Numerical solution of a system of fractional ordinary differential equations by a modified variational iteration procedure

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Abstract: - In this paper, a robust modification of the variational iteration method that gives a numerical solution for a system of linear/nonlinear differential equations of fractional order was proposed. This technique does not need the perturbation theory or linearization. The conformable fractional derivative initiated by the authors Khalil et al. is considered. The efficiency of the modified method is established via illustrative examples. For linear and nonlinear systems, the approximate solutions are in excellent agreement with the exact solutions.

Key-Words: - Conformable fractional derivative; Variational iteration method; Lagrange multiplier.

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1 Introduction

The differential equations of fractional order, as well as their exact and approximate solutions, have fundamental significance in several branches of science and engineering [12, 24, 23, 22, 7, 12, 28, 10, 21, 29]. The analytic and approximate solutions of linear and nonlinear systems of ordinary differential equations of fractional order have been discussed by several authors, see [9, 11, 3, 20, 26, 27].
The authors Khalil et al. defined a new fractional derivative in [17]. It is based on the definition of the basic limit of the derivative. The new simple fractional derivative is named the conformable fractional derivative, which then had been the focus of many studies [1, 2, 5, 6, 8, 19, 18, 25].
The variational iteration approach was developed for the first time by He [14]. This technique and its modifications [15, 16] have potentially been used to solve nonlinear differential equations. In [31], a comparative study between the Adomian decomposition method and the variational iteration method has been presented. The method has been used in [13] to provide an approximate solution for fractional differential equations with modified Riemann–Liouville fractional derivative.
The purpose of this paper is to extend the analysis of the variational iteration method to solve the system of fractional ordinary differential equations which is as follows:

$$\begin{aligned} D^{\alpha_1} x_1(t) &= f_1(t, x_1, x_2, \dots, x_n),\\ D^{\alpha_2} x_2(t) &= f_2(t, x_1, x_2, \dots, x_n),\\ &\vdots\\ D^{\alpha_n} x_n(t) &= f_n(t, x_1, x_2, \dots, x_n), \end{aligned} \tag{1}$$

where $D^{\alpha_i} = \frac{d^{\alpha_i}}{dt^{\alpha_i}}$ is the conformable fractional derivative of order $\alpha_i \in (0,1]$, for $i = 1, 2, \dots, n$. The system is subject to the initial conditions

$$x_1(0) = c_1,\ x_2(0) = c_2, \dots,\ x_n(0) = c_n.$$

The article is organized as follows: In Section 2 we discuss the basic definitions and properties of the conformable fractional derivative. The variational iteration method is presented in Section 3. Section 4 provides a series of examples to demonstrate the efficiency of the implemented method. Section 5 concludes.

2 Preliminaries and Notations

In this section, we introduce the main concepts and properties of the conformable fractional derivative.

Definition 2.1. The conformable fractional derivative of order α, $0 < \alpha \leq 1$ of $f : (0, \infty) \to \mathbb{R}$ is defined by

$$D_\alpha(f)(x) := \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon}$$

for all $x > 0$. If the limit exists, we say that f is $\alpha-$differentiable at x. Moreover, if f is $\alpha-$differentiable in some $(0, a)$, $a > 0$, and $\lim_{x\to 0^+} D_\alpha(f)(x)$ exists, then define

$$D_\alpha(f)(0) := \lim_{x\to 0^+} D_\alpha(f)(x).$$

Remark 1. Let $\alpha \in (0, 1]$ and f be differentiable and $\alpha-$differentiable for all $x > 0$. Then

$$D_\alpha(f)(x) = x^{1-\alpha}\frac{df}{dx}(x).$$

The fractional exponential function, denoted by $e^{\frac{1}{\alpha}x^\alpha}$, is defined by

$$e^{\frac{1}{\alpha}x^\alpha} = \sum_{j=0}^{\infty} \frac{x^{\alpha j}}{\alpha^j j!}.$$

Remark 2. The conformable fractional derivative of common functions are

- $D_\alpha(c) = 0$, for any constant c.
- $D_\alpha(x^r) = rx^{r-\alpha}, r \in \mathbb{R}$.
- $D_\alpha(\sin \frac{1}{\alpha}x^\alpha) = \cos \frac{1}{\alpha}x^\alpha$.
- $D_\alpha(\cos \frac{1}{\alpha}x^\alpha) = -\sin \frac{1}{\alpha}x^\alpha$.
- $D_\alpha(e^{\frac{1}{\alpha}x^\alpha}) = e^{\frac{1}{\alpha}x^\alpha}$.

Next, we introduce the fractional integral of order α.

Definition 2.2. Let $\alpha \in (0, 1]$ and $x \in [a, \infty)$, $a \geq 0$. The conformable fractional integral of order α is given by

$$(I^a_\alpha f)(x) := \int_a^x f(t)d\alpha(t, a) = \int_a^x (t-a)^{\alpha-1} f(t)dt.$$

When $a = 0$, we use I_α and $d\alpha(t)$.

Theorem 2.1. *Let $f : [0, \infty) \to \mathbb{R}$ be differentiable and $\alpha \in (0, 1]$. Then for all $t > 0$ we have*

$$I_\alpha D_\alpha(f)(t) = f(t) - f(0).$$

Proof. From definition and since f is differentiable we obtain

$$\begin{aligned} I_\alpha D_\alpha(f)(t) &= \int_0^t x^{\alpha-1} D_\alpha f(x)dx \\ &= \int_0^t x^{\alpha-1}x^{1-\alpha}f'(x)dx = f(t) - f(0). \end{aligned}$$

□

3 Variational iteration method

In 1999, He [14] proposed an analytical approach for a non-linear problem based on a general Lagrange multiplier. The method is called the variational iteration method, where no perturbation or linearization are needed, has been used effectively to solve a vast class of nonlinear problems.
To demonstrate the main concept of the method, we examine the following general nonlinear system:

$$Lu + Nu = g(t), \tag{2}$$

where L and N are a linear and nonlinear operators, respectively.
A correctional functional can be given by

$$u^{n+1}(t) = u^n(t) + \int_o^t \lambda[Lu^n(\tau) + N\tilde{u}^n(\tau) - g(\tau)]d\tau, \tag{3}$$

where λ is a general Lagrange multiplier which can be determined optimally through variational theory, and $\tilde{u}^n$ is considered such that $\delta\tilde{u}^n = 0$.
Accordingly, the exact solution can be given by

$$x(t) = \lim_{k\to\infty} x^k(t).$$

Now, consider the general fractional differential equation:

$$D^\alpha x(t) = Lx(t) + Nx(t) + g(t), \tag{4}$$

where $\alpha \in (0, 1]$, D^α is the conformable fractional derivative of $x(t)$ of order α.
The author has modified the above iteration method into:

$$x^{k+1}(t) = x^k(t) +$$

$$\int_0^t \lambda(\tau)[D^\alpha x^k(\tau) - (Lx^k(\tau) + N\tilde{x}^k(\tau) + g(\tau)]d\alpha(\tau),$$

which can be rewritten as

$$x^{k+1}(t) = x^k(t) + \tag{5}$$

$$\int_0^t \lambda(\tau)\tau^{\alpha-1}[D^\alpha x^k(\tau) - (Lx^k(\tau) + N\tilde{x}^k(\tau) + g(\tau)]d\tau,$$

where any selective function can be used for x^0, we usually use the initial condition $x(0)$. To find the optimal value of Lagrange multiplier λ, we perform the following, we take the variation of (5) with respect to $x(t)$

$$\delta x^{k+1}(t) = \delta x^k(t) +$$

$$\delta \int_0^t \lambda(\tau)\tau^{\alpha-1}[D^\alpha x^k(\tau) - (Lx^k(\tau) + N\tilde{x}^k(\tau) + g(\tau))]d\tau.$$

This yields the stationary conditions

$$\lambda'(\tau) = 0,$$
$$1 + \lambda(\tau) = 0.$$

Hence, we obtain

$$\lambda(\tau) = -1. \tag{6}$$

The correction functionals for system (1) can be expressed as

$$\begin{aligned}
x_1^{k+1}(t) &= x_1^k(t) + \\
&\int_0^t \lambda_1(\tau)(D^{\alpha_1} x_1^k(\tau) - f_1(\tau, \tilde{x}_1^k(\tau), \ldots, \tilde{x}_n^k(\tau)))d\alpha_1(\tau), \\
x_2^{k+1}(t) &= x_2^k(t) + \\
&\int_0^t \lambda_2(\tau)(D^{\alpha_2} x_2^k(\tau) - f_2(\tau, \tilde{x}_1^k(\tau), \ldots, \tilde{x}_n^k(\tau)))d\alpha_2(\tau), \\
&\vdots \\
x_n^{k+1}(t) &= x_n^k(t) + \\
&\int_0^t \lambda_n(\tau)(D^{\alpha_n} x_n^k(\tau) - f_n(\tau, \tilde{x}_1^k(\tau), \ldots, \tilde{x}_n^k(\tau)))d\alpha_n(\tau),
\end{aligned} \tag{7}$$

Substituting (6) into (7) gives

$$\begin{aligned}
x_1^{k+1}(t) &= x_1^k(t) - \\
&\int_0^t \tau^{\alpha_1 - 1}(D^{\alpha_1} x_1^k(\tau) - f_1(\tau, \tilde{x}_1^k(\tau), \ldots, \tilde{x}_n^k(\tau)))d\tau, \\
x_2^{k+1}(t) &= x_2^k(t) - \\
&\int_0^t \tau^{\alpha_2 - 1}(D^{\alpha_2} x_2^k(\tau) - f_2(\tau, \tilde{x}_1^k(\tau), \ldots, \tilde{x}_n^k(\tau)))d\tau, \\
&\vdots \\
x_n^{k+1}(t) &= x_n^k(t) - \\
&\int_0^t \tau^{\alpha_n - 1}(D^{\alpha_n} x_n^k(\tau) - f_n(\tau, \tilde{x}_1^k(\tau), \ldots, \tilde{x}_n^k(\tau)))d\tau.
\end{aligned} \tag{8}$$

4 Applications

In this section, we illustrate the efficiency of the our modified version of VIM by presenting four examples. The first two examples are considered for linear systems of fractional ordinary differential equations. The accuracy of the proposed method is appraised by comparison with the exact solutions. The third and fourth examples are considered for nonlinear systems. All computations are performed by Mathematica.

Example 4.1. Consider the linear system of fractional differential equations

$$\begin{aligned}
D^{\alpha_1} x(t) &= x(t) - y(t), \\
D^{\alpha_2} y(t) &= x(t) + y(t),
\end{aligned} \tag{9}$$

where $0 < \alpha_1, \alpha_2 \leq 1$ and $D^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ is the conformable fractional derivative, subject to the initial conditions

$$x(0) = 1, \quad y(0) = 0. \tag{10}$$

The exact solution of system (9) with initial conditions (10), when $\alpha_1 = \alpha_2 = \alpha$, refer to [4] for more details, is

$$\begin{aligned}
x(t) &= e^{\frac{t^\alpha}{\alpha}} \cos \frac{t^\alpha}{\alpha}, \\
y(t) &= e^{\frac{t^\alpha}{\alpha}} \sin \frac{t^\alpha}{\alpha}.
\end{aligned} \tag{11}$$

According to the formulas (8), the variational iteration formulas for system (9) are given by

$$\begin{aligned}
x^{k+1}(t) &= x^k(t) - \int_0^t \tau^{\alpha_1 - 1}[D^{\alpha_1} x^k(\tau) - x^k(\tau) + y^k(\tau))]d\tau, \\
y^{k+1}(t) &= y^k(t) - \int_0^t \tau^{\alpha_2 - 1}[D^{\alpha_2} y^k(\tau) - x^k(\tau) - y^k(\tau))]d\tau,
\end{aligned} \tag{12}$$

where $x^0(t) = 1$ and $y^0(t) = 0$.
Consequently, we obtain the following approximations

$$\begin{aligned}
x^1(t) =& 1 + \frac{t^{\alpha_1}}{\alpha_1}, \\
y^1(t) =& \frac{t^{\alpha_2}}{\alpha_2}, \\
x^2(t) =& 1 + \frac{t^{\alpha_1}}{\alpha_1} + \frac{t^{2\alpha_1}}{2\alpha_1^2} - \frac{t^{\alpha_1 + \alpha_2}}{\alpha_2(\alpha_1 + \alpha_2)}, \\
y^2(t) =& \frac{t^{\alpha_2}}{\alpha_2} + \frac{t^{\alpha_1 + \alpha_2}}{\alpha_1(\alpha_1 + \alpha_2)} + \frac{t^{2\alpha_2}}{2\alpha_2^2}, \\
x^3(t) =& 1 + \frac{t^{\alpha_1}}{\alpha_1} + \frac{t^{2\alpha_1}}{2\alpha_1^2} + \frac{t^{3\alpha_1}}{6\alpha_1^3} - \frac{t^{\alpha_1 + \alpha_2}}{\alpha_2(\alpha_1 + \alpha_2)} - \\
& \frac{t^{2\alpha_1 + \alpha_2}}{\alpha_1 \alpha_2(2\alpha_1 + \alpha_2)} - \frac{t^{\alpha_1 + 2\alpha_2}}{2\alpha_2^2(\alpha_1 + 2\alpha_2)},
\end{aligned}$$

$$
\begin{aligned}
y^3(t) =&\frac{t^{\alpha_2}}{\alpha_2}+\frac{t^{2\alpha_2}}{2\alpha_2^2}+\frac{t^{3\alpha_2}}{6\alpha_2^3}+\frac{t^{\alpha_1+\alpha_2}}{\alpha_1(\alpha_1+\alpha_2)}+\\
&\frac{t^{2\alpha_1+\alpha_2}}{2\alpha_1^2(2\alpha_1+\alpha_2)}-\frac{t^{\alpha_1+2\alpha_2}}{\alpha_2(\alpha_1+\alpha_2)(\alpha_1+2\alpha_2)}+\\
&\frac{t^{\alpha_1+2\alpha_2}}{\alpha_1(\alpha_1+\alpha_2)(\alpha_1+2\alpha_2)},\\
x^4(t) =&1+\frac{t^{\alpha_1}}{\alpha_1}+\frac{t^{2\alpha_1}}{2\alpha_1^2}+\frac{t^{3\alpha_1}}{6\alpha_1^3}+\frac{t^{4\alpha_1}}{24\alpha_1^4}-\\
&\frac{t^{\alpha_1+\alpha_2}}{\alpha_2(\alpha_1+\alpha_2)}-\frac{t^{2\alpha_1+\alpha_2}}{\alpha_1\alpha_2(2\alpha_1+\alpha_2)}-\\
&\frac{t^{\alpha_1+2\alpha_2}}{2\alpha_2^2(\alpha_1+2\alpha_2)}-\frac{t^{\alpha_1+3\alpha_2}}{6\alpha_2^3(\alpha_1+3\alpha_2)}-\\
&\frac{t^{3\alpha_1+\alpha_2}}{\alpha_1\alpha_2(2\alpha_1+\alpha_2)(3\alpha_1+\alpha_2)}-\\
&\frac{t^{2\alpha_1+2\alpha_2}}{4\alpha_2^2(\alpha_1+\alpha_2)(\alpha_1+2\alpha_2)}-\\
&\frac{t^{3\alpha_1+\alpha_2}}{2\alpha_1^2(2\alpha_1+\alpha_2)(3\alpha_1+\alpha_2)},\\
y^4(t) =&\frac{t^{\alpha_2}}{\alpha_2}+\frac{t^{2\alpha_2}}{2\alpha_2^2}+\frac{t^{3\alpha_2}}{6\alpha_2^3}+\frac{t^{4\alpha_2}}{24\alpha_2^4}+\\
&\frac{t^{\alpha_1+\alpha_2}}{\alpha_1(\alpha_1+\alpha_2)}+\frac{t^{3\alpha_1+\alpha_2}}{6\alpha_1^3(3\alpha_1+\alpha_2)}+\\
&\frac{t^{2\alpha_1+\alpha_2}}{2\alpha_1(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)}+\\
&\frac{\alpha_2 t^{2\alpha_1+\alpha_2}}{2\alpha_1^2(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)}+\\
&+\frac{t^{2\alpha_1+2\alpha_2}}{4\alpha_1^2(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)}-\\
&\frac{t^{2\alpha_1+2\alpha_2}}{2\alpha_1\alpha_2(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)}-\\
&\frac{t^{\alpha_1+3\alpha_2}}{2\alpha_2^2(\alpha_1^2+5\alpha_1\alpha_2+6\alpha_2^2)},\\
&\vdots
\end{aligned}
\tag{13}
$$

If $\alpha_1=\alpha_2=\alpha$, then (13) become

$$
\begin{aligned}
x^1(t) &= 1+\frac{t^\alpha}{\alpha},\\
y^1(t) &= \frac{t^\alpha}{\alpha},\\
x^2(t) &= 1+\frac{t^\alpha}{\alpha},\\
y^2(t) &= \frac{t^\alpha}{\alpha}+\frac{t^{2\alpha}}{\alpha^2},\\
x^3(t) &= 1+\frac{t^\alpha}{\alpha}-\frac{t^{3\alpha}}{3\alpha^3},\\
y^3(t) &= \frac{t^\alpha}{\alpha}+\frac{t^{2\alpha}}{\alpha^2}+\frac{t^{3\alpha}}{3\alpha^3},\\
x^4(t) &= 1+\frac{t^\alpha}{\alpha}-\frac{t^{3\alpha}}{3\alpha^3}-\frac{t^{4\alpha}}{6\alpha^4},\\
y^4(t) &= \frac{t^\alpha}{\alpha}+\frac{t^{2\alpha}}{\alpha^2}+\frac{t^{3\alpha}}{3\alpha^3},\\
&\vdots
\end{aligned}
\tag{14}
$$

The solution of (9) in series form is given by

$$
\begin{aligned}
x(t) &= 1+\frac{t^\alpha}{\alpha}-\frac{t^{3\alpha}}{3\alpha^3}-\frac{t^{4\alpha}}{6\alpha^4}+\dots,\\
y(t) &= \frac{t^\alpha}{\alpha}+\frac{t^{2\alpha}}{\alpha^2}+\frac{t^{3\alpha}}{3\alpha^3}+\dots,
\end{aligned}
\tag{15}
$$

which converges to the exact solution (11).
Figures 1 and 2 show the exact solution $x(t)$ and $y(t)$ and the approximate solution $x^4(t)$ and $y^4(t)$ for system (9) for different values of α. The figures show that our approximate solutions are in good agreement with the exact solutions.

Example 4.2. Consider the linear system of fractional order differential equations

$$
\begin{aligned}
D^{\alpha_1}x(t) &= x(t)-y(t)+4z(t),\\
D^{\alpha_2}y(t) &= 3x(t)+2y(t)-z(t),\\
D^{\alpha_3}z(t) &= 2x(t)+y(t)-z(t),
\end{aligned}
\tag{16}
$$

subject to the initial conditions

$$x(0)=-1,\quad y(0)=7,\quad z(0)=3. \tag{17}$$

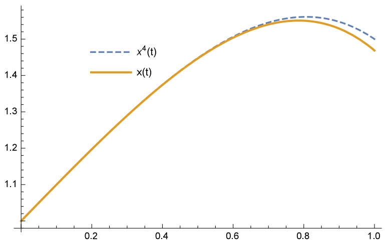
The exact solution of the system (16), when $\alpha_1=\alpha_2=\alpha_3=\alpha$, is

$$
\begin{aligned}
x(t) &= -e^{\frac{t^\alpha}{\alpha}}+e^{3\frac{t^\alpha}{\alpha}}-e^{-2\frac{t^\alpha}{\alpha}},\\
y(t) &= 4e^{\frac{t^\alpha}{\alpha}}+2e^{3\frac{t^\alpha}{\alpha}}+e^{-2\frac{t^\alpha}{\alpha}},\\
z(t) &= e^{\frac{t^\alpha}{\alpha}}+e^{3\frac{t^\alpha}{\alpha}}+e^{-2\frac{t^\alpha}{\alpha}}.
\end{aligned}
\tag{18}
$$

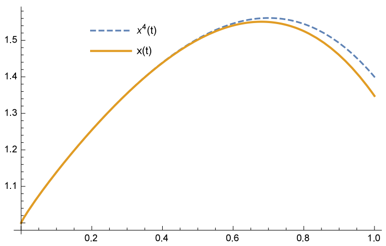
According to the formulas (8), the variational iteration formulas for system (16) are given by

$$x^{k+1}(t)=x^k(t)-$$

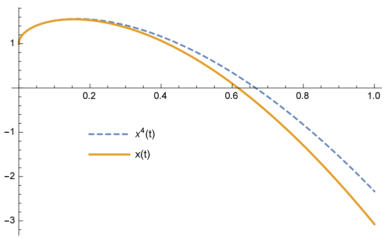
$$\int_0^t \tau^{\alpha_1-1}[D^{\alpha_1}x^k(\tau)-x^k(\tau)+y^k(\tau)-4z^k(\tau)]d\tau,$$



(a)



(b)



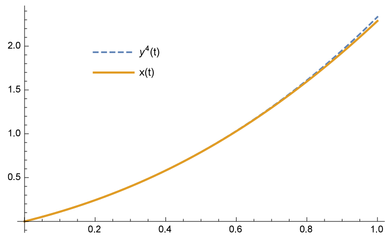
(c)

Figure 1: A Comparison for exact $x(t)$ and the approximate solution $x^4(t)$ in Example (4.1) where $\alpha_1 = \alpha_2 = 1,\ 0.9,\ 0.5.$

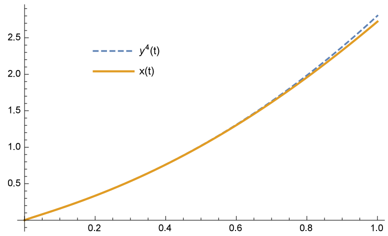
$$y^{k+1}(t) = y^k(t) -$$
$$\int_0^t \tau^{\alpha_2 - 1}[D^{\alpha_2} y^k(\tau) - 3x^k(\tau) - 2y^k(\tau) + z^k(\tau)]d\tau,$$
$$z^{k+1}(t) = z^k(t) -$$
$$\int_0^t \tau^{\alpha_3 - 1}[D^{\alpha_3} z^k(\tau) - 2x^k(\tau) - y^k(\tau) + z^k(\tau)]d\tau.$$

Begin with $x^0(t) = -1,\ y^0(t) = 7$ and $z^0(t) = 3$, we obtain

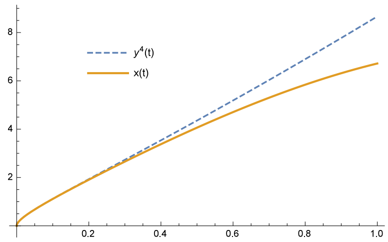
$$x^1(t) = -1 + 4\frac{t^{\alpha_1}}{\alpha_1},$$
$$y^1(t) = 7 + 8\frac{t^{\alpha_2}}{\alpha_2},$$
$$z^1(t) = 3 + 2\frac{t^{\alpha_3}}{\alpha_3},$$
$$x^2(t) = -1 + 4\frac{t^{\alpha_1}}{\alpha_1} + 2\frac{t^{2\alpha_1}}{\alpha_1^2} - 8\frac{t^{\alpha_1+\alpha_2}}{\alpha_2(\alpha_1+\alpha_2)} + 8\frac{t^{\alpha_1+\alpha_3}}{\alpha_3(\alpha_1+\alpha_3)},$$



(a)



(b)



(c)

Figure 2: A Comparison for exact $y(t)$ and the approximate solution $y^4(t)$ in Example (4.1) where $\alpha_1 = \alpha_2 = 1,\ 0.9,\ 0.5.$

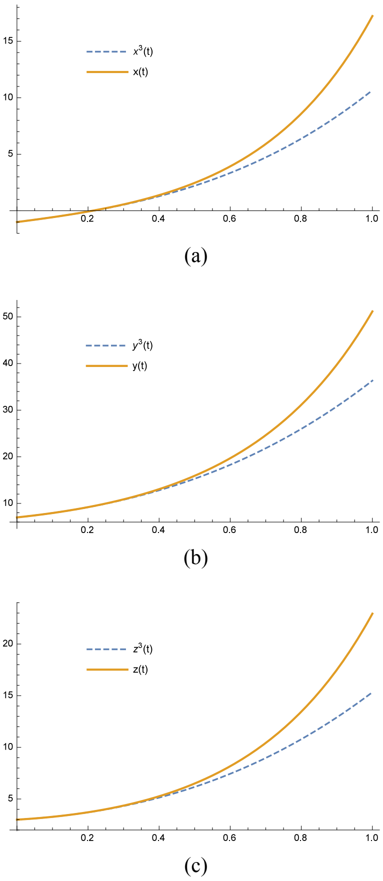
$$y^2(t) = 7 + \frac{8t^{\alpha_2}}{\alpha_2} + \frac{8t^{2\alpha_2}}{\alpha_2^2} + \frac{12t^{\alpha_1+\alpha_2}}{\alpha_1(\alpha_1+\alpha_2)} - \frac{2t^{\alpha_2+\alpha_3}}{\alpha_3(\alpha_2+\alpha_3)},$$
$$z^2(t) = 3 + \frac{2t^{\alpha_3}}{\alpha_3} - \frac{t^{2\alpha_3}}{\alpha_3^2} + \frac{8t^{\alpha_1+\alpha_3}}{\alpha_1(\alpha_1+\alpha_3)} + \frac{8t^{\alpha_2+\alpha_3}}{\alpha_2(\alpha_2+\alpha_3)},$$
$$x^3(t) = -1 + \frac{4t^{\alpha_1}}{\alpha_1} + \frac{2t^{2\alpha_1}}{\alpha_1^2} + \frac{2t^{3\alpha_1}}{3\alpha_1^3} - \frac{8t^{\alpha_1+\alpha_2}}{\alpha_2(\alpha_1+\alpha_2)} - \frac{12t^{2\alpha_1+\alpha_2}}{\alpha_1(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)} + \frac{8t^{\alpha_1+\alpha_3}}{\alpha_3(\alpha_1+\alpha_3)} - \frac{8t^{2\alpha_1+\alpha_2}}{\alpha_2(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)} - \frac{8t^{\alpha_1+2\alpha_2}}{\alpha_2^2(\alpha_1+2\alpha_2)} + \frac{32t^{2\alpha_1+\alpha_3}}{\alpha_1(\alpha_1+\alpha_3)(2\alpha_1+\alpha_3)} + \frac{8t^{2\alpha_1+\alpha_3}}{\alpha_3(\alpha_1+\alpha_3)(2\alpha_1+\alpha_3)} + \frac{32t^{\alpha_1+\alpha_2+\alpha_3}}{\alpha_2(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)} + \frac{2t^{\alpha_1+\alpha_2+\alpha_3}}{\alpha_3(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)} - \frac{4t^{\alpha_1+2\alpha_2}}{\alpha_3^2(\alpha_1+2\alpha_3)},$$
$$y^3(t) = 7 + \frac{8t^{\alpha_2}}{\alpha_2} + \frac{8t^{2\alpha_2}}{\alpha_2^2} + \frac{16}{3}\frac{t^{3\alpha_2}}{3\alpha_2^3} + \frac{12t^{\alpha_1+\alpha_2}}{\alpha_1(\alpha_1+\alpha_2)} + \frac{6t^{2\alpha_1+\alpha_2}}{\alpha_1^2(2\alpha_1+\alpha_2)} + \frac{24t^{\alpha_1+2\alpha_2}}{\alpha_1(\alpha_1+\alpha_2)(\alpha_1+2\alpha_2)} -$$

$$
\begin{aligned}
&\frac{24t^{\alpha_1+2\alpha_2}}{\alpha_2(\alpha_1+\alpha_2)(\alpha_1+2\alpha_2)}-\frac{2t^{\alpha_2+\alpha_3}}{\alpha_3(\alpha_2+\alpha_3)}-\\
&\frac{8t^{\alpha_1+\alpha_2+\alpha_3}}{\alpha_1(\alpha_1+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)}+\\
&\frac{24t^{\alpha_1+\alpha_2+\alpha_3}}{\alpha_3(\alpha_1+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)}-\\
&\frac{8t^{2\alpha_2+\alpha_3}}{\alpha_2(\alpha_2+\alpha_3)(2\alpha_2+\alpha_3)}-\\
&\frac{4t^{2\alpha_2+\alpha_3}}{\alpha_3(\alpha_2+\alpha_3)(2\alpha_2+\alpha_3)}+\frac{t^{\alpha_2+2\alpha_3}}{\alpha_3^2(\alpha_2+2\alpha_3)},\\
z^3(t)=3+{}&\frac{2t^{\alpha_3}}{\alpha_3}-\frac{t^{2\alpha_3}}{\alpha_3^2}+\frac{t^{3\alpha_3}}{3\alpha_3^3}+\frac{8t^{\alpha_1+\alpha_3}}{\alpha_1(\alpha_1+\alpha_3)}+\\
&\frac{4t^{2\alpha_1+\alpha_3}}{\alpha_1^2(2\alpha_1+\alpha_3)}+\frac{8t^{\alpha_2+\alpha_3}}{\alpha_2(\alpha_2+\alpha_3)}-\\
&\frac{16t^{\alpha_1+\alpha_2+\alpha_3}}{\alpha_2(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)}+\frac{8t^{2\alpha_2+\alpha_3}}{\alpha_2^2(2\alpha_2+\alpha_3)}-\\
&\frac{8t^{\alpha_1+2\alpha_3}}{\alpha_1(\alpha_1+\alpha_3)(\alpha_1+2\alpha_3)}+\\
&\frac{16t^{\alpha_1+2\alpha_3}}{\alpha_3(\alpha_1+\alpha_3)(\alpha_1+2\alpha_3)}-\\
&\frac{8t^{\alpha_2+2\alpha_3}}{\alpha_2(\alpha_2+\alpha_3)(\alpha_2+2\alpha_3)}-\\
&\frac{2t^{\alpha_2+2\alpha_3}}{\alpha_3(\alpha_2+\alpha_3)(\alpha_2+2\alpha_3)}+\\
&\frac{12t^{\alpha_1+\alpha_2+\alpha_3}}{\alpha_1(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)},\\
&\vdots
\end{aligned}
\tag{19}
$$

If $\alpha_1=\alpha_2=\alpha_3=\alpha$, then the solution of (16) in series form is given by

$$
\begin{aligned}
x(t)&=-1+4\frac{t^{\alpha}}{\alpha}+2\frac{t^{2\alpha}}{\alpha^2}+\frac{17t^{3\alpha}}{3\alpha^3}+\ldots,\\
y(t)&=7+8\frac{t^{\alpha}}{\alpha}+13\frac{t^{2\alpha}}{\alpha^2}+\frac{25t^{3\alpha}}{3\alpha^3}+\ldots,\\
z(t)&=3+2\frac{t^{\alpha}}{\alpha}+7\frac{t^{2\alpha}}{\alpha^2}+\frac{10t^{3\alpha}}{3\alpha^3}+\ldots,
\end{aligned}
\tag{20}
$$

which converges to the exact solution (18).
The graphics of the exact and approximate solutions are given in Figure 3.



(a) (b) (c)

Figure 3: A Comparison for exact $x(t)$, $y(t)$, $z(t)$ and the approximate solution $x^3(t)$, $y^3(t)$, $z^3(t)$, respectively in Example (4.2) where $\alpha_1=\alpha_2=\alpha_3=1$.

Example 4.3. Consider the nonlinear predator-prey system of fractional order differential equations

$$
\begin{aligned}
D^{\alpha_1}x(t)&=x(t)+y^2(t),\\
D^{\alpha_2}y(t)&=\frac{y(t)}{2},
\end{aligned}
\tag{21}
$$

subject to the initial conditions

$$x(0)=0,\quad y(0)=1. \tag{22}$$

The exact solution of the system (21), when $\alpha_1=\alpha_2=\alpha$, is

$$
\begin{aligned}
x(t)&=\frac{t^{\alpha}}{\alpha}\,e^{\frac{t^{\alpha}}{\alpha}},\\
y(t)&=e^{\frac{t^{\alpha}}{2\alpha}}.
\end{aligned}
\tag{23}
$$

According to the formulas (8), the variational iteration formulas for system (21) are given by

$$\begin{aligned} x^{k+1}(t) &= x^k(t) - \\ &\int_0^t \tau^{\alpha_1-1}[D^{\alpha_1}x^k(\tau) - x^k(\tau) - (y^k(\tau))^2]d\tau, \\ y^{k+1}(t) &= y^k(t) - \int_0^t \tau^{\alpha_2-1}[D^{\alpha_2}y^k(\tau) - \frac{y^k(\tau)}{2}]d\tau. \end{aligned} \tag{24}$$

Begin with $x^0(t) = 0$ and $y^0(t) = 1$, we obtain

$$\begin{aligned} x^1(t) &= \frac{t^{\alpha_1}}{\alpha_1}, \\ y^1(t) &= 1 + \frac{t^{\alpha_2}}{2\alpha_2}, \\ x^2(t) &= \frac{t^{\alpha_1}}{\alpha_1} + \frac{t^{2\alpha_1}}{2\alpha_1^2} + \frac{t^{\alpha_1+\alpha_2}}{\alpha_2(\alpha_1+\alpha_2)} + \frac{t^{\alpha_1+2\alpha_2}}{4\alpha_2^2(\alpha_1+2\alpha_2)}, \\ y^2(t) &= 1 + \frac{t^{\alpha_2}}{2\alpha_2} + \frac{t^{2\alpha_2}}{8\alpha_2^2}, \\ x^3(t) &= \frac{t^{\alpha_1}}{\alpha_1} + \frac{t^{2\alpha_1}}{2\alpha_1^2} + \frac{t^{3\alpha_1}}{6\alpha_1^3} + \frac{t^{\alpha_1+\alpha_2}}{\alpha_2(\alpha_1+\alpha_2)} + \\ &\frac{t^{2\alpha_1+\alpha_2}}{\alpha_2(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)} + \frac{\alpha_1 t^{\alpha_1+2\alpha_2}}{4\alpha_2^2(\alpha_1+2\alpha_2)^2} + \\ &\frac{3\alpha_1 t^{\alpha_1+2\alpha_2}}{4\alpha_2(\alpha_1+\alpha_2)(\alpha_1+2\alpha_2)^2} + \frac{t^{\alpha_1+3\alpha_2}}{8\alpha_1^3(\alpha_1+3\alpha_2)} + \\ &\frac{t^{\alpha_1+2\alpha_2}}{(\alpha_1+\alpha_2)(\alpha_1+2\alpha_2)^2} + \frac{t^{\alpha_1+4\alpha_2}}{64\alpha_2^4(\alpha_1+4\alpha_2)} + \\ &\frac{\alpha_1 t^{\alpha_1+2\alpha_2}}{4\alpha_2^2(\alpha_1+\alpha_2)(\alpha_1+2\alpha_2)} + \\ &\frac{t^{2\alpha_1+2\alpha_2}}{8\alpha_2^2(\alpha_1+\alpha_2)(\alpha_1+2\alpha_2)}, \\ y^3(t) &= 1 + \frac{t^{\alpha_2}}{2\alpha_2} + \frac{t^{2\alpha_2}}{8\alpha_2^2} + \frac{t^{3\alpha_2}}{48\alpha_2^3}, \\ &\vdots \end{aligned} \tag{25}$$

If $\alpha_1 = \alpha_2 = \alpha$, then

$$\begin{aligned} x^3(t) &= \frac{t^\alpha}{\alpha} + \frac{t^{2\alpha}}{\alpha^2} + \frac{37}{72}\frac{t^{3\alpha}}{\alpha^3}, \\ y^3(t) &= 1 + \frac{t^\alpha}{2\alpha} + \frac{t^{2\alpha}}{8\alpha^2} + \frac{t^{3\alpha}}{48\alpha^3}, \end{aligned} \tag{26}$$

Figures 4 and 5 show the approximate solutions $x^3(t)$ and $y^3(t)$ and the exact solutions $x(t)$ and $y(t)$, respectively for different values of $\alpha_1 = \alpha_2 = 1, 0.9, 0.5$.

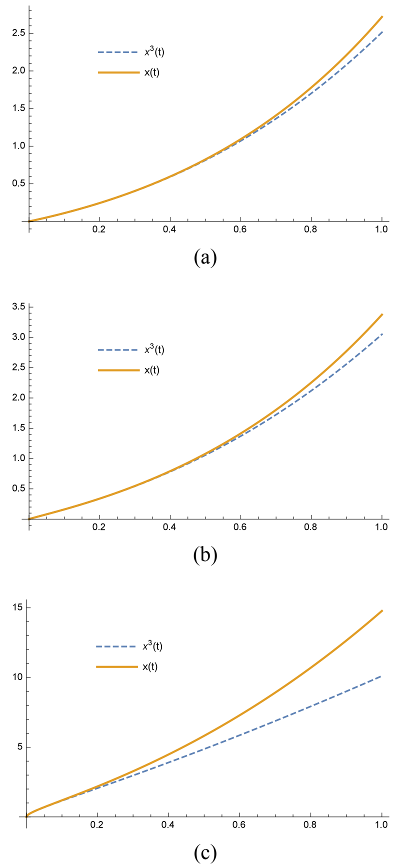


Figure 4: A Comparison for exact $x(t)$ and the approximate solution $x^3(t)$ in Example (4.3) where $\alpha_1 = \alpha_2 = 1,\ 0.9,\ 0.5$.

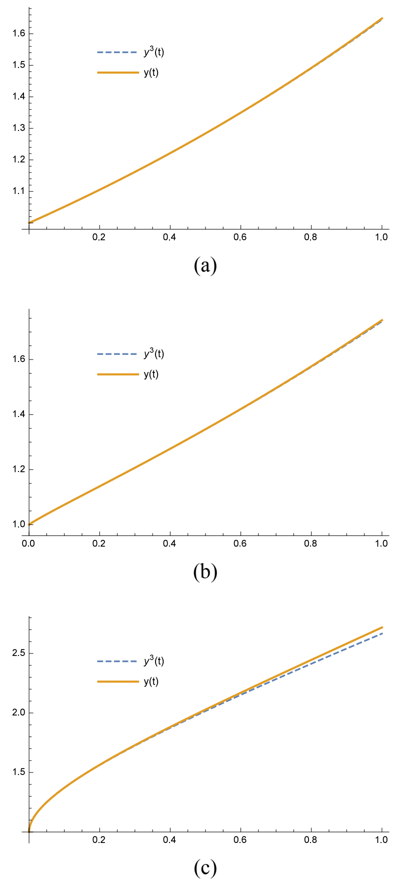


Figure 5: A Comparison for exact $y(t)$ and the approximate solution $y^3(t)$ in Example (4.3) where $\alpha_1 = \alpha_2 = 1,\ 0.9,\ 0.5$.

Example 4.4. Consider the nonlinear system of fractional order differential equations

$$\begin{aligned} D^{\alpha_1}x(t) &= t\; z(t), \\ D^{\alpha_2}y(t) &= x(t)\; y(t), \\ D^{\alpha_3}z(t) &= 2x^2(t), \end{aligned} \tag{27}$$

subject to the initial conditions

$$x(0) = 1, \quad y(0) = 1, \quad z(0) = 0. \tag{28}$$

According to the formulas (8), the variational iteration formulas for system (27) are given by

$$\begin{aligned} x^{k+1}(t) &= x^k(t) - \int_0^t \tau^{\alpha_1-1}[D^{\alpha_1}x^k(\tau) - \tau\; z^k(\tau)]d\tau, \\ y^{k+1}(t) &= y^k(t) - \\ &\int_0^t \tau^{\alpha_2-1}[D^{\alpha_2}y^k(\tau) - x^k(\tau)\; y^k(\tau)]d\tau, \\ z^{k+1}(t) &= z^k(t) - \int_0^t \tau^{\alpha_3-1}[D^{\alpha_3}z^k(\tau) - 2(x^k(\tau))^2]d\tau. \end{aligned} \tag{29}$$

Begin with $x^0(t) = 1,\ y^0(t) = 1$ and $z^0(t) = 0$, we obtain

$$\begin{aligned} x^1(t) &= 1, \\ y^1(t) &= 1 + \frac{t^{\alpha_2}}{\alpha_2}, \\ z^1(t) &= 2\frac{t^{\alpha_3}}{\alpha_3}, \\ x^2(t) &= 1 + \frac{2t^{\alpha_1+\alpha_3+1}}{\alpha_3(\alpha_1+\alpha_3+1)}, \\ y^2(t) &= 1 + \frac{t^{\alpha_2}}{\alpha_2} + \frac{t^{2\alpha_2}}{2\alpha_2^2}, \\ z^2(t) &= 2\frac{t^{\alpha_3}}{\alpha_3}, \\ x^3(t) &= 1 + \frac{2t^{\alpha_1+\alpha_3+1}}{\alpha_3(\alpha_1+\alpha_3+1)}, \\ y^3(t) &= 1 + \frac{t^{\alpha_2}}{\alpha_2} + \frac{t^{2\alpha_2}}{2\alpha_2^2} - \frac{t^{3\alpha_2}}{6\alpha_2^3} + \\ &\frac{2t^{\alpha_1+\alpha_2+\alpha_3+1}}{\alpha_3(\alpha_1+\alpha_3+1)(\alpha_1+\alpha_2+\alpha_3+1)} + \\ &\frac{2t^{\alpha_1+2\alpha_2+\alpha_3+1}}{\alpha_2\alpha_3(\alpha_1+\alpha_3+1)(\alpha_1+2\alpha_2+\alpha_3+1)} + \\ &\frac{t^{\alpha_1+3\alpha_2+\alpha_3+1}}{\alpha_2^2\alpha_3(\alpha_1+\alpha_3+1)(\alpha_1+3\alpha_2+\alpha_3+1)}, \\ z^3(t) &= 2\frac{t^{\alpha_3}}{\alpha_3} + 4\frac{t^{\alpha_1+2\alpha_3+1}}{\alpha_3(\alpha_1+\alpha_3+1)(\alpha_1+2\alpha_3+1)}, \end{aligned} \tag{30}$$

5 Conclusions

In this article, we presented a modified technique of the variational iteration method that approximate the solutions of linear and nonlinear systems of differential equations of fractional order. Several examples were examined to show the efficiency of our new method. We have solved three systems: linear and nonlinear of fractional differential equations by the proposed technique. The numerical solutions are given in series form, that converge to the exact solution obtained by conformable Laplace transform method modified by the author. We observed that our approach is effective in obtaining numerical solutions for linear and nonlinear systems.
Future research might apply our procedure to obtain numerical solution for nonlinear fractional differential equations arise in physics and engineering.

Conflict of Interests

The author declares no conflict of interest with regards to the publication of this paper.

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