen subsets, then the open cover $\{u_\alpha: \alpha \in \Lambda\}$ consists of pairwise open subsets since each preopen subset is open. So, $\exists\{\alpha_1, \alpha_2, ...\}$ a countable subset such that $X = \bigcup_{k=1}^{\infty} u_{\alpha_k}$.Thus X is pairwise Lindelöf.

Proposition 1: A subspace of a pairwise strongly C-closed bitopological space is strongly C-closed.

Proposition 2: Every subspace of a pairwise strongly Lindelöf bitopological space is pairwise strongly Lindelöf.

Proposition 3: Every subspace of a pairwise strongly C-Lindelöf bitopological space is pairwise strongly C-Lindelöf.

Proposition 4: A pairwise Lindelöf bitopological space is a pairwise C-Lindelöf.
Proof: Let K be a τ_1 −preclosed subset of a bitopological space (X, τ_1, τ_2), if $\{u_\alpha: \alpha \in \Lambda\}$ is a cover of K consisting of τ_2 −open subsets of X, then $\tilde{u} = \{u_\alpha: \alpha \in \Lambda\} \cup (X - K)$ is an open cover of X that admits a countable subcover $= \{u_{\alpha_n}: n \in \mathbb{N}\} \cup (X - K)$. So, K is covered by a countable subcover. Thus, K is τ_2 −Lindelöf. Similarly, if we assume that K is a τ_2 −closed subset of X, we will get that K is $\tau_2 -$ Lindelöf.

A bitopological space (X, τ_1, τ_2) is called p-Hausdorff [4] if $\forall$ x $\neq$ y in X, there $\exists$ v a τ_i −open subset and a τ_j −open subset w disjoint from v such that x $\in$ v and y $\in$ w $\forall$i, j = 1,2 i $\neq$ j.

Definition 2: In a bitoplogical space (X, τ_1, τ_2), τ_i *is regular with respect to* τ_j [8] if $\forall x \in X$ and each τ_i −closed subset K such that $x \notin K$, there exists a τ_i −open subset W and a τ_j −open subset V disjoint from W such that $x \in W$ and $K \in V$ $\forall i, j = 1,2$ $i \neq j$.

A bitopological space (X, τ_1, τ_2) is *p-regular* [3] if τ_1 is regular with respect to τ_2 and τ_2 is regular with respect to τ_1.

Definition 3: A subset M of (X, τ_1, τ_2) is said to be (i,j) *−regular open (resp.* (i,j) *−regular closed)* if $M = i - int(j - cl(M)$ (resp. $M = i - cl(j - int(M))$. If M is (1,2) −regular open and (2,1) −regular open, then M is *pairwise regular open*. The complement of a pairwise regular open is also *pairwise regular closed.*

Proposition 5: If the bitopological space (X, τ_1, τ_2) is p-regular pairwise C- Lindelöf, then X is pairwise Lindelöf.
Proof: Let (X, τ_1, τ_2) be a p-regular pairwise C-Lindelöf, that is not pairwise Lindelöf. Let $\widetilde{U} = \{u_\alpha: \alpha \in \Lambda\}$ be an open cover of X that has no countable subcover, but X is C- Lindelöf, so a subcover consisting of pairwise preclosed subsets of it admits a countable subcover , i.e there exists a countable subset $\{\alpha_1, \alpha_2, ...\}$ such that $X = \bigcup_{k \in \mathbb{N}} F_{\alpha_k}$ where $F_\alpha$is pairwise preclosed subset of X. Hence, X is pairwise Lindelöf.

Proposition 6: Every pairwise C- Lindelöf p-regular bitopological space is pairwise strongly Lindelöf.

Proposition 7: A pairwise strongly C- Lindelöf p-regular is a pairwise strongly Lindelöf.

A bitopological space (X, τ_1, τ_2) is said to be *pairwise sequential* [1] if every non τ_i −closed subset F of X contains a sequence that converges to a point in $X - F$ $\forall i = 1,2$.

Proposition 8: The p-Hausdorff pairwise sequential bitopological space is pairwise strongly C-closed.
Proof: Suppose that K is a τ_1 −countably compact subset of X which is not τ_2 −preclosed, then $\exists x \in X$ such that $x \in \tau_2 - (precl(K) - K)$.
Assume that $G = K \cup \{x\}$, then G is τ_1 −countably compact. Since G is a pairwise sequential, $\exists(x_k)$ in a sequence in K such that (x_k) converges to $G - K = \{x\}$, that is (x_k) does not have τ_1 −cluster points in K which is a contradiction.

Proposition 9: A pairwise strongly C-Lindelöf p-regular bitopological space is pairwise Lindelöf.

Corollary 1: A pairwise strongly Lindelöf bitopological space is a pairwise C-Lindelöf.

Proposition 10:[1] Let (X, τ_1, τ_2) be a pairwise p-Hausdorff bitopological space, let (x_k) be a convergent sequence in X, then it has exactly one limit point.

Proposition 11: Let (X, τ_1, τ_2) be a p-Hausdorff, if every pairwise countably compact subset of X is pairwise sequential, then X is pairwise strongly C-closed.
Proof: Suppose that A is a τ_1 −countably compact subset of X and that A is not τ_2 −preclosed. If $x \in \tau_2 - precl(A) - A$, and $B = A \cup \{x\}$, then B is τ_1 −countably compact. A is not τ_2 −closed in B, but B is sequential, then there exists a sequence (x_k)in A such that $x_k \longrightarrow (B - A) = \{x\}$. Hence, $(x_k)$is a sequence in A that has no τ_1 −limit points in A which is a contradiction.

Consider the bitopological spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2), the function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is *pairwise continuous* provided that it is continuous both as a map from (X, τ_1) to (Y, σ_1) and as map from (X, τ_2) to (Y, σ_2).

Proposition 12: If a p-Hausdoff bitopological space (X, τ_1, τ_2) admits a pairwise continuous surjective mapping into a pairwise C-closed space (Y, σ_1, σ_2), then X is pairwise C-closed.
Proof: Suppose the bitopological space (X, τ_1, τ_2) is p-Housdorff a pairwise and that (Y, σ_1, σ_2) is a pairwise C-closed space, let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise continuous surjective function, if K_1 is a countably compact subset of (X, τ_1), then its image under $f_1: (X, \tau_1) \to (Y, \sigma_1)$ is a countably compact subset of (Y, σ_1), similarly for K_2 in (X, τ_2) under $f_2: (X, \tau_2) \to (Y, \sigma_2)$. Now, since Y is a pairwise C-closed space, $f_i(K_i)$ is closed in $(Y, \sigma_i)\ \forall i = 1,2$. Thus, $(f_i^{-1}(f_i(K_i)$ is closed in $(X, \tau_i)\ \forall i = 1,2$.

Proposition 13: If (X, τ_1, τ_2) is a p-regular bitopological space, then if every point $x \in X$ has a pairwise C-closed neighbourhood, then X is pairwise C-closed**.**
Proof. Suppose that K be a pairwise countably compact subset of X. For any point x in X, there exists a p-open subset W of X containing x. Since X is p-regular, there exists a p-open subset V of X such that $x \in V$ and $cl_i(V) \subseteq W$. Now, $cl(V) \cap K$ is a pairwise countably compact subset of U, $x \in cl(cl(V) \cap K)$, hence K is a neighbourhood of x.

Proposition 14: If (X, τ_1, τ_2) is a p-regular pairwise countably compact, then each pairwise F_σ −subset is pairwise closed in X.

Proof: Suppose that $\tilde{F} = \{F_i: i \in \mathbb{N}\}$ be a family of pairwise closed subsets of X, and assume that $K = \bigcup_{i=1}^{\infty} F_i$ is not pairwise closed, if $x \in cl(K) - K$ and $M = K \cup \{x\}$, then M is pairwise countably compact first countable at x since x is pairwise G_δ. There is a sequence (x_n) in K converging to x, hence this sequence has no cluster points in K which contradicts the assumption.

Proposition 15: If (X, τ_1, τ_2) is a p-regular bitopological space and each of its pairwise countably compact subsets is an F_σ, then X is a pairwise C-closed space.
Proof: By proposition 14, we get the result.

Proposition 16: If (X, τ_1, τ_2) is a p-regular bitopological space and X is the countable union of its pairwise C-closed subspaces, then X is pairwise C-closed.

Proposition 17: If (X, τ_1, τ_2) is a p-regular bitopological space and its points are G_δ , then X is pairwise C-closed.
Proof: Suppose that K is a pairwise countably compact subset of X and that K is not pairwise closed. If $\forall x \in X$, $x \in \overline{K} - K$ and $A = K \bigcup \{x\}$
is a first countable pairwise countably compact subset of X. Consequently there exists a sequence (x_n) converging to x, that is x is not a cluster point which is a contradiction. Thus assumed result is hold.

Proposition 18: If (X, τ_1, τ_2) is a p-Hausdorff bitopological space and each pairwise countably compact subset of X is sequential, then X is pairwise C-closed [9].

Corollary 2: Each pairwise countably compact subset of the bitopological space (X, τ_1, τ_2) is sequential and X is not sequential.

Corollary 3: Each subspace of the pairwise C-closed space is pairwise C-closed.

Lemma 2: If (X, τ_1, τ_2) is a pairwise countably compact C-closed space, then its cardinality is less than or equal to $d(X)^{\omega_0}$ where the density of X is denoted by $d(X)$.
Proof: Consider the subset A of X, if $e^{\omega_0}(A) = \{u \subseteq A: \text{l}u\text{l} \leq \omega_0\}$, let x be a limit point of u and

$\beta(x) = \{x \in u: u \subseteq e^{\omega_0}(A)\}$, then $|\beta(x)| \leq |A|^{\omega_0}$. If u_0 is a dense subset of X and $\alpha < \beta < \omega_1$, then $|u_\beta| \leq d(X)^{\omega_0}$and hence $|u_\alpha| \leq d(X)^{\omega_0}$. If $K = \bigcup_{\alpha<\omega_1} u_\alpha$, then K is pairwise countably compact and $|K| \leq d(X)^{\omega_0}$and u_0 is a pairwise dense subset of X, thus $X = K$.

4 Conclusion

Every pairwise strongly Lindelöf bitopological space is pairwise strongly C- Lindelöf and each subspace of a pairwise strongly C-closed bitopological space is strongly C-closed. For the subspaces of a pairwise strongly Lindelöf bitopological spaces, each subspace is pairwise strongly Lindelöf and the subspace of a pairwise strongly C-Lindelöf bitopological space is pairwise strongly C-Lindelöf. For the pairwise Lindelöf bitopological space, they are pairwise C-Lindelöf spaces. Pairwise C- Lindelöf p-regular bitopological space is pairwise strongly Lindelöf and strongly C-Lindelöf p-regular spaces are pairwise strongly Lindelöf.

References:

[1] Kelly J. C., Bitopological spaces, London Math. Soc. Proc. (1965), 15, pp. 71-89.

[2] Ismail M., Nyikos P. On spaces in which countably compact sets are closed, and hereditary properties. Topology and its Applications,(1980), 11(3), pp. 281-292.

[3] Omar N., Hdeib H. On pairwise C-closed space in bitopological space, Journal of Semigroup Theory and Applications, (2019), 3, pp. 1-7.

[4] Arhangel'skii A.V. Bicompact sets and the topology of spaces. Trudy Moskov. Mat. Qbsc. (1965), 13, pp. 3-55.

[5] Chaber J. Conditions which imply compactness in countably compact spaces. Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. (1976), 24(11), pp. 993-998.

[6] Ismail M. A note on a theorem of Arhangel'skii. General Topology and Appl, (1978), 3, pp. 217-220.

[7] Ismail M. Generalizations of compactness and cardinal invariants of spaces. Dissertation, University of Pittsburgh, 1979.

[8] Olson R.S. Biquotient maps, Countably Biquotient spaces and related topics, General Topology and Appl. (1974), 4, pp. l-28.

[9] Omar N., Hdeib H., Almuhur E., Al-labadi M. On P_2-C-Closed Space in Bitopological Space. Advances in Mathematics: Scientific Journal, (2021), 10(3), pp. 1839–1843.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The correspondence author carried out the all propositions and their proofs.

Creative Commons Attribution License 4.0 (Attribution 4.0 International, CC BY 4.0)

This article is published under the terms of the Creative Commons Attribution License 4.0
https://creativecommons.org/licenses/by/4.0/deed.en_US