Doubly Truncated Power- Hazard Rate Distribution via Generalized Order Statistics

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Abstract: - The paper highlights the moments characteristics of the doubly truncated power hazard rate distribution via generalized order statistics. The particular cases and several deductions are explained. The characterization result has also deliberated. Additionally, some numerical computations through R software are listed.

Key-Words: - Single and product moments, Truncation, and Characterization.

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1 Introduction

The behavior of any probability distribution depends on its hazard functions. Several hazard functions are available to deal with the different data. The power hazard function has one of them to receive attention among researchers. The power hazard function has suggested by [1]. This model is adaptable to befit all classical structures, including increasing, constant, and decreasing.

The hazard function (HF), probability density function (PDF) and cumulative density function (CDF) for the power hazard rate distribution $(PHRD)$ are stated respectively as observes

$$h(x) = \alpha x^{\theta}, x > 0, \alpha > 0 \text{ and } \theta > -1 \qquad (1)$$

$$f(x) = \alpha x^{\theta} e^{-\left\{\frac{\alpha}{\theta+1} x^{\theta+1}\right\}}, x > 0, \qquad (2)$$

$$F(x) = 1 - e^{-\left\{\frac{\alpha}{\theta+1} x^{\theta+1}\right\}},\ x > 0, \qquad (3)$$

where α and θ are scale and shape parameters.

The $PHRD$ is still getting a lot of attention by several authors due to its flexible properties of hazard rate function (HRF). The model given in this article generalizes various important distributions, (see, Weibull, exponential, Rayleigh, and linear failure rate distribution). More detail information, see [2].

1.1 Doubly Truncated Power Hazard Rate Distribution

This sub-section describes the formulation of doubly $TPHRD$ as follows

For stated P_1 and Q_1

$\int_0^{Q_1} f_D(x)dx = Q$ and $\int_0^{P_1} f_D(x)dx = P$.

The pdf of doubly $TPHRD$ is

$$f_D(x) = \frac{\alpha x^{\theta} e^{-\left\{\frac{\alpha}{\theta+1} x^{\theta+1}\right\}}}{P-Q}, x \in (Q_1, P_1), \qquad (4)$$

and the df $F_D(x)$ of (4) is

$$\bar{F}_D(x) = -P_2 + \frac{1}{\alpha x^{\theta}} f_D(x), \qquad (5)$$

$$f_D(x) = \alpha x^{\theta}[P_2 + \bar{F}_D(x)] \qquad (6)$$

where

$$P_2 = \frac{1-p}{p-Q}, \quad Q_2 = \frac{1-Q}{p-Q}$$

$$P = 1 - e^{-\left\{\frac{\alpha}{\theta+1} P_1^{\theta+1}\right\}},\ Q = 1 - e^{-\left\{\frac{\alpha}{\theta+1} Q_1^{\theta+1}\right\}}.$$

The doubly truncated distributions have a significant contribution in many domains of science such as hydrology, economics, biology, cosmology engineering psychology, etc. ([3-4]). After a detailed search, we notice that the moment properties of doubly truncated $PHRD$ remain unknown, which is the theme of the findings.

1.2 Generalized Order Statistics

This sub-section reviews some basic definitions of generalized order statistics (GOS).

The GOS has been reported in literature by [5]. It is a well-developed model for ascendingly ordered random variables (RV). This concept has become an indispensable tool in the field of mathematical and applied statistics.

Let $X_1, \dots, X_n$ be RVs having CDF $F(.)$ and PDF $f(.)$, if it contains the joint PDF of $n\ GOS$ as the following form

$$f_{(1:,\dots,n,\tilde{m},k)}(x_1, \dots, x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right) \times \left(\prod_{i=1}^{n-1}[\bar{F}(x_i)]^{m_i} f(x_i)\right)[\bar{F}(x_n)]^{k-1} f(x_n) \qquad (7)$$

where $\bar{F}(x) = 1 - F(x)$
and $\gamma_i = k + (n-i)(m+1),\ i = 1,2,\dots,n$

From (7) the PDF of the r^{th} GOS is

$$f_{r:n,m,k}(x) = \frac{C_{r-1}}{(r-1)!} f(x)[\bar{F}(x)]^{\gamma_r - 1} g_m^{r-1}[F(x)],$$

$-\infty < x < \infty$ (8)

The joint PDF of the r^{th} and s^{th} GOS is

$$f_{r,s:n,m,k}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x)$$

$$\times g_m^{r-1} F(x)[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(y), \quad (9)$$

$1 \le r < s \le n$ and $-\infty < x < y < \infty$ are needed for (9). Further, we note that

$$C_{s-1} = \prod_{i=1}^{s} \gamma_i,$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1} & , m \ne -1 \\ -\log(1-x) & , m = -1 \end{cases}$$

and $g_m(x) = h_m(x) - h_m(0),\ x \in [0,1)$.

Ordinary order statistics ($O.O.S.$), sequential $O.S.$, progressively Type -II censoring $O.S.$, and record values are main examples of the GOS model. For more details [6-7].

The doubly truncated distribution of GOS develops from GOS when a sample is from non-truncated distribution. Many authors have developed the moment properties of GOS for doubly truncated distribution. Detailed information can be noticed in, [8-14]and among others.

Reducing the number of direct computations is the main characteristic of recurrence relations. The characterization outcomes play an essential part to finds out the probability distributions. This article addresses the moments of doubly truncated $PHRD$ using GOS, which are unseen in the literature.

The remainder of the manuscript is as follows: Section 2 contains the recurrence relations for single moments and numerical computations for mean and variance for several values of parameters. Product moments are elaborated in Section 3. Characterization result from doubly truncated $PHRD$ based on GOS is in Section 4. Section 5 ends with conclusion.

2 Single Moments

Here use, $E[X^{\delta}(r:m,n,k)] = \mu_{r:m,n,k}^{(\delta)}$

Theorem 2.1. For reported pdf in (4) and $n \in N$, $m \in \Re$, $2 \le r \le n$, $\delta = 0,1,2 \dots$

$$\mu_{r:m,n,k}^{(\delta)} = P_2 A\left\{\frac{\alpha}{\delta+\theta+1}\left[\mu_{r:n-1,m,k+m}^{\delta+\theta+1}\right] - \left[\mu_{r-1:n-1,m,k+m}^{\delta+\theta+1}\right]\right\} + \frac{\alpha}{\delta+\theta+1}\left\{\gamma_r\left[\mu_{r:n,m,k}^{\delta+\theta+1}\right] - \left[\mu_{r-1:n,m,k}^{\delta+\theta+1}\right]\right\} \quad (10)$$

where

$$A = \frac{C_{r-2}}{C_{r-2}^{(n-1,k+m)}} = \prod_{i=1}^{r-1}\left(\frac{\gamma_i}{\gamma_i - 1}\right),$$

$C_{r-2}^{(n-1,k+m)} = \prod_{i=1}^{r-1} \gamma_i^{(n-1,k+m)}$ and

$\gamma_i^{(n-1,k+m)} = \gamma_i - 1.$

Proof: Applying (6) in (8), we have

$$\mu_{r:m,n,k}^{(\delta)} = \frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{\delta}[\bar{F}_D(x)]^{\gamma_r - 1}\{(\alpha x^{\theta}) \times [P_2 + \bar{F}_D(x)]\} g_m^{r-1}[F_D(x)]dx.$$

Next, one can write the above expression as

$$\mu_{r:m,n,k}^{(\delta)} = \frac{C_{r-1}}{(r-1)!}\left[P_2\left\{\alpha\int_{Q_1}^{P_1} x^{\delta+\theta}[\bar{F}_D(x)]^{\gamma_r^{(n-1,k+m)}} g_m^{r-1}[F_D(x)]dx + \alpha\int_{Q_1}^{P_1} x^{\delta+\theta}[\bar{F}_D(x)]^{\gamma_r} g_m^{r-1}[F_D(x)]dx\right]$$

$$\mu_{r:m,n,k}^{(\delta)} = \frac{C_{r-1}}{(r-1)!}\left[P_2\left\{\alpha B_{\delta+\theta}^{(n-1,k+m)}(x)\right\} + \alpha B_{\delta+\theta}^{(n,k)}(x)\right]$$

(11)

where

$$B_{\delta+\theta}^{(n-1,k+m)}(x) = \int_{Q_1}^{P_1} x^{\delta+\theta}[\bar{F}_D(x)]^{\gamma_r^{(n-1,k+m)}} g_m^{r-1}[F_D(x)]dx$$

$$B_{\delta+\theta}^{(n,k)}(x) = \int_{Q_1}^{P_1} x^{\delta+\theta}[\bar{F}_D(x)]^{\gamma_r} g_m^{r-1}[F_D(x)]dx.$$

Integrating by parts taking $x^{\delta+\theta}$ for integration, we obtain

$$B_{\delta+\theta}^{(n-1,k+m)}(x) = \frac{(r-1)!}{(\delta+\theta+1)C_{r-2}^{(n-1,k+m)}}\left[\mu_{r:n-1,m,k+m}^{\delta+\theta+1} - \mu_{r-1:n-1,m,k+m}^{\delta+\theta+1}\right].$$

Similarly

$$B_{\delta+\theta}^{(n,k)}(x) = \frac{(r-1)!}{(\delta+\theta+1)C_{r-2}}\left[\mu_{r:n,m,k}^{\delta+\theta+1} - \mu_{r-1:n,m,k}^{\delta+\theta+1}\right]$$

Inserting the terms of $B_{\delta+\theta}^{(n-1,k+m)}(x)$ and $B_{\delta+\theta}^{(n,k)}(x)$ in (10) and solving, the Theorem 2.1 is proved.

Some corollaries and remarks based on single moments of GOS, when sample from doubly truncated $PHRD$ is described as follows.

2.1 Corollary

(i) For $(m = 0, k = 1)$, Theorem 2.1 reduces to single moments of $O.S.$, as

$$\mu_{r:m}^{(\delta)} = P_2\left\{\frac{\alpha}{\delta+\theta+1}\left[\mu_{r:n-1,}^{\delta+\theta+1}\right] - \left[\mu_{r-1:n-1}^{\delta+\theta+1}\right]\right\} + \frac{\alpha}{\delta+\theta+1}\left\{(n-r+1)\left[\mu_{r:n}^{\delta+\theta+1}\right] - \left[\mu_{r-1:n}^{\delta+\theta+1}\right]\right\}$$

(ii) Single moments of k^{th}record can be given from Theorem 2.1. $(at\ m = -1)$

(iii) Setting $P = 1$ and $Q = 0$ (for non-truncated case) in Theorem 2.1,

$\mu^{(\delta)}_{r:m,n,k} = \frac{\alpha}{\delta+\theta+1}\{\gamma_r[\mu^{\delta+\theta+1}_{r:n,m,k}] - [\mu^{\delta+\theta+1}_{r-1:n,m,k}]\}$

as reported by similar result in [15] for $PHRD$.

(iv) As stated in Corollary (iii) and setting $m = -1, k = 1$ in (10), the corresponding result is same as obtained by [16] for $PHRD$.

Remark 2.1

(i) Doubly truncated exponential distribution can be obtained at $\theta = 0$ in Theorem 2.1,

(ii) Setting $\theta = \alpha - 1$, in Theorem 2.1, we get doubly truncated Weibull distribution as discussed by [10].

(iii) Setting $\theta = 1$ in Theorem 2.1, it gives doubly truncated linear exponential distribution, as established by [12].

(iv) Setting $\alpha = \frac{1}{\beta^2}$ and $\theta = 1$ in Theorem 2.1, it yields, doubly truncated Rayleigh distribution.

For arbitrarily selected values of (α, θ) and sample sizes $n = 10, 20, \ldots,$ Table 1-2, represents the numerical computations of first four moments and variances of $O.S.$ from $PHRD$.

Table 1. Moments of $O.S.$ for $PHRD$.

n	**r**	$\alpha = 1, \theta = 2$				$\alpha = 2, \theta = 1$			
		$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$
10	1	0.1381	0.0648	0.0333	0.0184	0.0992	0.0250	0.0074	0.0025
20	1	0.1097	0.0408	0.0167	0.0073	0.0701	0.0125	0.0026	0.0006
	2	2.0835	0.7760	0.3167	0.1389	1.3312	0.2375	0.0499	0.0119
30	1	0.0958	0.0312	0.0111	0.0043	0.0572	0.0083	0.0014	0.0003
	2	2.7781	0.9038	0.3222	0.1235	1.6599	0.2417	0.0415	0.0081
	3	19.4466	6.3269	2.2556	0.8643	11.6127	1.6917	0.29032	0.0564
40	1	0.0870	0.0257	0.0083	0.0029	0.0495	0.0062	0.0009	0.0002
	2	3.3944	1.0034	0.3250	0.1131	1.9321	0.2438	0.0362	0.0061
	3	32.2469	9.5322	3.0875	1.0749	18.3551	2.3156	0.3442	0.0579
	4	132.5707	39.1879	12.6931	4.4190	75.4601	9.5198	1.4149	0.2380
50	1	0.0808	0.0222	0.0067	0.0022	0.0443	0.0050	0.0007	1e-04
	2	3.960	1.0864	0.3267	0.1056	2.1712	0.2450	0.0326	0.0049
	3	47.5088	13.0369	3.92	1.2669	26.0551	2.9400	0.3908	0.0588
	4	248.1016	68.0816	20.4711	6.6160	136.0654	15.3533	2.0410	0.3071
	5	713.2921	195.7347	58.8544	19.0211	391.188	44.1408	5.8678	0.8828
		$\alpha = 3, \theta = 4$				$\alpha = 4, \theta = 3$			
n	**r**	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$
10	1	0.0931	0.0455	0.0232	0.0122	0.0901	0.0350	0.0144	0.0062
20	1	0.0810	0.0345	0.0153	0.0070	0.0758	0.0248	0.0086	0.0031
	2	1.5384	0.6555	0.2911	0.1338	1.4396	0.4706	0.1632	0.0594
30	1	0.0746	0.0293	0.0120	0.0051	0.0685	0.0202	0.0063	0.0021
	2	2.1652	0.8507	0.3483	0.1477	1.9854	0.5865	0.1838	0.0604
	3	15.1565	5.955	2.4383	1.0334	13.8983	4.1057	1.2865	0.4229
40	1	0.0705	0.0261	0.0101	0.0040	0.0637	0.0175	0.0051	0.0016
	2	2.7490	1.0197	0.3942	0.1577	2.4848	0.6831	0.1992	0.0609
	3	26.1159	9.6872	3.7447	1.4983	23.6058	6.4895	1.8923	0.5789
	4	107.3652	39.8252	15.3947	6.1597	97.0462	26.6792	7.7793	2.3799
50	1	0.0674	0.0239	0.0088	0.0033	0.0603	0.0157	0.0043	0.0013
	2	3.3032	1.1718	0.4332	0.1658	2.9526	0.76775	0.2117	0.0612
	3	39.6380	14.0612	5.1982	1.9891	35.4308	9.2119	2.5403	0.7350
	4	206.9984	73.4309	27.1463	10.3877	185.0275	48.1064	13.2662	3.8383
	5	595.1203	211.1139	78.0455	29.8646	595.1203	211.1139	78.0455	29.8646

Table 2. Variances of $O.S.$ for $PHRD$.

n	r	$\alpha = 1$		$\alpha = 2$		$\alpha = 3$		$\alpha = 4$	
		$\theta = 3$	$\theta = 4$	$\theta = 3$	$\theta = 4$	$\theta = 5$	$\theta = 6$	$\theta = 5$	$\theta = 6$
10	1	0.0538	0.0572	0.0381	0.0434	0.0402	0.0419	0.0365	0.0386
20	1	0.0381	0.0434	0.0269	0.0329	0.0319	0.0344	0.0289	0.0316
	2	5.0862	4.6901	3.5965	3.5544	2.9269	2.8471	2.6592	2.5416

30	1	0.0311	0.0369	0.0220	0.0279	0.0279	0.0306	0.0253	0.0282
	2	9.0573	8.5955	6.4045	6.5142	5.4497	5.1827	4.9513	4.7738
	3	378.1142	347.2498	267.3671	263.1662	213.0991	196.2422	193.6136	180.7573
40	1	0.0269	0.0329	0.0190	0.0250	0.0253	0.0282	0.0230	0.0260
	2	13.7149	13.3101	9.6979	10.0871	8.5490	8.1934	7.7673	7.3853
	3	1101.492	1043.385	778.872	790.7382	654.1655	612.1221	594.3487	563.8213
	4	18889.27	17950.38	13356.74	13603.85	11291.1	10600.43	9763.988	8274.752
50	1	0.0241	0.0306	0.0170	0.0228	0.0235	0.0228	0.0214	0.0244
	2	18.9706	19.8566	13.4142	14.2102	12.1759	11.7492	11.0626	10.8430
	3	2492.259	2416.392	1762.294	1831.283	1539.214	1457.132	1398.469	1342.153
	4	68566.56	66607.95	48483.87	50479.38	42512.04	40325.07	38624.75	37143.15
	5	565673.6	416202.2	399991.6	416281.5	350430.3	332261.5	318387	306043.6

3 Product Moments

Here use,

$$E\left[X^{\delta_1}(r:n,m,k)\,X^{\delta_2}(s:n,m,k)\right]=\mu_{r,s:n,m,k}^{(\delta_1,\delta_2)}$$

Theorem 3.1. For outlined pdf in (4) and $1\le r<s\le n-1,\ \delta_1,\delta_2\ge 0$

$$\mu_{r,s:n,m,k}^{(\delta_1,\delta_2)}=P_2A^*\left[\left\{\frac{\alpha}{\delta_2+\theta+1}\left[\mu_{r,s:n-1,m,k+m}^{(\delta_1,\delta_2+\theta+1)}\right]-\left[\mu_{r,s-1:n-1,m,k+m}^{(\delta_1,\delta_2+\theta+1)}\right]\right\}\right]+\frac{\alpha}{\delta_2+\theta+1}\times\left\{\gamma_s\left[\mu_{r,s:n,m,k}^{(\delta_1,\delta_2+\theta+1)}\right]-\left[\mu_{r,s-1:n,m,k}^{(\delta_1,\delta_2+\theta+1)}\right]\right\} \quad (12)$$

where

$$A^*=\frac{C_{s-2}}{C_{s-2}^{(n-1,k+m)}}=\prod_{i=1}^{s-1}\left(\frac{\gamma_i}{\gamma_i-1}\right).$$

Proof: Using (8), we have

$$\mu_{r,s:n,m,k}^{(\delta_1,\delta_2)}=\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{Q_1}^{P_1}x^{\delta_1}[\bar{F}_D(x)]^m\times f_D(x)g_m^{r-1}[F_D(x)]L(x)dx \quad (13)$$

where

$$L(x)=\int_x^{P_1}y^{\delta_2}[h_m(F_D(y))-h_m(F_D(x))]^{s-r-1}[\bar{F}_D(y)]^{\gamma_s-1}f_D(y)dy \quad (14)$$

Next, using (5) in (13), we get

$$L(x)=P_2\left\{\alpha\int_x^{P_1}y^{\delta_2+\theta}[h_m(F_D(y))-h_m(F_D(x))]^{s-r-1}[\bar{F}_D(y)]^{\gamma_s^{(n-1,k+m)}}dy\right\}+\alpha\int_x^{P_1}y^{\delta_2+\theta}[h_m(F_D(y))-h_m(F_D(x))]^{s-r-1}[\bar{F}_D(y)]^{\gamma_s}dy$$

$$L(x)=P_2\left\{\alpha L_{\delta_2+\theta}^{(n-1,k+m)}(x)+\alpha L_{\delta_2+\theta}^{(n,k)}(x)\right\} \quad (15)$$

where

$$L_{\delta_2+\theta}^{(n-1,k+m)}(x)=\int_x^{P_1}y^{\delta_2+\theta}\left[h_m\left(F_D(y)\right)-h_m(F_D(x))\right]^{s-r-1}[\bar{F}_D(y)]^{\gamma_s^{(n-1,k+m)}}dy$$

and

$$L_{\delta_2+\theta}^{(n,k)}(x)=\int_x^{P_1}y^{\delta_2+\theta}\left[h_m\left(F_D(y)\right)-h_m(F_D(x))\right]^{s-r-1}[\bar{F}_D(y)]^{\gamma_s}dy.$$

Integrating by parts taking $y^{\delta_2+\theta}$ for integration, we get

$$L_{\delta_2+\theta}^{(n,k)}(x)=\frac{1}{(\delta_2+\theta+1)}\left\{\gamma_s\int_x^{P_1}y^{\delta_2+\theta+1}\left[h_m(F_D(y))-h_m\left(F_D(x)\right)\right]^{s-r-1}[\bar{F}_D(y)]^{\gamma_s-1}f_D(y)dy-(s-r-1)\int_x^{P_1}y^{\delta_2+\theta+1}\left[h_m(F_D(y))-h_m\left(F_D(x)\right)\right]^{s-r-2}[\bar{F}_D(y)]^{\gamma_s+m}f_D(y)dy\right\}$$

Similarly

$$L_{\delta_2+\theta}^{(n-1,k+m)}(x)=\frac{1}{(\delta_2+\theta+1)}\left\{\gamma_s\int_x^{P_1}y^{\delta_2+\theta+1}\left[h_m(F_D(y))-h_m\left(F_D(x)\right)\right]^{s-r-1}[\bar{F}_D(y)]^{\gamma_s^{(n-1,k+m)-1}}f_D(y)dy-(s-r-1)\int_x^{P_1}y^{\delta_2+\theta+1}\left[h_m(F_D(y))-h_m\left(F_D(x)\right)\right]^{s-r-2}[\bar{F}_D(y)]^{\gamma_{s-1}^{(n-1,k+m)-1}}f_D(y)dy\right\}.$$

Following the same steps as reported in Theorem 2.1. Hence the Theorem 3.1 is complete.

Note: Product moments is reduced to single moments at $\delta_1=0$. The corollaries and remarks as listed in Section 2 are the same for the product moments.

4 Characterization

The characterization of doubly $TPHRD$ is addressed in this section.

Theorem 4.1: Let X follows doubly $TPHRD$, then the necessary and sufficient condition for X is listed below

$$\mu_{r:n,m,k}^{(\delta)}=P_2A\left\{\frac{\alpha}{\delta+\theta+1}\left[\mu_{r:n-1,m,k+m}^{(\delta+\theta+1)}\right]-\left[\mu_{r-1:n-1,m,k+m}^{(\delta+\theta+1)}\right]\right\}+\frac{\alpha}{\delta+\theta+1}\left\{\gamma_r\left[\mu_{r:n,m,k}^{(\delta+\theta+1)}\right]-\left[\mu_{r-1:n,m,k}^{(\delta+\theta+1)}\right]\right\}$$

(16)

where A is defined in Section 2.

Proof: Using (10), it is easy to determine the necessary part.

For sufficiency part: On rearranging the terms in (16), and after some simplification, it gives

$$\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{\delta}[\bar{F}_D(x)]^{\gamma_r-1} g_m^{r-1}[F_D(x)] f_D(x)dx = P_2\left\{\frac{\alpha}{(\delta+\theta+1)}\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{\delta+\theta+1} k'_{\gamma_r-1}(x)dx\right\} + \frac{\alpha\gamma_r}{(\delta+\theta+1)}\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{\delta+\theta+1} k'_{\gamma_r}(x)dx,$$
(17)

where

$$k_{\gamma_r}(x) = -[\bar{F}_D(x)]^{\gamma_r} g_m^{r-1}[F_D(x)] \qquad (18)$$

and

$$k'_{\gamma_r}(x) = -[\bar{F}_D(x)]^{\gamma_r} g_m^{r-2}[F_D(x)] \times f_D(x)\left[\frac{g_m[F_D(x)]}{[\bar{F}_D(x)]} - \frac{(r-1)[\bar{F}_D(x))]^m}{\gamma_r}\right]$$

Now, integrating RHS in (16) by parts. Utilizing the values of $k_{\gamma_r}(x)$ and $k_{\gamma_r-1}(x)$ from (18), we get

$$\frac{C_{r-1}}{(r-1)!}\int_{Q_1}^{P_1} x^{\delta-1}[\bar{F}_D(x)]^{\gamma_r-1} g_m^{r-1}[F_D(x)]\left[f_D(x) - \alpha x^{\theta}(P_2 + \bar{F}_D(x))\right]dx = 0. \qquad (19)$$

Next, generalization of the Müntz-Szász theorem [17] apply to (19), we get

$$f_D(x) = \alpha x^{\theta}[P_2 + \bar{F}_D(x)]$$

5 Conclusion

Moment's properties of GOS from doubly $TPHRD$ are investigated. For selected values, means and variances for order statistics are enumerated. Characterization of doubly $TPHRD$ via GOS is given.

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References:

[1] Mugdadi, A. R., The least squares type estimation of the parameters in the power hazard function, *Applied Mathematics Computation*, Vol. **169,** 2005, pp. 737–748.

[2] Ismail, K., Estimation of $P(Y < X)$ for distribution having power hazard function, *Pakistan Journal of Statistics*, Vol. **30,** 2014, pp. 57–70.

[3] Mailhot, L., Some properties of truncated distributions connected with log-concavity of distribution functions, *Applicationes Mathematicae,* Vol. **20,** No. 4, 1988, pp.531–

[4] Salah, H. A., Properties of doubly truncated Fréchet distribution, *American Journal of Applied Mathematics and Statistics,* Vol. **4,** No.1, 2016, pp. 9-15.

[5] Kamps, U., A *Concept of Generalized Order Statistics*. B.G. Teubner Stuttgart, Germany, 1995.

[6] Ahsanullah, M. and Nevzorov, V. B., *Ordered Random Variables*, Nova Science Publishers, USA, 2001.

[7] Shahbaz, M. Q., Ahsanullah, M., Hanif Shahbaz, S. and Al-Zahrani, B., *Ordered Random Variables: Theory and Applications*, Atlantis Studies in Probability and Statistics, Springer, 2016.

[8] Ahmad, A. A. and Fawzy, A. M., Recurrence relations for single moments of generalized order statistics from doubly truncated distributions, *Journal of Statistical and Planning Inference,* Vol. **117,** 2013, pp. 241-249.

[9] Ahmad, A. A., Recurrence relations for single and product moments of generalized order statistics from doubly truncated Burr type XII distribution, *Journal of Egyptian Mathematical Society,* Vol.**15,** 2007, pp. 117-128.

[10] Khan, R. U., Anwar, Z. and Athar, H., Recurrence relations for single and product moments of generalized order statistics from doubly truncated Weibull distribution, *Aligarh Journal of Statistics*, Vol. **27,** 2007, pp. 69–79.

[11] Kumar, D. and Khan, M. I., Relations for generalized order statistics from doubly truncated generalized exponential distribution and its characterization, *Mathematical Science Letter,* Vol. **2,** No. 1, 2013, pp. 9-18.

[12] Khan, R. U. and Zia, B., Generalized order statistics of doubly truncated linear exponential distribution and a characterization, *Journal of Applied Probability and Statistics*, Vol. **9,** No.1, 2014, pp. 53-65

[13] Jamal, F. and Chesneau, C., The Moment Properties of Order, Reversed Order and Upper Record Statistics for the Power Ailamujia Distribution, *WSEAS Transactions on Mathematics*, Vol. **20,** 2021, pp. 607-614.

[14] Khan, M.I., Moments of generalized order statistics from doubly truncated power-linear hazard rate distribution, *Statistics Optimization, and Information Computing*, 2022, (Accepted).

[15] Khan, M. I., The distribution having power hazard function based on ordered random variable, *Journal of Statistics Applications & Probability Letter*, Vol. **4,** No.1, 2017, pp. 33-36.

[16] Khan, M. I. and Khan, M. A. R, Generalized record values from distributions having power hazard function and characterization, *Journal of Statistics Applications & Probability*, Vol. **8,** No. 2, 2019, pp.103-111.

[17] Hwang, J. S. and Lin, G. D., Extensions of Muntz-Szasz theorems and application, *Analysis,* Vol. **4,** No. 1-2, 1984, pp.143–160.