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Optimal number of required points for Evaluation - Interpolation technique in complex basis

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Abstract: - According to polynomial evaluation-interpolation technique, first of all, the set of required points, in which interpolation will be executed, must be defined in evaluation part. Taking into account that working in complex domain provides as with the advantage of conjugate properties, the number of evaluations at these points could be reduced. This paper presents the appropriate form of required points, aiming to an optimal reduction of their amount. The appropriate form of points for two variable polynomials depends on the degree of each variable. As a repercussion the number of evaluations could be reduced even up to the half.

Key-Words: - complex domain, Evaluation - Interpolation technique, two-variable polynomials, polynomial degrees

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1 Introduction

Through years, many numerical computational methods have been developed in the need of solving systems and control theory problems, especially those that consists of polynomials in one or even in several variables, [1], [2]. Such problems are calculation of the determinant of a polynomial matrix [3], [4], computation of the generalised inverse [5], [6], Moore-Penrose inverse [7] and Drazin inverse of a polynomial matrix [8], as well as calculation of the great common divisor among polynomials [9], transfer function computation for multidimensional systems [10] and solution of diophantine and matrix equations [11].

Newton evaluation - interpolation technique is one of these methods. In this method instead of finding coefficients of the polynomial by solving a system of equations (analytical solution), numerical analysis is used. It consists of two parts. In evaluation part, first of all, a set of distinct required points is defined. The number of points depends on the degree of polynomial. In interpolation part, the unique polynomial that passes through these points is defined [12].

According to [13] choosing the appropriate form of points is less expensive both in memory and in computational cost. Taking into account that working in complex domain provides us with the opportunity of using conjugate properties and according to [4], we can reduce the number of evaluations in the first part of this technique, in the means of choosing the form of the required points by setting abscissa or ordinate as complex number. Points that are defined by conjugate numbers give conjugate results as well, when they are evaluated. That is why the number of the needed operations in evaluation part can be reduced.

The purpose of this work is to indicate the most appropriate form of points according to the degree of each variable of the polynomial. That's why we study two variable polynomials. The upper bound of the degree of each variable is already known, so we use rectangular basis. First of all an example is given to show the significance of choosing the appropriate form of points in complex domain. After that, the cases depending on the degrees of each variable are presented. In addition to that, we prove our claim with a short proof and examples for each case.

2 The advantage of working in complex domain

2.1 Real domain

Let the determinant of a polynomial matrix $A(x,y)$ which is a polynomial $p(x,y)$ whose degrees are $k_1 = deg_x(p(x,y)) = 3$ and $k_2 = deg_y(p(x,y)) = 2$ respectively. According to [12], working in rectangular basis, we need to define $N = (k_1+1)\cdot(k_2+1) = 12$ required fixed points. The set of these points is

$$S^{(3,2)} = \{(x_{\ell_1}, y_{\ell_2}) | \ell_1 = 0, \ldots, 3 \ \ell_2 = 0, \ldots, 2\}$$

and a set of $S^{(3,2)}$ is showing in figure 1.

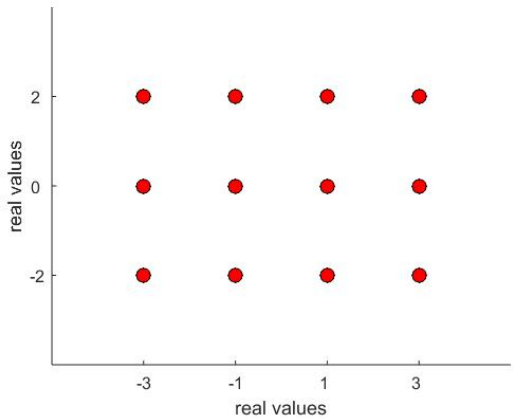

Figure 1: Real domain

We evaluate these points to the matrix $A(x, y)$ to find the constant determinants at the fixed points and construct the zero order table $(k = 0)$ of divided differences. (12 evaluations)

	$y_0 = -2$	$y_1 = 0$	$y_2 = 2$
$x_0 = -3$	$P_{0,0}{}^{(0)} = -24$	$P_{0,1}{}^{(0)} = -28$	$P_{0,2}{}^{(0)} = -38$
$x_1 = -1$	$P_{1,0}{}^{(0)} = -4$	$P_{1,1}{}^{(0)} = -2$	$P_{1,2}{}^{(0)} = -8$
$x_2 = 1$	$P_{2,0}{}^{(0)} = -6$	$P_{2,1}{}^{(0)} = 0$	$P_{2,2}{}^{(0)} = -2$
$x_3 = 3$	$P_{3,0}{}^{(0)} = 16$	$P_{3,1}{}^{(0)} = 26$	$P_{3,2}{}^{(0)} = 28$

For $k = 1$ to n, where $n = max\{k_1, k_2\} = 3$ we compute the k-th order table of divided differences according to [14]. The Newton polynomial (the determinant of $A(x, y)$) is:

$$p(x, y) = x^3 + x \cdot y - y^2 - 1$$

It occurs from the following:

$$p(x, y) = X^T \cdot P \cdot Y$$

where

$$X = \begin{bmatrix} 1 \\ x+3 \\ (x+3)(x+1) \\ (x+3)(x+1)(x-1) \end{bmatrix},$$

$$Y = \begin{bmatrix} 1 \\ y+2 \\ y(y+2) \end{bmatrix},$$

$$P = \begin{bmatrix} -26 & -1 & -1 \\ 11 & 1 & 0 \\ -3 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

P is the n-th order table of divided differences.

2.2 Complex domain

Let the same determinant of a polynomial matrix $A(x, y)$ which is a polynomial $p(x, y)$ and lets define the 12 required points in complex domain. The set of these points is

$$\widetilde{S}^{(3,2)} = \{(a_{\ell_1}, b_{\ell_2}) | \ell_1 = 0, \dots, 3,\ \ell_2 = 0, \dots, 2\}$$

According to [4] and due to the conjugate properties, the number of evaluations can be reduced. We can distinguish two subcases.

2.2.1 First case

Points of the form $(a_{\ell_1} \cdot i, b_{\ell_2})$ where $\ell_1 = 0, \dots, 3$, $\ell_2 = 0, \dots, 2$ and i is the imaginary unit. We have to evaluate only the circle points as they are showing in figure 2. The square ones can occur as conjugate numbers of them. (6 evaluations) We evaluate these

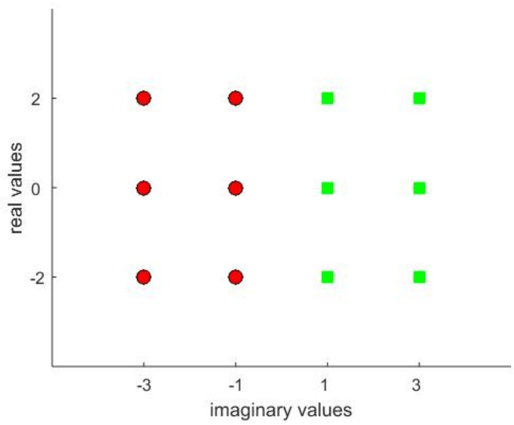

Figure 2: Complex domain, points of the form $(a_{\ell_1} \cdot i, b_{\ell_2})$, circles and squares

points to the matrix $A(x, y)$ to find the constant determinants at the fixed points and construct the zero order table $(k = 0)$ of divided differences.

	$b_0 = -2$	$b_1 = 0$	$b_2 = 2$
$a_0 = -3i$	$P_{0,0}{}^{(0)} = -5 + 33i$	$P_{0,1}{}^{(0)} = -1 + 27i$	$P_{0,2}{}^{(0)} = -5 + 21i$
$a_1 = -i$	$P_{1,0}{}^{(0)} = -5 + 3i$	$P_{1,1}{}^{(0)} = -1 + i$	$P_{1,2}{}^{(0)} = -5 - i$
$a_2 = i$	$P_{2,0}{}^{(0)} = \mathbf{-5 - 3i}$	$P_{2,1}{}^{(0)} = \mathbf{-1 - i}$	$P_{2,2}{}^{(0)} = \mathbf{-5 + i}$
$a_3 = 3i$	$P_{3,0}{}^{(0)} = \mathbf{-5 - 33i}$	$P_{3,1}{}^{(0)} = \mathbf{-1 - 27i}$	$P_{3,2}{}^{(0)} = \mathbf{-5 - 21i}$

Conjugate numbers are denoted by bold

For $k = 1$ to n, where $n = max\{k_1, k_2\} = 3$ we compute the k-th order table of divided differences. The Newton polynomial (the determinant of $A(x, y)$) is:

$$p(x, y) = x^3 + x \cdot y - y^2 - 1$$

It occurs from the following:

$$p(x, y) = \widetilde{X}^T \cdot \widetilde{P} \cdot \widetilde{Y}$$

where

$$\widetilde{X} = \begin{bmatrix} 1 \\ x+3i \\ (x+3i)(x+i) \\ (x+3i)(x+i)(x-i) \end{bmatrix},$$
$$\widetilde{Y} = \begin{bmatrix} 1 \\ y+2 \\ y(y+2) \end{bmatrix},$$
$$\widetilde{P} = \begin{bmatrix} -5+33i & 2-3i & -1 \\ -15 & 1 & 0 \\ -3i & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

P is the n-th order table of divided differences.

2.2.2 Second case

Points of the form $(a_{\ell_1}, b_{\ell_2} \cdot i)$ where $\ell_1 = 0, \ldots, 3$, $\ell_2 = 0, \ldots, 2$ and i is the imaginary unit. We have to evaluate both round and asterisk points as they are showing in figure 3. The square points can occur as conjugate numbers of the round ones. (8 evaluations) We evaluate these points to the matrix $A(x, y)$ to

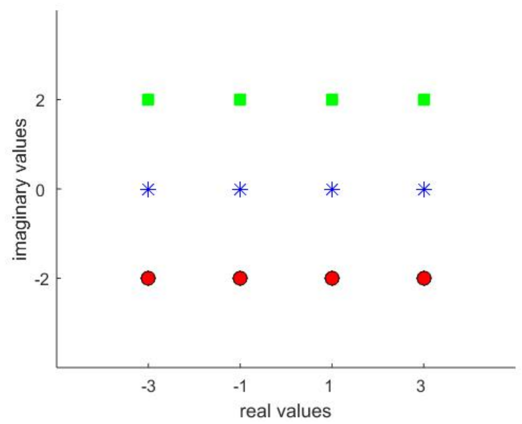

Figure 3: Complex domain,points of the form $(a_{\ell_1}, b_{\ell_2} \cdot i)$

find the constant determinants at the fixed points and construct the zero order table ($k = 0$) of divided differences.

$k=0$	$b_0 = -2i$	$b_1 = 0$	$b_2 = 2i$
$a_0 = -3$	$P_{0,0}{}^{(0)} = -24+6i$	$P_{0,1}{}^{(0)} = -28$	$P_{0,2}{}^{(0)} = \mathbf{-24-6i}$
$a_1 = -1$	$P_{1,0}{}^{(0)} = 2+2i$	$P_{1,1}{}^{(0)} = -2$	$P_{1,2}{}^{(0)} = \mathbf{2-2i}$
$a_2 = 1$	$P_{2,0}{}^{(0)} = 4-2i$	$P_{2,1}{}^{(0)} = 0$	$P_{2,2}{}^{(0)} = \mathbf{4+2i}$
$a_3 = 3$	$P_{3,0}{}^{(0)} = 30-6i$	$P_{3,1}{}^{(0)} = 26$	$P_{3,2}{}^{(0)} = \mathbf{30+6i}$

For $k = 1$ to n, where $n = max\{k_1, k_2\} = 3$ we compute the k-th order table of divided differences. The Newton polynomial (the determinant of $A(x, y)$) is:

$$p(x, y) = \widetilde{X}^T \cdot \widetilde{P} \cdot \widetilde{Y}$$

It occurs from the following:

$$\widetilde{X} = \begin{bmatrix} 1 \\ x+3 \\ (x+3)(x+1) \\ (x+3)(x+1)(x-1) \end{bmatrix}, \widetilde{Y} = \begin{bmatrix} 1 \\ y+2i \\ y(y+2i) \end{bmatrix},$$
$$\widetilde{P} = \begin{bmatrix} -24+6i & -3+2i & -1 \\ 13-2i & 1 & 0 \\ -3 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

the n-th order table of divided differences.
From the above we can conclude that the number of necessary evaluations that have to be computed in the evaluation part, depends on the form of points. Whether abscissa or ordinate are complex number,a different number of evaluations must be executed in each case. That's why we are going to indicate the best form, depending on the degree of each variable.

3 Analysing cases.

For every polynomial p(x,y) where $k_1 = degP_x, k_2 = degP_y$. We have the following cases:

- k_1 even, k_2 odd
  - If we choose points of the form $(a \cdot i, b)$ we have $\widetilde{N} = N/2 + (k_2 + 1)/2$
  - If we choose points of the form $(a, b \cdot i)$ we have $\widetilde{N} = N/2$
- k_1 odd, k_2 even
  - If we choose points of the form $(a \cdot i, b)$ we have $\widetilde{N} = N/2$
  - If we choose points of the form $(a, b \cdot i)$ we have $\widetilde{N} = N/2 + (k_1 + 1)/2$
- k_1 odd, k_2 odd
  - If we choose points of the form $(a \cdot i, b)$ we have $\widetilde{N} = N/2$
  - If we choose points of the form $(a, b \cdot i)$ we have $\widetilde{N} = N/2$
- k_1 even, k_2 even

- if $k_1 < k_2$
  * If we choose points of the form $(a \cdot i, b)$ we have $\widetilde{N} = (N - (k_2+1))/2 + k_2 + 1 = N/2 + (k_2+1)/2$
  * If we choose points of the form $(a, b \cdot i)$ we have $\widetilde{N} = (N - (k_1+1))/2 + k_1 + 1 = N/2 + (k_1+1)/2$
- if $k_1 > k_2$
  * If we choose points of the form $(a \cdot i, b)$ we have $\widetilde{N} = (N - (k_2+1))/2 + k_2 + 1 = N/2 + (k_2+1)/2$
  * If we choose points of the form $(a, b \cdot i)$ we have $\widetilde{N} = (N - (k_1+1))/2 + k_1 + 1 = N/2 + (k_1+1)/2$

The cases are summarized in the following scheme:

$$\begin{cases} k_1 \quad odd & \begin{cases} k_2 \quad odd & (a \cdot i, b) \; or \; (a, b \cdot i) \\ k_2 \quad even & (a \cdot i, b) \end{cases} \\ k_1 \quad even & \begin{cases} k_2 \quad odd & (a, b \cdot i) \\ k_2 \quad even & \begin{cases} k_1 < k_2 & (a, b \cdot i) \\ k_1 > k_2 & (a \cdot i, b) \end{cases} \end{cases} \end{cases}$$

4 Examples

4.1 k_1 even, k_2 odd

- Let the determinant of a polynomial matrix $A(x,y)$ which is a polynomial $p(x,y)$ whose degrees are $k_1 = deg_x(p(x,y)) = 2$ and $k_2 = deg_y(p(x,y)) = 9$ respectively.
  $N = (k_1+1) \cdot (k_2+1) = 30$ the number of required points.
  We choose the required points from real domain. The set of these points is:

  $$S^{(2,9)} = \{(x_{\ell_1}, y_{\ell_2}) | \ell_1 = 0, \ldots, 2 \quad \ell_2 = 0, \ldots, 9\}$$

  $S^{(2,9)}$ is showing in figure 4. In this case we need to evaluate 30 points.

- Let the same polynomial, and lets define now the required points in complex domain.
  The set of these points is:

  $$\widetilde{S}^{(2,9)} = \{(a_{\ell_1}, b_{\ell_2}) | \ell_1 = 0, \ldots, 2 \quad \ell_2 = 0, \ldots, 9\}$$

  There are two subcases.

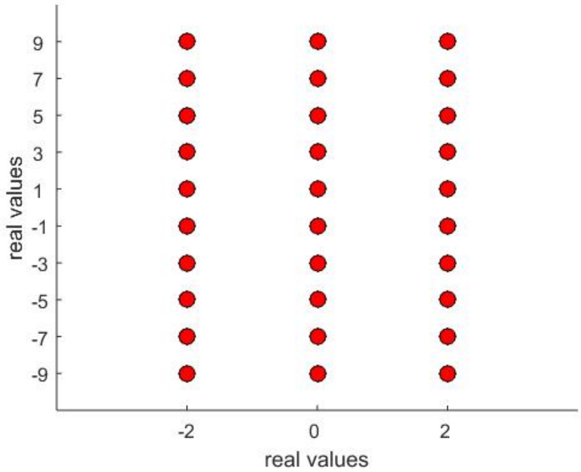

Figure 4: Real domain

- Points of the form $(a_{\ell_1} \cdot i, b_{\ell_2})$ where $\ell_1 = 0, \ldots, 2 \quad \ell_2 = 0, \ldots, 9$, i is the imaginary unit. We need to evaluate both asterisk and round points as they are showing in figure 5. The square ones occurs automatically due to the conjugate properties. In this case we need to evaluate 20 points.

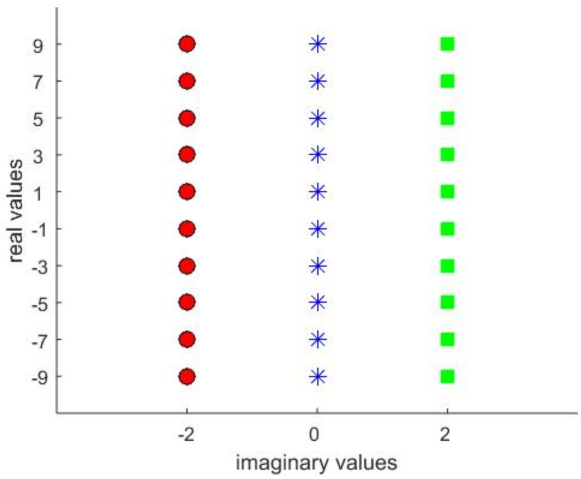

Figure 5: Complex domain,points of the form $(a_{\ell_1} \cdot i, b_{\ell_2})$

- Points of the form $(a_{\ell_1}, b_{\ell_2} \cdot i)$ where $\ell_1 = 0, \ldots, 2 \quad \ell_2 = 0, \ldots, 9$, i is the imaginary unit. We need to evaluate only the round points as they are showing in figure 6. The square ones occurs automatically due to the conjugate properties. In this case we need to evaluate 15 points.

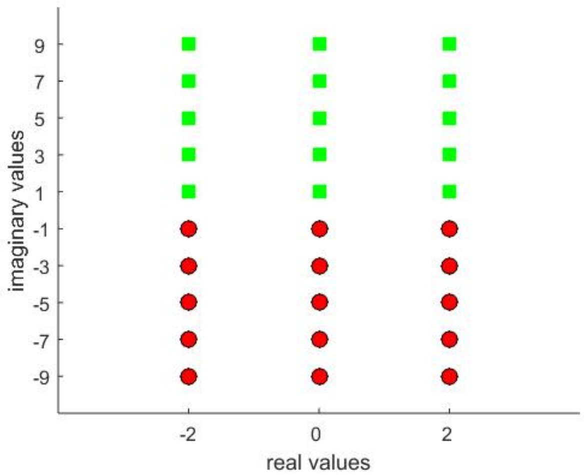

Figure 6: Complex domain,points of the form $(a_{\ell_1}, b_{\ell_2} \cdot i)$

4.2 k_1 odd, k_2 even

- Let the determinant of a polynomial matrix $A(x, y)$, which is a polynomial $p(x, y)$ whose degrees are $k_1 = deg_x(p(x, y)) = 9$ and $k_2 = deg_y(p(x, y)) = 2$ respectively.
  $N = (k_1 + 1) \cdot (k_2 + 1) = 30$ the number of required points.
  We choose the required points from real domain. The set of these points is:

  $$S^{(9,2)} = \{(x_{\ell_1}, y_{\ell_2})|\ell_1 = 0, \dots, 9 \quad \ell_2 = 0, \dots, 2\}$$

  $S^{(9,2)}$ is showing in figure 7. In this case we need to evaluate 30 points.

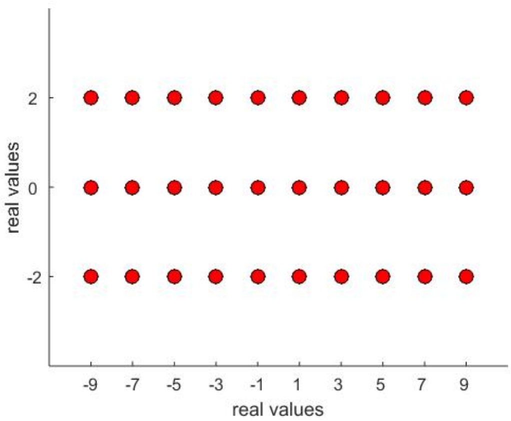

Figure 7: Real domain

- Let the same polynomial, and lets define now the required points in complex domain.
  The set of these points is:

  $$\widetilde{S}^{(9,2)} = \{(a_{\ell_1}, b_{\ell_2})|\ell_1 = 0, \dots, 9 \quad \ell_2 = 0, \dots, 2\}$$

  There are two subcases.

  – Points of the form $(a_{\ell_1} \cdot i, b_{\ell_2})$ where $\ell_1 = 0, \dots, 9 \quad \ell_2 = 0, \dots, 2$, i is the imaginary unit. We need to evaluate only the round points as they are showing in figure 8. The square points occurs automatically due to the conjugate properties. In this case we need to evaluate 15 points.

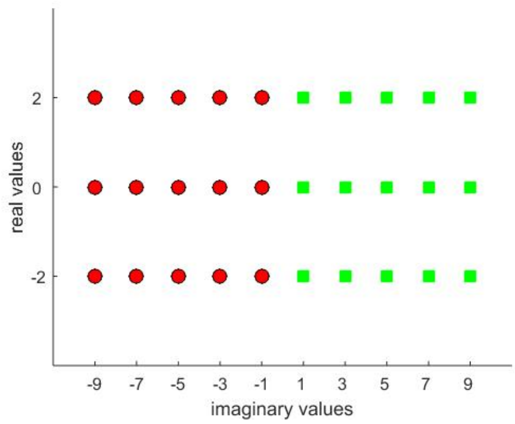

Figure 8: Complex domain,points of the form $(a_{\ell_1} \cdot i, b_{\ell_2})$

  – Points of the form $(a_{\ell_1}, b_{\ell_2} \cdot i)$ where $\ell_1 = 0, \dots, 9 \quad \ell_2 = 0, \dots, 2$, i is the imaginary unit. We need to evaluate both asterisk and round points as they are showing in figure 9. The square ones occurs automatically due to the conjugate properties. In this case we need to evaluate 20 points.

4.3 k_1 odd, k_2 odd

- Let the determinant of a polynomial matrix $A(x, y)$, which is a polynomial $p(x, y)$ whose degrees are $k_1 = deg_x(p(x, y)) = 5$ and $k_2 = deg_y(p(x, y)) = 7$ respectively.
  $N = (k_1 + 1) \cdot (k_2 + 1) = 48$ the number of required points.
  We choose the required points from real domain. The set of these points is:

  $$S^{(5,7)} = \{(x_{\ell_1}, y_{\ell_2})|\ell_1 = 0, \dots, 5 \quad \ell_2 = 0, \dots, 7\}$$

  $S^{(5,7)}$ is showing in figure 10. In this case we need to evaluate 48 points.

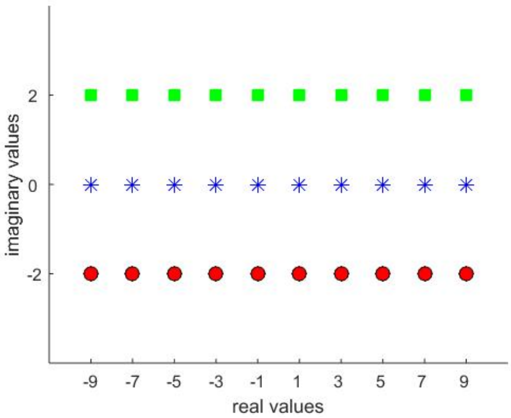

Figure 9: Complex domain,points of the form $(a_{\ell_1}, b_{\ell_2} \cdot i)$

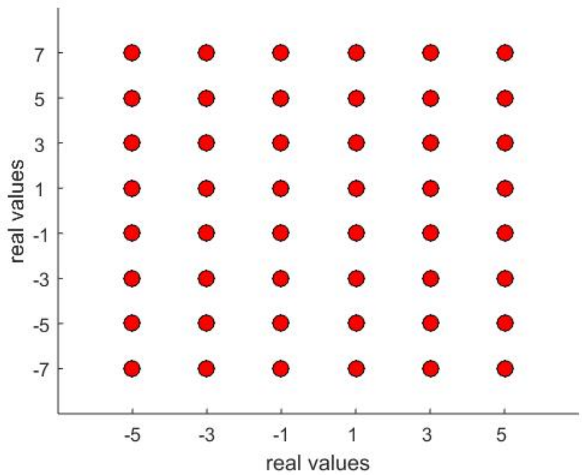

Figure 10: Real domain

- Let the same polynomial, and lets define now the required points in complex domain.
  The set of these points is:

  $$\widetilde{S}^{(5,7)} = \{(a_{\ell_1}, b_{\ell_2}) | \ell_1 = 0, \dots, 5 \quad \ell_2 = 0, \dots, 7\}$$

  There are two subcases.

  - Points of the form $(a_{\ell_1} \cdot i, b_{\ell_2})$ where $\ell_1 = 0, \dots, 5 \quad \ell_2 = 0, \dots, 7$, i is the imaginary unit. We need to evaluate only the round points as they are showing in figure 11. The square ones occurs automatically due to the conjugate properties. In this case we need to evaluate 24 points.
  - Points of the form $(a_{\ell_1}, b_{\ell_2} \cdot i)$ where $\ell_1 = 0, \dots, 5 \quad \ell_2 = 0, \dots, 7$, i is the imaginary unit. We need to evaluate only the round points as they are showing in figure 12. The square ones occurs automatically due to the conjugate properties. In this case we need to evaluate 24 points.

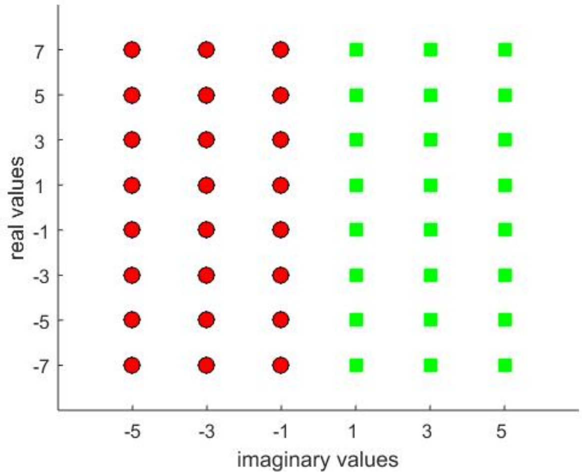

Figure 11: Complex domain, points of the form $(a_{\ell_1} \cdot i, b_{\ell_2})$

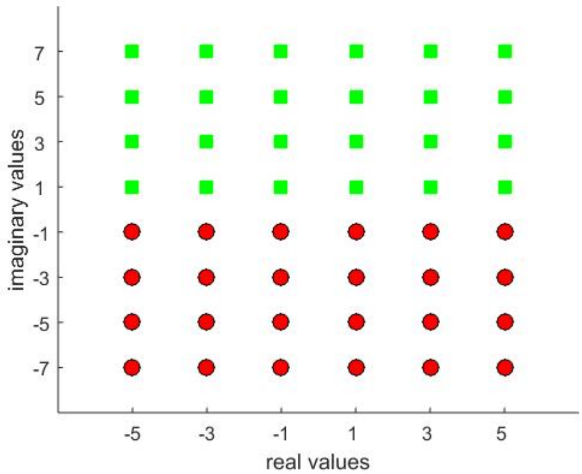

Figure 12: Complex domain, points of the form $(a_{\ell_1}, b_{\ell_2} \cdot i)$

As we can see, if both k_1 and k_2 are odd we can reduce the number of evaluated points even up to the half by choosing points of either the form $(a \cdot i, b)$ or $(a, b \cdot i)$ where $a, b \,\epsilon\, \mathbf{R}$ and i is the imaginary unit.

4.4 k_1 even, k_2 even, $k_1 < k_2$

- Let the determinant of a polynomial matrix $A(x, y)$, which is a polynomial $p(x, y)$ whose degrees are $k_1 = deg_x(p(x, y)) = 4$ and $k_2 =$

$deg_y(p(x,y)) = 12$ respectively.
$N = (k_1+1)\cdot(k_2+1) = 65$ the number of required points.
We choose the required points from real domain. The set of these points is:

$$S^{(4,12)} = \{(x_{\ell_1}, y_{\ell_2}) | \ell_1 = 0, \dots, 4 \quad \ell_2 = 0, \dots, 12\}$$

$S^{(4,12)}$ is showing in figure 13 In this case we

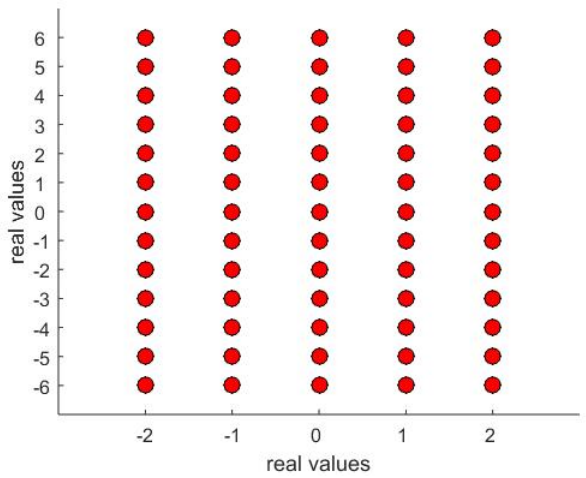

Figure 13: Real domain

need to evaluate 65 points.

- Let the same polynomial, and lets define now the required points in complex domain.
  The set of these points is:

  $$\widetilde{S}^{(4,12)} = \{(a_{\ell_1}, b_{\ell_2}) | \ell_1 = 0, \dots, 4 \quad \ell_2 = 0, \dots, 12\}$$

  There are two subcases.

  - Points of the form $(a_{\ell_1}\cdot i, b_{\ell_2})$ where $\ell_1 = 0, \dots, 4 \quad \ell_2 = 0, \dots, 12$, i is the imaginary unit. We need to evaluate both round and asterisk points as they are showing in figure 14. The square points occurs automatically due to the conjugate properties. In this case we need to evaluate 39 points.
  - Points of the form $(a_{\ell_1}, b_{\ell_2}\cdot i)$ where $\ell_1 = 0, \dots, 4 \quad \ell_2 = 0, \dots, 12$, i is the imaginary unit. We need to evaluate both round and asterisk points as they are showing in figure 15. The square points occurs automatically due to the conjugate properties. In this case we need to evaluate 35 points.

4.5 k_1 even, k_2 even, $k_1 > k_2$

- Let the determinant of a polynomial matrix $A(x,y)$, which is a polynomial $p(x,y)$ whose degrees are $k_1 = deg_x(p(x,y)) = 12$ and $k_2 =$

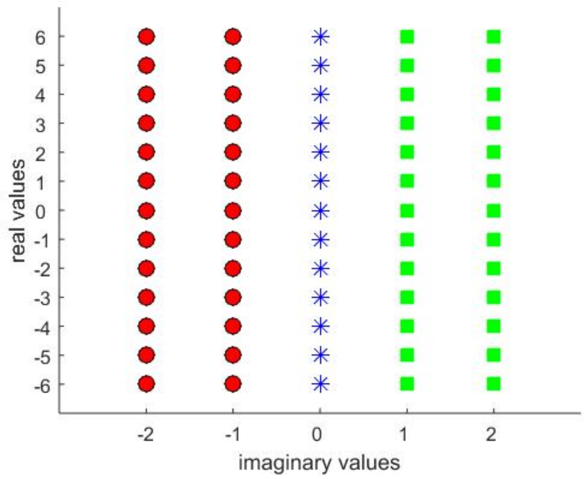

Figure 14: Complex domain, points of the form $(a_{\ell_1}\cdot i, b_{\ell_2})$

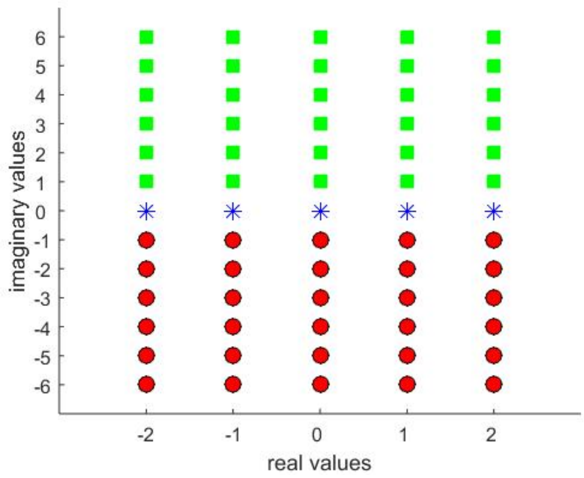

Figure 15: Complex domain, points of the form $(a_{\ell_1}, b_{\ell_2}\cdot i)$

$deg_y(p(x,y)) = 4$ respectively.
$N = (k_1+1)\cdot(k_2+1) = 65$ the number of required points.
We choose the required points from real domain. The set of these points is:

$$S^{(12,4)} = \{(x_{\ell_1}, y_{\ell_2}) | \ell_1 = 0, \dots, 12 \quad \ell_2 = 0, \dots, 4\}$$

$S^{(12,4)}$ is showing in figure 16. In this case we need to evaluate 65 points.

- Let the same polynomial, and lets define now the required points in complex domain.
  The set of these points is:

  $$\widetilde{S}^{(12,4)} = \{(a_{\ell_1}, b_{\ell_2}) | \ell_1 = 0, \dots, 12 \quad \ell_2 = 0, \dots, 4\}$$

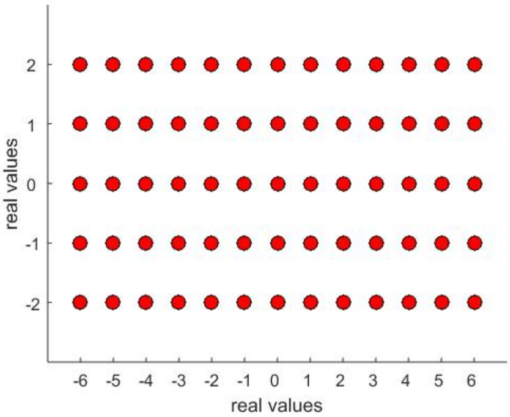

Figure 16: Real domain

There are two subcases.

– Points of the form $(a_{\ell_1} \cdot i, b_{\ell_2})$ where $\ell_1 = 0, \dots, 12 \quad \ell_2 = 0, \dots, 4$, i is the imaginary unit. We need to evaluate both round and asterisk points as they are showing in figure 17. The square points occurs automatically due to the conjugate properties. In this case we need to evaluate 35 points.

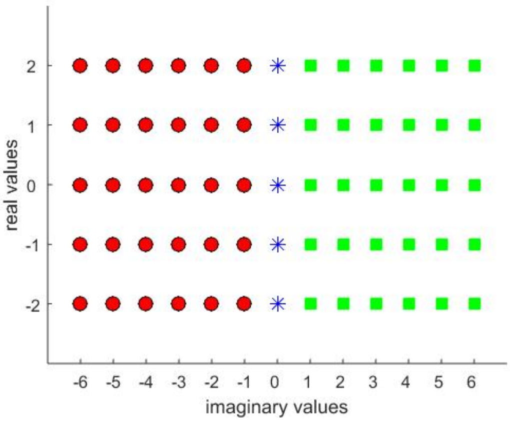

Figure 17: Complex domain,points of the form $(a_{\ell_1} \cdot i, b_{\ell_2})$

– Points of the form $(a_{\ell_1}, b_{\ell_2} \cdot i)$ where $\ell_1 = 0, \dots, 12 \quad \ell_2 = 0, \dots, 4$, i is the imaginary unit. We need to evaluate both round and asterisk points as they are showing in figure 18. The square points occurs automatically due to the conjugate properties. In this case we need to evaluate 39 points.

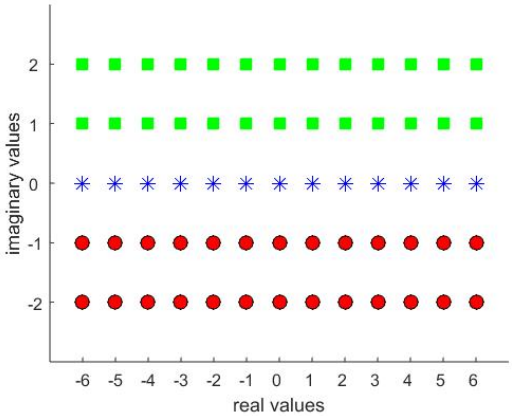

Figure 18: Complex domain,points of the form $(a_{\ell_1}, b_{\ell_2} \cdot i)$

5 Results

To put it concisely, we present the number of calculations that have to be made in evaluation part, according to the degrees of each variable k_1, k_2. The number in parentheses represents the number of required evaluations while working in real domain. The number in brackets represents the number of required evaluations in complex domain, when points are of the form $(a \cdot i, b)$, where $a, b \in \mathbb{R}$ and i is the imaginary unit. The number in curly brackets represents the number of required evaluations in complex domain, when points are of the form $(a, b \cdot i)$, where $a, b \in \mathbb{R}$ and i is the imaginary unit.:

- $k_1 = 10, k_2 = 10 : (121)[66]\{66\}$
- $k_1 = 10, k_2 = 15 : (176)[96]\{88\}$
- $k_1 = 10, k_2 = 20 : (231)[126]\{121\}$
- $k_1 = 15, k_2 = 10 : (176)[88]\{96\}$
- $k_1 = 15, k_2 = 15 : (256)[128]\{128\}$
- $k_1 = 15, k_2 = 20 : (336)[168]\{176\}$
- $k_1 = 20, k_2 = 10 : (231)[121]\{126\}$
- $k_1 = 20, k_2 = 15 : (336)[176]\{168\}$
- $k_1 = 20, k_2 = 20 : (441)[231]\{231\}$

In the following matrix, all these cases are presented. Numbers in bold indicate the optimal case according to the degree of each variable.

	$k_2 = 10$	$k_2 = 15$
$k_1 = 10$	(121)[**66**]{**66**}	(176)[96]{**88**}
$k_1 = 15$	(176)[**88**]{96}	(256)[**128**]{**128**}
$k_1 = 20$	(231)[**121**]{126}	(336)[176]{**168**}

	$k_2 = 20$
$k_1 = 10$	(231)[126]{**121**}
$k_1 = 15$	(336)[**168**]{176}
$k_1 = 20$	(441)[**231**]{**231**}

6 Conclusion

Taking advantage of working in complex domain, a way to reduce the number of evaluations in evaluation-interpolation technique has been proposed. This way is to determine the specific form of required points. Choosing the best form depends on the degree of each variable, whether it is even or odd. As we can see it can lead to an optimal reduction of computations in evaluation part, even up to the half. Consequently, less necessary operations have to be made and the execution time of the process will also be reduced.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Author Contributions: Please, indicate the role and the contribution of each author:
Example
In this paper the authors has working equally

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