Asymptotic Series Evaluation of Integrals Arising in the Particular Solutions to Airy's Inhomogeneous Equation with Special Forcing Functions

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Abstract: - In this work, particular and general solutions to Airy's inhomogeneous equation are obtained when the forcing function is one of Airy's functions of the first and second kind, and the standard Nield-Kuznetsov function of the first kind. Particular solutions give rise to special integrals that involve products of Airy's and Nield-Kuznetsov functions. Evaluation of the resulting integrals is facilitated by expressing their integrands in asymptotic series. Corresponding expressions for the Nield-Kuznetsov function of the second kind are obtained.

Key-Words: - Airy's inhomogeneous equation, special integrals, asymptotic series, Nield-Kuznetsov functions

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1 Introduction

The objective of this work is to consider solutions to the inhomogeneous Airy's ordinary differential equation, ode, [1], when the right-hand-side forcing functions is a special function. In particular, the interest is in the right-hand-side being an Airy's function of the first and of the second kind, $A_i(x)$ and $B_i(x)$, respectively, [2,3], and the standard Nield-Kuznetsov function of the first kind, $N_i(x)$, [4], discussed below.

Airy's ode and Airy's functions are some of the mathematical gems that arise in mathematical physics due to the fact that many problems in this field can be reduced to Airy's ode, and a number of special functions are rooted in Airy's functions, [3, 5]. Furthermore, solutions to Airy's ode give rise to interesting integral functions and special integrals that lead to advancements of modern day mathematics. Inhomogeneity in Airy's ode due to the functions chosen in this work give rise to important integrals that involve products of Airy's functions and other functions, [6,7]. Some of these products represent solutions to interesting differential equations, as discussed in this work.

In the following sections, an overview of Airy's ode is provided together with its forms of solution.

This will be followed with solutions to Airy's ode with special forcing functions. A discussion and evaluations of the arising integrals then follows.

2 Airy's Equation and its Solutions

The celebrated Airy's (ode) is rooted in the nineteenth century and has various practical applications and theoretical implications in mathematical physics, [2]. Its homogeneous part takes the form

$$y'' - xy = 0 \quad (1)$$

where "prime" notation denotes ordinary differentiation. General solution of (1) is given by, [2,3]:

$$y = a_1 A_i(x) + a_2 B_i(x) \quad (2)$$

where a_1 and a_2 are arbitrary constants, $A_i(x)$ and $B_i(x)$ are the linearly independent Airy's functions of the first and second kind, respectively, and defined by the following integrals, [2,3]:

$$A_i(x) = \frac{1}{\pi}\int_0^\infty \cos\left(xt + \frac{t^3}{3}\right)dt \quad (3)$$

$$B_i(x) = \frac{1}{\pi}\int_0^\infty [\sin\left(xt + \frac{t^3}{3}\right) + \exp\left(xt - \frac{t^3}{3}\right)]dt \quad (4)$$

The non-zero Wronskian of $A_i(x)$ and $B_i(x)$ is given by, [2]:

$$W\big(A_i(x), B_i(x)\big) = A_i(x)B'_i - B_i(x)A'_i = \frac{1}{\pi} \quad (5)$$

The twentieth century witnessed an interest in the inhomogeneous Airy's ode due to applications in systems theory, solutions to Schrodinger and Stark equations, and in fluid mechanics, among others (*cf*. Scorer, [7],; Khanmamedov *et.al*., [8]; Alzahrani *et.al*., [9]; Nield and Kuznetsov, [10]; Lee, [11]; Dunster, [12,13]; and the references therein).

The literature shows that particular solutions to the inhomogeneous Airy's equation of the form:

$$y'' - xy = R \quad (6)$$

are given by

$$y = b_1 A_i(x) + b_2 B_i(x) + H_i(x) \quad (7)$$

when $R = \frac{1}{\pi}$, and by

$$y = c_1 A_i(x) + c_2 B_i(x) + G_i(x) \quad (8)$$

when $R = -\frac{1}{\pi}$, where b_1 and b_2, and c_1 and c_2 are arbitrary constants. The functions $G_i(x)$ and $H_i(x)$ are known as the Scorer functions, [7,11], with integral representation given by:

$$G_i(x) = \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{t^3}{3}\right)dt \quad (9)$$

$$H_i(x) = \frac{1}{\pi}\int_0^\infty \exp\left(xt - \frac{t^3}{3}\right)dt \quad (10)$$

The literature also shows that writing solution to ode (6) in terms of the Scorer functions for any constant R, requires non-trivial mathematical manipulations, [3].

The twenty first century, however, witnessed the introduction of a general methodology to find the general solution to equation (6) for any constant R. This solution is given by Hamdan and Kamel, [4], as:

$$y = d_1 A_i(x) + d_2 B_i(x) - \pi R N_i(x) \quad (11)$$

where d_1, d_2 are arbitrary constants, and the integral function $N_i(x)$ is called the Standard Nield-Kuznetsov Function of the First Kind, and is given by, [4]:

$$N_i(x) = A_i(x)\int_0^x B_i(t)dt - B_i(x)\int_0^x A_i(t)dt \quad (12)$$

When $R = \frac{1}{\pi}$, solution (11) reduces to (8), and $N_i(x) = -H_i(x)$, and when $R = -\frac{1}{\pi}$, solution (11) reduces to (9), and $N_i(x) = G_i(x)$.

Clearly, relationship between $N_i(x)$ and the Scorer functions is given by

$$N_i(x) = \frac{2}{3}G_i(x) - \frac{1}{3}H_i(x) \quad (13)$$

with integral representation obtained from (9), (10) and (13) as

$$N_i(x) = \frac{2}{3\pi}\int_0^\infty \sin\left(xt + \frac{t^3}{3}\right)dt - \frac{1}{3\pi}\int_0^\infty \exp\left(xt - \frac{t^3}{3}\right)dt \quad (14)$$

The main properties of the Standard Nield-Kuznetsov function of the first kind, $N_i(x)$, and its efficient computations have been discussed by previously discussed, [4,14,15]. The case when the right-hand-side of Airy's inhomogeneous equation is a function of x, namely the ode

$$y'' - xy = f(x) \quad (15)$$

was elegantly discussed in the mid-twentieth century work of Miller and Mursi, [16].

They have shown that (15) might be solved when

$$f(x) = u(x)g(x) + h(x)u'(x) \tag{16}$$

with

$$u = a_1 A_i(x) + a_2 B_i(x) \tag{17}$$

and where $h(x)$ and $g(x)$ are expressed as power series. The solution may be expressed in the same form or as a series of derivatives of $u(x)$.

The solution is also given in the case where $f(x)$ is itself expressed as a power series; in this case it is of the form

$$y = k(x) + v(x)l(x) \tag{18}$$

where

$$v(x) = u(x) + \pi G_i(x) \tag{19}$$

and $k(x)$ and $l(x)$ are expressed as power series. The series solution terminates if $g(x)$ and $h(x)$ are polynomials, or if $f(x)$ is a polynomial.

It is clear that the method of Miller and Mursi, [16], above, has some restrictions on the function $f(x)$ in addition to being time consuming in its application.

A decade ago, a method was introduced by Hamdan and Kamel, [4], to find the general solution of (15) when $f(x)$ is a differentiable function of x. They showed that the general solution to (15) can be expressed as:

$$y = c_1 A_i(x) + c_2 B_i(x) + \pi K_i(x) - \pi f(x) N_i(x) \tag{20}$$

where $N_i(x)$ in the Standard Nield-Kuznetsov function of the first kind, defined in (14), and $K_i(x)$ has been referred to as the Standard Nield-Kuznetsov function of the second kind, defined by the following equivalent forms:

$$K_i(x) = A_i(x) \int_0^x \left\{\int_0^t B_i(\tau) d\tau\right\} f'(t) dt - B_i(x) \int_0^x \left\{\int_0^t A_i(\tau) d\tau\right\} f'(t) \tag{21}$$

$$K_i(x) = f(x) N_i(x) - \{A_i(x) \int_0^x f(t) B_i(t)\, dt - B_i(x) \int_0^x f(t) A_i(t)\, dt\} \tag{22}$$

3 Forms of Particular Solutions

Solution (20) indicates that the particular solution to (15) is written as:

$$y_p = \pi K_i(x) - \pi f(x) N_i(x) \tag{23}$$

This has proved to be convenient for computations involving many forms of $f(x)$, [14,15].

Using (22) in (23), equation (23) in written the following equivalent form:

$$y_p = \pi\{B_i(x) \int_0^x f(t) A_i(t)\, dt - A_i(x) \int_0^x f(t) B_i(t)\, dt\} \tag{24}$$

Equations (23) and (24) reflect the dependence of the particular solution on the forcing function $f(x)$ and on integrability of the product of $f(x)$ and Airy's functions. Clearly, when $f(x)$ is itself an Airy's function, then the integrals involve products of Airy's functions. In order to illustrate the arising integrals and their evaluations, the following three examples of $f(x)$ are discussed and the particular solution is obtained using both forms, (23) and (24), which produce the same integrals.

3.1. Case 1: Using Form (24)

Example 1: If $f(x) = A_i(x)$, equation (15) takes the form

$$y'' - xy = A_i(x) \tag{25}$$

Particular integral (24) for ode (25) takes the form

$$y_p = \pi\{B_i(x) \int_0^x [A_i(t)]^2\, dt - A_i(x) \int_0^x A_i(t) B_i(t)\, dt\} \tag{26}$$

and the general solution to (25) is written as

$$y = c_1 A_i(x) + c_2 B_i(x) + \pi\{B_i(x) \int_0^x [A_i(t)]^2\, dt - A_i(x) \int_0^x A_i(t) B_i(t)\, dt\} \tag{27}$$

Example 2: If $f(x) = B_i(x)$, equation (15) takes the form

$$y'' - xy = B_i(x) \tag{28}$$

Particular integral (24) for ode (28) takes the form

$$y_p = \pi\{B_i(x) \int_0^x A_i(t)\, B_i(t) dt - A_i(x) \int_0^x [B_i(t)]^2\, dt\} \tag{29}$$

and the general solution to (25) is written as

$$y = c_1 A_i(x) + c_2 B_i(x) + \pi\{B_i(x)\int_0^x A_i(t)\, B_i(t)dt - A_i(x)\int_0^x [B_i(t)]^2\, dt\} \quad (30)$$

Example 3: If $f(x) = N_i(x)$, (15) takes the form

$$y'' - xy = N_i(x) \quad (31)$$

Particular integral (24) for ode (31) takes the form

$$y_p = \pi\{B_i(x)\int_0^x N_i(t)A_i(t)\, dt - A_i(x)\int_0^x N_i(t)B_i(t)\, dt\} \quad (32)$$

and the general solution to (31) is written as

$$y = c_1 A_i(x) + c_2 B_i(x) + \pi\{B_i(x)\int_0^x N_i(t)A_i(t)\, dt - A_i(x)\int_0^x N_i(t)B_i(t)\, dt\} \quad (33)$$

Equations (27), (30) and (33), involve the five integrals:

$\int_0^x [A_i(t)]^2\, dt, \int_0^x [B_i(t)]^2\, dt, \quad \int_0^x A_i(t)\, B_i(t)dt,$
$\int_0^x N_i(t)A_i(t)\, dt$ and $\int_0^x N_i(t)B_i(t)\, dt.$

Their method of evaluation will be discussed below. It is worth noting here that the functions $[A_i(x)]^2$, $[B_i(x)]^2$ and $A_i(x)B_i(x)$ are three linearly independent solutions of the homogeneous third-order ode $y''' - 4xy' - 2y = 0$, with Wronskian $W([A_i(x)]^2, [B_i(x)]^2 \text{ and } A_i(x)B_i(x)) = \frac{2}{\pi^3}$ (*cf.* Vallée and Soares, [3], Page 30).

3.2. Case 2: Using Form (23)

Equation (20) gives the general solution in terms of $N_i(x)$ and $K_i(x)$. Using the same three example, above, we obtain the following general solutions.

Example 1: When $f(x) = A_i(x)$ in (15), general solution to (15) takes the form

$$y = c_1 A_i(x) + c_2 B_i(x) + \pi K_i(x) - \pi A_i(x) N_i(x) \quad (34)$$

where $K_i(x)$ is evaluated using (21) or (22), respectively, as

$$K_i(x) = A_i(x)\int_0^x \left\{\int_0^t B_i(\tau)d\tau\right\} A'(t)dt - B_i(x)\int_0^x \left\{\int_0^t A_i(\tau)d\tau\right\} A'(t)dt \quad (35)$$

or

$$K_i(x) = A_i(x)\, N_i(x) - \{A_i(x)\int_0^x A_i(t)\, B_i(t)\, dt - B_i(x)\int_0^x [A_i(t)\,]^2\, dt\} \quad (36)$$

Example 2: When $f(x) = B_i(x)$ in (15), general solution to (15) takes the form

$$y = c_1 A_i(x) + c_2 B_i(x) + \pi K_i(x) - \pi B_i(x) N_i(x) \quad (37)$$

where $K_i(x)$ is evaluated using (21) or (22), respectively, as

$$K_i(x) = A_i(x)\int_0^x \left\{\int_0^t B_i(\tau)d\tau\right\} B'(t)dt - B_i(x)\int_0^x \left\{\int_0^t A_i(\tau)d\tau\right\} B'(t)dt \quad (38)$$

or

$$K_i(x) = B_i(x)\, N_i(x) - \{A_i(x)\int_0^x [B_i(t)]^2\, dt - B_i(x)\int_0^x A_i(t)B_i(t)\, dt\} \quad (39)$$

Example 3: When $f(x) = N_i(x)$ in (15), general solution to (15) takes the form

$$y = c_1 A_i(x) + c_2 B_i(x) + \pi K_i(x) - \pi [N_i(x)]^2 \quad (40)$$

where $K_i(x)$ is evaluated using (21) or (22), respectively, as

$$K_i(x) = A_i(x)\int_0^x \left\{\int_0^t B_i(\tau)d\tau\right\} N'(t)dt - B_i(x)\int_0^x \left\{\int_0^t A_i(\tau)d\tau\right\} N'(t)dt \quad (41)$$

or

$$K_i(x) = [N_i(x)]^2 - \{A_i(x)\int_0^x f(t)B_i(t)\, dt - B_i(x)\int_0^x f(t)A_i(t)\, dt\} \quad (42)$$

5 Asymptotic Series Representation of Arising Integrals

In order to evaluate the integrals and the Nield-Kuznetsov functions arising in the solutions above, the following asymptotic series expressions are used when $x \gg 1$, [3-5]:

$$A_i(x) \approx \frac{\exp(-\mu)}{2\sqrt{\pi}x^{\frac{1}{4}}} \quad (43)$$

$$B_i(x) \approx \frac{\exp(\mu)}{\sqrt{\pi}x^{\frac{1}{4}}} \quad (44)$$

wherein $\mu = \frac{2}{3}x^{3/2}$. Hamdan and Kamel, [4], obtained the following asymptotic series for the Nield-Kuznetsov functions:

$$N_i(x) \approx -\frac{\exp(\mu)}{3\sqrt{\pi}x^{1/4}} \tag{45}$$

$$Ki(x) \approx \frac{\exp(-\mu)}{2\sqrt{\pi}x^{\frac{1}{4}}}\int_0^x \left\{\frac{\exp(\varphi)}{\sqrt{\pi}t^{\frac{3}{4}}}\right\} f'(t)dt - \frac{\exp(\mu)}{3\sqrt{\pi}x^{\frac{1}{4}}}f(x) \tag{46}$$

wherein $\varphi = \frac{2}{3}x^{2/3}$.

Using (43)-(45), the following values of the integrals appearing in (36), (39), and (42) are obtained, where some have been evaluated using *Wolfram Alpha*:

$$\int_0^x A_i(t)B_i(t)\,dt = \frac{\sqrt{x}}{\pi} \tag{47}$$

$$\int_0^x A_i(t)N_i(t)\,dt = -\frac{\sqrt{x}}{3\pi} \tag{48}$$

$$\int_0^x [A_i(t)\,]^2\,dt = \frac{(x^{\frac{3}{2}})^{\frac{3}{2}}\left\{\Gamma\left(\frac{1}{3}\right)-\Gamma\left(\frac{1}{3},\frac{4}{3}x^{\frac{3}{2}}\right)\right\}}{2\left(6^{\frac{2}{3}}\right)\pi x}; \quad Re\left(x^{\frac{3}{2}}\right) > 0 \tag{49}$$

$$\int_0^x [B_i(t)]^2\,dt = \frac{\sqrt[3]{2}(-x^{\frac{3}{2}})^{\frac{3}{2}}\left\{\Gamma\left(\frac{1}{3},-\frac{4}{3}x^{\frac{3}{2}}\right)-\Gamma\left(\frac{1}{3}\right)\right\}}{\left(3^{\frac{2}{3}}\right)\pi x};$$

$$Re\left(x^{\frac{3}{2}}\right) < 0 \tag{50}$$

$$\int_0^x B_i(t)N_i(t)\,dt = \frac{\sqrt[3]{2}(-x^{\frac{3}{2}})^{\frac{3}{2}}\left\{\Gamma\left(\frac{1}{3}\right)-\Gamma\left(\frac{1}{3},-\frac{4}{3}x^{\frac{3}{2}}\right)\right\}}{3\left(3^{\frac{2}{3}}\right)\pi x};$$

$$Re\left(x^{\frac{3}{2}}\right) < 0 \tag{51}$$

where $\Gamma(x)$ is the gamma function, $\Gamma(\alpha, x)$ is the incomplete gamma function, and $Re(z)$ is the real part of z.

6 Expressions for the Nield-Kuznetsov Function of the Second Kind

Using asymptotic series exressions (43)-(45), and integrals (47)-(51), the following expressions are obtained for the particular solution (32) for each of three examples considered. Furthermore, using (36), (39), and (42), expressions for $Ki(x)$ are obtained.

Example 1: Using (36), the following expression is obtained for $Ki(x)$:

$$K_i(x) = \frac{1}{2\pi\sqrt{\pi}}\left\{x\frac{\exp(\mu)}{6^{\frac{2}{3}}}\left\{\Gamma\left(\frac{1}{3}\right) - \Gamma\left(\frac{1}{3},\frac{4}{3}x^{\frac{3}{2}}\right)\right\} - x^{\frac{1}{4}}\exp(-\mu)\right\} - \frac{1}{6\pi x^{\frac{1}{2}}} \tag{52}$$

Particular solution (26) and general solution (27) take the following forms, respectively:

$$y_p = \frac{1}{2\sqrt{\pi}}\left\{x\frac{\exp(\mu)}{6^{\frac{2}{3}}}\left\{\Gamma\left(\frac{1}{3}\right) - \Gamma\left(\frac{1}{3},\frac{4}{3}x^{\frac{3}{2}}\right)\right\} - x^{\frac{1}{4}}\exp(-\mu)\right\} \tag{53}$$

$$y = c_1\frac{\exp(-\mu)}{2\sqrt{\pi}x^{\frac{1}{4}}} + c_2\frac{\exp(\mu)}{\sqrt{\pi}x^{\frac{1}{4}}} + \frac{1}{2\sqrt{\pi}}\left\{x\frac{\exp(\mu)}{6^{\frac{2}{3}}}\left\{\Gamma\left(\frac{1}{3}\right) - \Gamma\left(\frac{1}{3},\frac{4}{3}x^{\frac{3}{2}}\right)\right\} - x^{\frac{1}{4}}\exp(-\mu)\right\} \tag{54}$$

Example 2: Using (39), the following expression is obtained for $Ki(x)$:

$$K_i(x) = \frac{1}{\pi\sqrt{\pi}}\left\{x^{\frac{1}{4}}\exp(\mu) - \frac{\exp(-\mu)}{6^{\frac{2}{3}}x^{\frac{5}{4}}}(-x^{\frac{3}{2}})^{\frac{3}{2}}\left\{\Gamma\left(\frac{1}{3},-\frac{4}{3}x^{\frac{3}{2}}\right) - \Gamma\left(\frac{1}{3}\right)\right\}\right\} - \frac{\exp(2\mu)}{3\pi x^{\frac{1}{2}}} \tag{55}$$

Particular solution (29) and general solution (30) take the following forms, respectively:

$$y_p = \frac{1}{\sqrt{\pi}}\left\{x^{\frac{1}{4}}\exp(\mu) - \frac{\exp(-\mu)}{6^{\frac{2}{3}}x^{\frac{5}{4}}}(-x^{\frac{3}{2}})^{\frac{3}{2}}\left\{\Gamma\left(\frac{1}{3},-\frac{4}{3}x^{\frac{3}{2}}\right) - \Gamma\left(\frac{1}{3}\right)\right\}\right\} \tag{56}$$

$$y = c_1\frac{\exp(-\mu)}{2\sqrt{\pi}x^{\frac{1}{4}}} + c_2\frac{\exp(\mu)}{\sqrt{\pi}x^{\frac{1}{4}}} + \frac{1}{\sqrt{\pi}}\left\{x^{\frac{1}{4}}\exp(\mu) - \frac{\exp(-\mu)}{6^{\frac{2}{3}}x^{\frac{5}{4}}}(-x^{\frac{3}{2}})^{\frac{3}{2}}\left\{\Gamma\left(\frac{1}{3},-\frac{4}{3}x^{\frac{3}{2}}\right) - \Gamma\left(\frac{1}{3}\right)\right\}\right\} \tag{57}$$

Example 3: Using (42), the following expression is obtained for $Ki(x)$:

$$K_i(x) = \frac{\exp(2\mu)}{9\pi x^{1/2}} + \frac{1}{3\pi\sqrt{\pi}}\left\{-x^{\frac{1}{4}}\exp(\mu) - \frac{\exp(-\mu)}{6^{\frac{2}{3}}x^{\frac{5}{4}}}(-x^{\frac{3}{2}})^{\frac{3}{2}}\left\{\Gamma\left(\frac{1}{3}\right) - \Gamma\left(\frac{1}{3},-\frac{4}{3}x^{\frac{3}{2}}\right)\right\}\right\} \tag{58}$$

Particular solution (32) and general solution (33) take the following forms, respectively:

$$y_p = \frac{1}{3\sqrt{\pi}}\left\{-x^{\frac{1}{4}}\exp(\mu) - \frac{\exp(-\mu)}{6^{\frac{2}{3}}x^{\frac{5}{4}}}(-x^{\frac{3}{2}})^{\frac{3}{2}}\left\{\Gamma\left(\frac{1}{3}\right) - \Gamma\left(\frac{1}{3}, -\frac{4}{3}x^{\frac{3}{2}}\right)\right\}\right\} \quad (59)$$

$$y = c_1\frac{\exp(-\mu)}{2\sqrt{\pi}x^{\frac{1}{4}}} + c_2\frac{\exp(\mu)}{\sqrt{\pi}x^{\frac{1}{4}}} + \frac{1}{3\sqrt{\pi}}\left\{-x^{\frac{1}{4}}\exp(\mu) - \frac{\exp(-\mu)}{6^{\frac{2}{3}}x^{\frac{5}{4}}}(-x^{\frac{3}{2}})^{\frac{3}{2}}\left\{\Gamma\left(\frac{1}{3}\right) - \Gamma\left(\frac{1}{3}, -\frac{4}{3}x^{\frac{3}{2}}\right)\right\}\right\} \quad (60)$$

7 Conclusion

In this work, a method of solving the inhomogeneous Airy's equation when the right-hand-side is a special function (such as one of Airy's functions or the Nield-Kuznetsov function of the first kind), was presented. Arising special integrals involve products of these special functions. The integrands have been expressed using asymptotic series, and integral evaluations were carried out using *Wolfram Alpha*. The particular and general solutions of Airy's inhomogeneous equation have been presented and a derivation of expressions for the Nield-Kuznetsov function of the second kind, corresponding to each forcing function, have been obtained. Significance of this work stems from the fact that Airy's equation is one of our mathematical gems and its solutions have given rise to many special functions since its inception. It also plays a role in the development and optimization of computational algorithms designed to provide efficient computations of Airy's functions. These same algorithms are of great value to the numerical analysis literature. The arising integrals in this work might find applications in mathematical physics.

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