Mass points, spaces of spheres, ”hyperbolae” and Dandelin and Quételet theorems

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Abstract: The use of mass points eases the definition of a branch of a hyperbola in the Euclidean plane based on a Rational Quadratic Bézier Curve. In the space of spheres, any circular cone, circular cylinder, torus, pencil of spheres or Dupin cyclide is represented by a rational quadratic Bézier curve that is conic arc seen as circle arc. The limit points of the Poncelet pencil or the singular points of the Dupin cyclide can be determined using the asymptotes of this circle. This article shows that the use of mass points simplifies the modelling of these pencils or these Dupin cyclides in the space of spheres. The determination of the Dandelin spheres ends this work as an application.

Key-Words: Mass points, Rational Quadratic Bézier curves, Conics, Space of spheres, Minkowski-Lorentz space

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1 Introduction

Among the three known problems coming from Antiquity, the trisection of an angle is one of them. It requires the construction of the cubic root of 2. And leads Ménechme to set up the intersection between a parabola and a hyperbola. Since the IV^{th} century B.C. Ménechme and Aristée knew that the section between a cone and a plane perpendicular to a generatrix of the cone provides different curves depending on the angle of the cone. This angle can be acute, right or obtuse and it gives an ellipse, parabola or hyperbola respectively as the conic section [1]. Apollonius was the first who noticed that all conic sections can be determined from the same oblique cone with a circular base. One must wait till XIX^{th} century and the Dandelin and Quételet theorems to link the different definitions of the conic sections [2]. The first one was the intersection between a cone and a plane, the second one was the definition with two foci and the third one was based on one focus and a directrix line. Depending on the problem, these definitions are used in the geometric modeling domain. An algorithm that provides the Dandelin spheres allows to get from a conic representation to another one. The invariants of any conic are thus completed by an algorithm.

The computer aided design and the computer aided manufacturing are dealt with the sketch of conic arcs. Their representation with control points is very useful when the dimension space is at least 3. A classical solution consists in the use of polynomial or rational quadratic Bézier curves. These curves were invented by Pierre Bézier [3] from Renault. Their iterative construction are based on the algorithm founded by Paul de Casteljau [4] from Citroën. See also [5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

Anyway, a rational quadratic Bézier curve, called BR curve is defined by the three Bernstein polynomial of degree 2 and three weighted points (P_0, ω_0), (P_1, ω_1) and (P_2, ω_2) with non-zero weights or non negative weights as the most frequent case. From a geometrical point of view, such an arc is defined as the set of barycentres of weighted points. The weight of this barycentre equals the sum of the product of the weights of control points times adequate Bernstein polynomials. Moreover, some problem exists with the non bounded curves as parabola or hyperbola arcs. In order to fix it, a smart and efficient solution consists in gathering in a same space the set of weighted points and the set of vectors considered as points assigned with a null weight. The space is thus denoted the mass points space [15, 16]. Thus, an adequate representation of rational quadratic Bézier curve allows to obtain the invariant of the defined conic [17, 18, 19]. Moreover, some researchers use homogenous coordinates to model a semi-circle [20, 21, 22], but L. Piegl mentions this problem in [21]: "A point in projective space is what mathematicians call an equivalence class. This means that $\overrightarrow{P_j}\,(x_j, y_j, z_j, 0)$ and $\overrightarrow{P_j}^*\,(\alpha x_j, \alpha y_j, \alpha z_j, 0)$ are two representations of the same point in projective space. However, substituting $\overrightarrow{P_j}$ and $\overrightarrow{P_j}^*$ into Eq (7.34) clearly results in two different curves." So, the use of homogeneous coordinates is not possible. On one hand, the metric structure , Euclidean or not of the plane can not be considered in its projective closure. On second hand, the choice of any representative element from an equivalent class is changing the status of the curve [9]. By the use of homogeneous coordinates, a proper conic not differ from another if the line of infinity is not known [23].

Using mass points, we choose the vector which allows to obtain a semi-circle, this circle can be Euclidean, Figure 1 or not, Figure 2. This vector is defined by using perpendicular conditions and pseudo-metric conditions to determine a Bézier curve which models a given hyperbola seen as a circle (for the non-degenerate indefinite quadratic form). Let us give the conditons to obtain a semi-circle.

Theorem 1 :

Let $(\omega_0, \omega_2) \in \mathbb{R}^* \times \mathbb{R}^*$. Let (P_0, ω_0), $\left(\overrightarrow{P_1}, 0\right)$ and (P_2, ω_2) be three control mass point of a quadratic Bézier curve γ.

The curve γ is a semi-circle if and only if

$$\left\{ \begin{array}{l} \overrightarrow{P_0P_2} \perp \overrightarrow{P_1} \\ \omega_0\ \omega_2\ \overrightarrow{P_0P_2}^2 = 4\ \overrightarrow{P_1}^2 \end{array} \right. \tag{1}$$

In the Figure 1, the basis vectors verify

$$\left\{ \begin{array}{l} \overrightarrow{\imath}^2 = \overrightarrow{\jmath}^2 = 1 \\ \overrightarrow{\imath} \cdot \overrightarrow{\jmath} = 0 \end{array} \right.$$

and the equation of the circle, in blue line, is

$$x^2 + y^2 = 1 \tag{2}$$

To obtain the semi-ellipse, the second control point is $\overrightarrow{Q_1} = -\frac{1}{2}\overrightarrow{P_1}$.

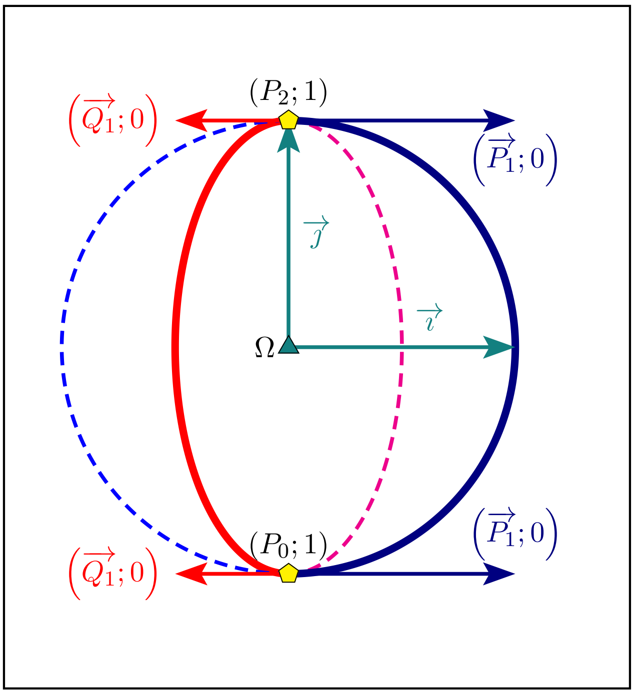

Figure 1: An Euclidean semi-circle and a semi-ellipse of center Ω.

In the Figure 2, consideing the quadratic form

$$\mathcal{Q}\,(x, y) = x^2 - y^2$$

the basis vectors satisfy to

$$\left\{ \begin{array}{l} \overrightarrow{\imath}^2 = 1 \\ \overrightarrow{\jmath}^2 = -1 \\ \overrightarrow{\imath} \cdot \overrightarrow{\jmath} = 0 \end{array} \right.$$

and the equation of the circle[1] is

$$x^2 - y^2 = 1 \tag{3}$$

From the examples given in Figure 1 and 2, the table 1 gives the computations of Formula (1).

The formalism in the description of Bézier curve based on mass control points is spreading to any dimension. It does not depend on the considered quadratic form. That facilitates their use in the space of spheres we are focusing on.

The space of spheres has been introduced in different ways. For example, in [24, 25] M. Berger is putting himself in the projectif space of quadratic

[1]The rigorous term is pseudo-circle which is an hyperbola with an Euclidean point of view.

Point, vector and calculus	Figure 1	Figure 2
P_0	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$
P_2	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -\sqrt{2} \\ -1 \end{pmatrix}$
$\overrightarrow{P_1}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$
$\overrightarrow{P_0P_2}$	$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$	$\begin{pmatrix} -2\sqrt{2} \\ -2 \end{pmatrix}$
$\overrightarrow{P_0P_2}^2$	4	$\left(-2\sqrt{2}\right)^2 - (-2)^2 = 4$
$\omega_0\omega_2\overrightarrow{P_0P_2}^2$	4	$1 \times -1 \times 4 = -4$
$\overrightarrow{P_1}^2$	1	$1^2 - \sqrt{2}^2 = -1$
$\overrightarrow{P_0P_2} \centerdot \overrightarrow{P_1}$	$0 \times 1 + 2 \times 0 = 0$	$-2\sqrt{2} \times 1 - (-2) \times \sqrt{2} = 0$
$\omega_0\omega_2\overrightarrow{P_0P_2}^2 - 4\overrightarrow{P_1}^2$	$4 - 4 \times 1 = 0$	$-4 - 4 \times -1 = 0$

Table 1: Formula 1 and circles generated by Bézier curves with control mass points $(P_0, 1)$, $\left(\overrightarrow{P_1}, 0\right)$ and $(P_2, 1)$ in Figure 1 and $(P_2, -1)$ in Figure 2.

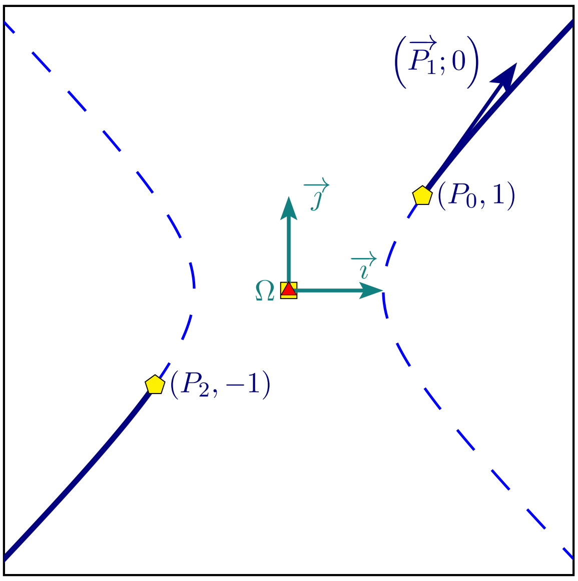

Figure 2: A non-Euclidean semi-circle of center Ω.

forms on the Euclidean space. M. Paluszny [26] is working in a 4-dimension projective space using the Mœbius hypersphere. U. Hertrich-Jeromin [27], T. Cecil [28], R. Langevin, J. O'Hara [29, 30] and P. Walczak [31] are using a 4-dimension quadric on the 5-dimension Lorentz space. This space generalises the Minkowski time-space. The following is based on this last theme. The inside or outside area of a sphere is defined by its orientation. This structure offers to solve unsolved problems in $\mathbb{R}^3$ as well as in others representations of the sphere space [32, 33, 34]. Moreover, the Lorentz space allows to use the powerful structure of geometric algebra [35, 36]. In the space of oriented spheres, a Dupin cyclide, a circular cone, a circular cylinder, a torus is represented by two conics on a 4-dimensional unit sphere of the Lorentz space. In this space, a pencil of spheres is an unit circle (for the Lorentz structure) on the previous unit sphere [37, 10, 35, 32, 31, 30]. This space eases the handling of surfaces. They are thus linearized because considered as the intersection of a unit sphere with a 2 dimension plane. The singularities of Dupin cyclides or the limit points of a spheres pencil are represented by isotropic vectors for the quadratic form of Lorentz. The use of a rational quadric Bézier curve for the representation of any circle arc in the spheres space will take account of vectors in the Bézier curve [38, 13, 14]. The article is composed as follows. The rational quadratic Bézier curve representation with mass control points, the modeling of hyperbola branch and the spheres space are proposed to the reader in paragraph 2. In section 3, oriented spheres and orthogonal spheres are introduced in the usual Euclidean 3D affine space. The Minkowski-Lorentz space and the space of the oriented spheres are defined in the paragraph 4. Rational quadratic Bezier curve with mass control points are thus applied to the representation of a spheres pencil with limit points or any Dupin cyclide with two singular points including a circular cone having a point at infinity, section 5. Before conclusion and future works, the Dandelin spheres using the Minkowski-Lorentz space and the space of spheres are computed in paragraph 6. Two appendices propose a proof of a lemma and numerical results.

2 Rational quadratic Bézier curves in the set of the mass point $\widetilde{\mathcal{P}}$

In the following $(O;\overrightarrow{\imath};\overrightarrow{\jmath})$ designates a direct reference frame in the usual Euclidean affine plane $\mathcal{P}$ and $\overrightarrow{\mathcal{P}}$ the set of the vector plane. The set of mass points is defined by :

$$\widetilde{\mathcal{P}} = (\mathcal{P} \times \mathbb{R}^*) \cup \left(\overrightarrow{\mathcal{P}} \times \left\{\overrightarrow{0}\right\}\right)$$

and $\left(\widetilde{\mathcal{P}}, \oplus, \odot\right)$ is a vector space [19]. So, a mass point is a weighted point (M,ω) with $\omega \neq 0$ or a vector $(\overrightarrow{u}, 0)$. The three quadratic Bernstein polynomials defined by :

$$\begin{array}{rcl} B_0(t) & = & (1-t)^2 \\ B_1(t) & = & 2\,t\,(1-t) \\ B_2(t) & = & t^2 \end{array} \tag{4}$$

where , $t \in [0,1]$. provide the definition of rational quadratic Bézier curve (BR curve) in $\widetilde{\mathcal{P}}$ given below.

Definition 1

Let ω_0, ω_1 and ω_2 be three real numbers.

Let $(P_0;\omega_0)$, $(P_1;\omega_1)$ and $(P_2;\omega_2)$ be three mass points in $\widetilde{\mathcal{P}}$, these points are not collinear.

Define two sets

$$I = \{i \mid \omega_i \neq 0\} \quad \text{and} \quad J = \{i \mid \omega_i = 0\}$$

Define the function ω_f as follows

$$\begin{array}{rccl} \omega_f : & [0;1] & \longrightarrow & \mathbb{R} \\ & t & \longmapsto & \omega_f(t) = \displaystyle\sum_{i\in I} \omega_i \times B_i(t) \end{array} \tag{5}$$

A mass point $(M;\omega)$ or $(\overrightarrow{u};0)$ belongs to the quadratic Bézier curve defined by the three control mass points $(P_0;\omega_0)$, $(P_1;\omega_1)$ and $(P_2;\omega_2)$, if there is a real t_0 in $[0;1]$ such that:

- if $\omega_f(t_0) \neq 0$ then

$$\left\{\begin{array}{rcl} \overrightarrow{OM} = & & \dfrac{1}{\omega_f(t_0)} \left(\displaystyle\sum_{i\in I} \omega_i\, B_i(t_0)\; \overrightarrow{OP_i}\right) \\ & + & \dfrac{1}{\omega_f(t_0)} \left(\displaystyle\sum_{i\in J} B_i(t_0)\; \overrightarrow{P_i}\right) \\ \omega = & & \omega_f(t_0) \end{array}\right. \tag{6}$$

- if $\omega_f(t_0) = 0$ then

$$\overrightarrow{u} = \sum_{i\in I} \omega_i B_i(t_0)\,\overrightarrow{OP_i} + \sum_{i\in J} B_i(t_0)\,\overrightarrow{P_i} \tag{7}$$

Such a curve is denoted $BR\{(P_0;\omega_0);(P_1;\omega_1);(P_2;\omega_2)\}$

If $J = \emptyset$, this definition leads to the usual rational quadratic Bézier curve.

This kind of curve can model a non-Euclidean circular arc as the example shown in the Monkowski-Lorentz space.

3 The oriented spheres in the 3D usual Euclidean affine space $\mathcal{E}_3$

In the Euclidean affine space $\mathcal{E}_3$, each sphere S, of centre C, with a non-negative radius r, defines two oriented spheres S^+ and S^-. We distinguish the inside space and the outside space of S. If the interior (resp. exterior) of the sphere is bounded (resp. non-bounded), the radius of the oriented sphere S^+ (resp. S^-) is $\rho = r$ (resp. $\rho = -r$). The sphere S^+ (resp. S^+) is such that the unit normal vector $\overrightarrow{N}$ at the point M is in the same direction (resp. opposite direction) as the vector $\overrightarrow{CM}$. Then, the same formula matches two cases

$$\overrightarrow{\Omega M} = \rho \overrightarrow{N} \tag{8}$$

where $|\rho| = r$. The power of a point with respect to a sphere is thus defined.

Definition 2 : Power of point with respect to a sphere

Let S be an oriented sphere of centre Ω and radius ρ. Let M be a point of $\mathcal{E}_3$.

The power of the point M with respect to the sphere S is

$$\chi_{\mathrm{S}}(M) = \Omega M^2 - \rho^2 \tag{9}$$

The points of the sphere S are the points M which verify $\chi_{\mathrm{S}}(M) = 0$. The points of $\mathcal{E}_3$ which belong to the open ball defined by the sphere S are the points M which verify $\chi_{\mathrm{S}}(M) < 0$ (resp. $\chi_{\mathrm{S}}(M) > 0$) if the algebraic radius is positive (resp. negative). The other points of $\mathcal{E}_3$ are the points M which verify $\chi_{\mathrm{S}}(M) > 0$ (resp. $\chi_{\mathrm{S}}(M) < 0$) if the algebraic radius is positive (resp. negative).

Definition 3 : Orthogonal spheres and power of two spheres
Let S_1 and S_2 be two oriented spheres of centres Ω_1 and Ω_2, $\Omega_1 \neq \Omega_2$, and radii ρ_1 and ρ_2. The power of the two spheres S_1 and S_2 are

$$\chi_{\mathrm{S}_1,\mathrm{S}_2} = \Omega_1\Omega_2^2 - \rho_1^2 - \rho_2^2$$

The spheres S_1 and S_2 are orthogonal iff we have

$$\chi_{\mathrm{S}_1,\mathrm{S}_2} = 0 \tag{10}$$

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4 The Minkowski-Lorentz space and the space of spheres Λ^4

The Minkowski-Lorentz space $\overrightarrow{\mathrm{L}_{4,1}}$ is the real space vector of dimension 5. A bilinear form $\mathcal{L}_{4,1}$ denoted by a dot product, is defined on the canonical basis $(\overrightarrow{e_-}, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_+})$ as follows

$$\left\{\begin{array}{rcll} \overrightarrow{e_i} \cdot \overrightarrow{e_j} & = & 0 & \text{if } i \neq j, \\ \overrightarrow{e_-} \cdot \overrightarrow{e_-} & = & -1 & \\ \overrightarrow{e_i} \cdot \overrightarrow{e_i} & = & 1 & \end{array}\right. \tag{11}$$

with $i \in \{1, 2, 3, +\}$ and $j \in \{-, 1, 2, 3, +\}$.

Let $\mathcal{Q}_{4,1}$ be the quadratic form associated to $\mathcal{L}_{4,1}$. The affine Minkowski-Lorentz space $\mathrm{L}_{4,1}$ is defined by the point $O_5 = (0, 0, 0, 0, 0)$ and $\overrightarrow{\mathrm{L}_{4,1}}$. A new basis $(\overrightarrow{e_o}, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_\infty})$ with

$$\left\{\begin{array}{rcl} \overrightarrow{e_o} & = & \overrightarrow{e_-} - \overrightarrow{e_+} \\ \overrightarrow{e_\infty} & = & \frac{1}{2}\left(\overrightarrow{e_-} + \overrightarrow{e_+}\right) \end{array}\right. \tag{12}$$

eases to embed the usual 3D Euclidean affine space $\mathcal{E}_3$ in the Minkowski-Lorentz space. The reader will check that

$$\overrightarrow{e_o} \cdot \overrightarrow{e_o} = \overrightarrow{e_\infty} \cdot \overrightarrow{e_\infty} = 0$$

and

$$\overrightarrow{e_o} \cdot \overrightarrow{e_\infty} = -1$$

The origin point O_3 of $\mathcal{E}_3$ is obtained by

$$\overrightarrow{e_o} = \overrightarrow{O_5 O_3}$$

and the vector $\overrightarrow{e_\infty}$ represents the point at infinity of $\mathcal{E}_3$.

According to the Minkowski definitions, any vector $\overrightarrow{u} \in \overrightarrow{\mathrm{L}_{4,1}}$ such that $\mathcal{Q}_{4,1}(\overrightarrow{u})$ is negative, positive or zero is qualified as a time-like, space-like or light-like vector respectively. In $\mathrm{L}_{4,1}$, the light cone C_l is defined by :

$$\mathrm{C}_l = \left\{ M \in \mathrm{L}_{4,1} \mid \overrightarrow{O_5 M}^2 = 0 \right\} \tag{13}$$

Let P be a point in $\mathcal{E}_3$ and $\overrightarrow{P} = \overrightarrow{O_3 P}$ its position vector. Then, the representation of the point P is the point p or the position vector $\overrightarrow{p}$ given by

$$\overrightarrow{O_5 p} = \overrightarrow{p} = \overrightarrow{e_o} + \overrightarrow{P} + \frac{1}{2}\left\|\overrightarrow{P}\right\|^2 \overrightarrow{e_\infty}. \tag{14}$$

It can be noticed that the point P defined by

$$\overrightarrow{O_5 P} = \overrightarrow{e_o} + \overrightarrow{P}$$

belongs to the embedding of $\mathcal{E}_3$ in $\mathrm{L}_{4,1}$. After some calculations, it yields

$$\overrightarrow{p} \cdot \overrightarrow{p} = 0 \tag{15}$$

thus $\overrightarrow{p}$ representing the point P in $\mathcal{E}_3$ lays on the light cone C_l. In fact, the set of these points p defines a 3-dimensional paraboloid P on the hyperplane defines by the point O_3 and the vectors $\overrightarrow{e_1}$, $\overrightarrow{e_2}$, $\overrightarrow{e_3}$, $\overrightarrow{e_\infty}$. The axis of P is the line defined by O_3 and the light-like vector $\overrightarrow{e_\infty}$ and then, the paraboloid P is isometric to $\mathcal{E}_3$. Conversely, the light-like vector $\overrightarrow{p}\,(x_o; x; y; z; x_\infty)$ represents the point $P\left(\dfrac{x}{x_o}; \dfrac{y}{x_o}; \dfrac{z}{x_o}\right)$ of $\mathcal{E}_3$ if $x_0 \neq 0$ or the point at infinity of $\mathcal{E}_3$ (i.e. $\overrightarrow{p} = \overrightarrow{e_\infty}$ if $x_0 = 0$. See in Figure 3.

In the rest of this paper, M designates a point of $\mathcal{E}_3$. Its representation in the Minkowski-Lorentz space is denoted m or $\overrightarrow{m}$ through misuse of language.

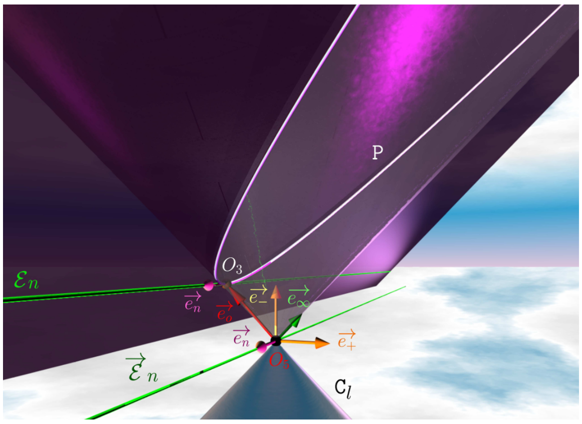
4.1 The space of spheres Λ^4

In the Minkowski-Lorentz space, the representation of the sphere with centre C and algebraic radius ρ in $\mathcal{E}_3$ is given by [36]:

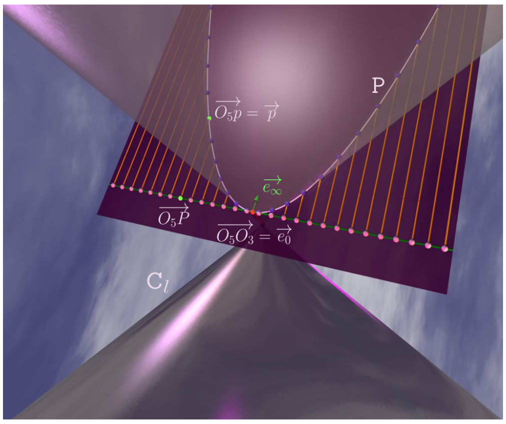
$$\begin{array}{rcl} \overrightarrow{\sigma} & = & \dfrac{1}{\rho}\left(\overrightarrow{c} - \dfrac{1}{2}\rho^2\overrightarrow{e_\infty}\right) \\ & = & \dfrac{1}{\rho}\left(\overrightarrow{e_o} + \overrightarrow{C} + \dfrac{1}{2}\left(\left\|\overrightarrow{C}\right\|^2 - \rho^2\right)\overrightarrow{e_\infty}\right) \end{array} \tag{16}$$

In the same way, the oriented plane $\mathcal{P}$ is defined by the point P and the unit normal vector $\overrightarrow{N}$ by

$$\overrightarrow{\pi} = \overrightarrow{N} + \left(\overrightarrow{N} \cdot \overrightarrow{P}\right)\overrightarrow{e_\infty}$$



(a)



(b)

Figure 3: Construction of the 3-dimensional paraboloid P isometric to the usual 3-dimensional Euclidean affine space $\mathcal{E}_3$. The type of $\overrightarrow{e_\infty}$, $\overrightarrow{e_o}$, $\overrightarrow{\mathcal{H}}$. The hyperplane $\overrightarrow{\mathcal{H}}$ is tangent to C_l. Each light-like vector $\overrightarrow{n_2}$, such as $\overrightarrow{n_2}$ and $\overrightarrow{e_\infty}$ are not collinear, defines a point of $\mathcal{E}_3$ via the paraboloid P: the point is the intersection between the line $(O;\overrightarrow{n_2})$ and the paraboloid P.

and $\overrightarrow{\pi}^2 = \overrightarrow{N}^2 = 1$. Thus, the unit sphere[2] of $\mathrm{L}_{4,1}$

$$\Lambda^4 = \left\{ \sigma \in \mathrm{L}_{4,1} \mid \overrightarrow{O_5\sigma}^2 = 1 \right\} \tag{17}$$

is called the space of spheres, representing the oriented spheres and the oriented planes of $\mathcal{E}_3$. The planes can be seen as spheres through the point at infinity of $\mathcal{E}_3$.

4.2 Relative positions of two spheres

Theorem 2 : Relative positions of two spheres

Let S and S_x be two oriented spheres in $\mathcal{E}_3$. Let σ and σ_x be their representations on Λ^4.

[2]Seen as a 4-dimensional one-sheeted hyperboloid.

Then, we have three cases:

- $\mathrm{S} \cap \mathrm{S}_x$ is a circle if and only if
$$\left| \overrightarrow{O_5\sigma} \cdot \overrightarrow{O_5\sigma_x} \right| < 1$$
- S and S_x are tangent if and only if
$$\left| \overrightarrow{O_5\sigma} \cdot \overrightarrow{O_5\sigma_x} \right| = 1$$
- $\mathrm{S} \cap \mathrm{S}_x = \emptyset$ if and only if
$$\left| \overrightarrow{O_5\sigma} \cdot \overrightarrow{O_5\sigma_x} \right| > 1$$

<u>Proof:</u> see [39].
■
Moreover, we can state

Corollary 1 : Let S and S_x be two tangent spheres in $\mathcal{E}_3$. Let σ and σ_x be their representations on Λ^4. If the orientations of the spheres S and S_x are the same, then

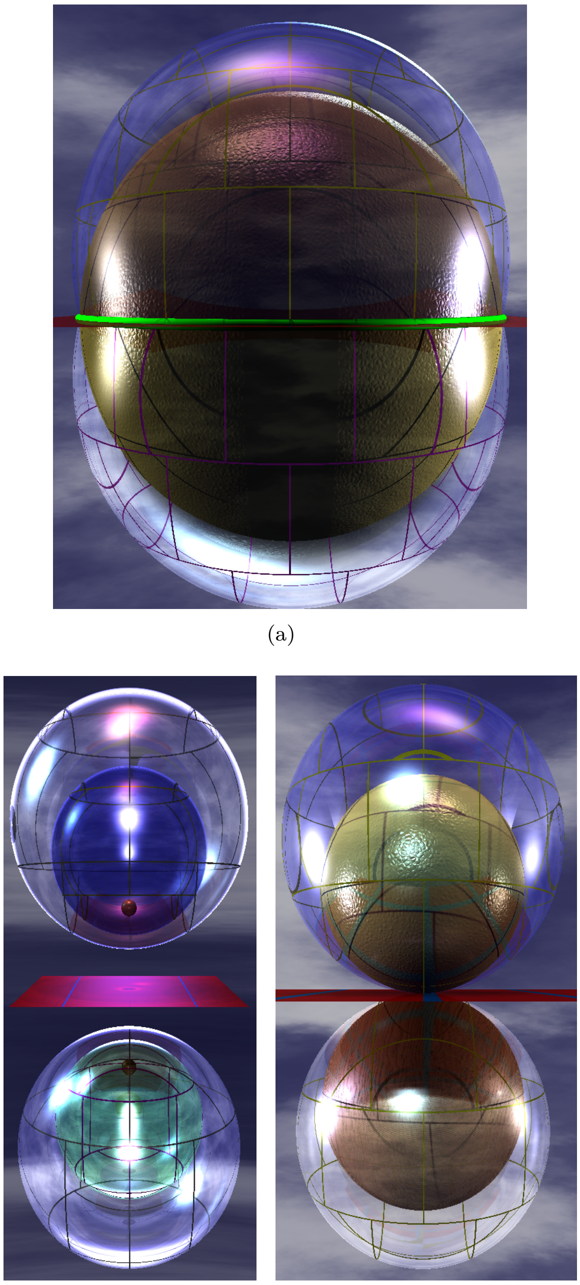
- $\overrightarrow{O_5\sigma} \cdot \overrightarrow{O_5\sigma_x} = 1$;
- $\overrightarrow{\sigma\sigma_x}^2 = 0$ and the point of tangency is defined by the light-like vector $\overrightarrow{\sigma\sigma_x}$ via the paraboloid.

4.3 Linear pencils of spheres and their correspondence in Λ^4

In [31], the authors recall that any linear pencil of spheres of $\mathcal{E}_3$ is represented in $\mathrm{L}_{4,1}$ by the section of the quadric Λ^4 by an affine 2-plane P passing through O_5. Given the type of the plane $\overrightarrow{P}$, we have different types of pencil of spheres.

Proposition 1 :

- The section of the quadric Λ^4 by a space-like plane P matches a pencil of spheres with a base circle i.e. all the spheres of the pencil have a common circle, Figure 4(a).
- The section of the quadric Λ^4 by a time-like plane P matches a Poncelet pencil, i.e. the balls, defined by the spheres, are contained in each other and tend to two limit points, Figures 4(b) and 5(a).
- The section of the quadric Λ^4 by a light-like plane P matches a pencil of tangent spheres in a point is the union of two light-like lines, Figures 4(c) and 5(b).



(a)

(b) (c)

Figure 4: Three pencils of spheres in $\mathcal{E}_3$. (a): A pencil of spheres with a base circle. (b): A Poncelet pencil. (c): A pencil of tangent spheres at a point.

5 Application in the space of spheres

5.1 Equation of a pencil of spheres with limit points

Consider two points as spheres with zeros radices belonging to the spheres pencil with limit points. A parameterisation of the pencil is given on Λ^4. In a first case an equation of a hyperbola in the frame formed by its two asymptotic lines is given (e.g. $y = \frac{1}{x}$). In a second case, an equation of the hyperbola with respect to its half axes is obtained.

Proposition 2 : Consider the two limit points M_1 and M_2 of a pencil of spheres $\mathrm{S}(t)$.

Let $\overrightarrow{m_1}$ and $\overrightarrow{m_2}$ the points representation in the Lorentz space.

Let $\varepsilon \in \{-1; 1\}$.

Any pencil of spheres gets two equivalent parametric representations

$$\overrightarrow{\sigma}(t) = t\,\frac{1}{M_2M_1}\,\overrightarrow{m_1} - \frac{1}{t}\,\frac{1}{M_2M_1}\,\overrightarrow{m_2} \tag{18}$$

with $t \in \mathbb{R}^*$ and

$$\begin{aligned}\overrightarrow{\sigma_{hyp}}(t) = {}& \varepsilon\ \mathrm{ch}(t)\ \frac{1}{M_2M_1}\ \left(\overrightarrow{m_1} - \overrightarrow{m_2}\right) \\ &+ \ \mathrm{sh}(t)\ \frac{1}{M_2M_1}\ \left(\overrightarrow{m_1} + \overrightarrow{m_2}\right)\end{aligned} \tag{19}$$

with $t \in \mathbb{R}$

Proof: left to the reader.

■

Consider to distinct points M_0 et M_2 with $\overrightarrow{m_0}$ et $\overrightarrow{m_2}$ as respective representations by two light vectors. In the spheres space, this type of pencil is represented by a circle arc of centre O_5 by the use of a BR curve with intermediate control mass point $(O_5; \omega_1)$ and $(\overrightarrow{m_0}; 0)$ and $(-\overrightarrow{m_2}; 0)$ as edge points. Let ω_1 be the O_5 weight. For $t \in \,]0; 1[$, any point of the Bézier curve is defined by the following equation

$$\begin{aligned}\overrightarrow{O_5\sigma(t)} &= \frac{1}{\omega_1\,B_1(t)}\left(B_0(t)\,\overrightarrow{m_0} - B_2(t)\,\overrightarrow{m_2}\right) \\ &+ \frac{1}{\omega_1\,B_1(t)}\left(\omega_1\,B_1(t)\,\overrightarrow{O_5O_5}\right) \\ &= \frac{1}{\omega_1\,B_1(t)}\left(B_0(t)\,\overrightarrow{m_0} - B_2(t)\,\overrightarrow{m_2}\right)\end{aligned}$$

The weight ω_1 is determined by

$$\overrightarrow{O_5\sigma(t)}^2 = 1 \tag{20}$$

that is equivalent to

$$\omega_1^2\,B_1^2(t)\ \overrightarrow{O_5\sigma(t)}^2 = \left(B_0(t)\,\overrightarrow{m_0} - B_2(t)\,\overrightarrow{m_2}\right)^2$$

Developping and simplifying the right hand side gives

$$\begin{aligned}\left(B_0(t)\,\overrightarrow{m_0} - B_2(t)\,\overrightarrow{m_2}\right)^2 &= -2\,B_0(t)\ B_2(t)\ \overrightarrow{m_0} \centerdot \overrightarrow{m_2} \\ &= -\frac{B_1^2(t)}{2}\,\overrightarrow{m_0} \centerdot \overrightarrow{m_2}\end{aligned}$$

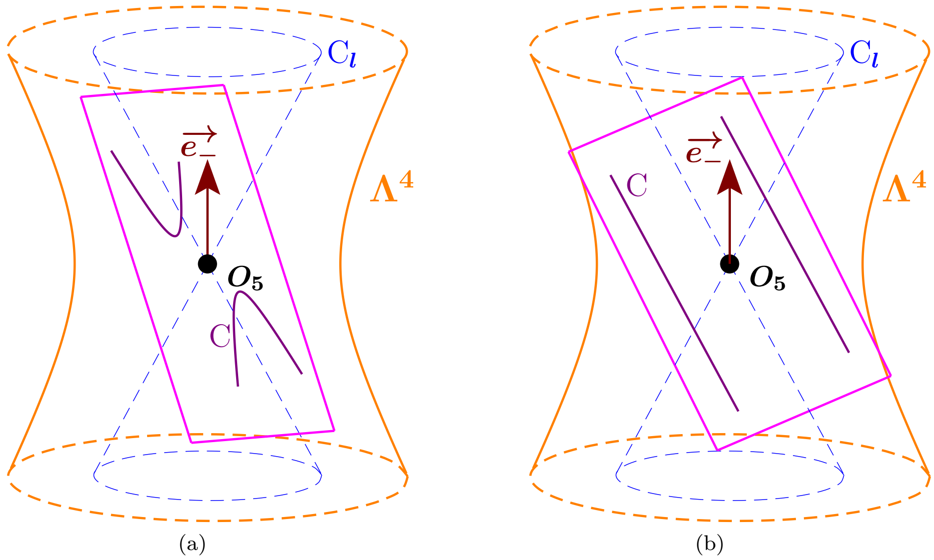


Figure 5: Representation of two pencils of spheres in Λ^4, each pencil is the section of Λ^4 by a 2-plane. (a): a Poncelet pencil and the type of the 2-plane is time-like. (b): A pencil of tangent spheres at a point and the type of the 2-plane is light-like.

thus the Formula (20) is equivalent to

$$\omega_1^2 = -\frac{1}{2}\overrightarrow{m_0} \cdot \overrightarrow{m_2} = \frac{1}{4}\overrightarrow{M_2M_0}^2$$

For any non zero real λ, it yields

$$(\lambda\,\overrightarrow{m_0})^2 = (\lambda\,\overrightarrow{m_2})^2 = \overrightarrow{m_0}^2 = \overrightarrow{m_2}^2 = 0$$

the condition $\overrightarrow{m_0}^2 = \overrightarrow{m_2}^2$ given in [17, 19] turns non valid and is formulated as follows

Condition 1 :
if a BR curve represents a circle arc drawn as an hyperbola, the edge mass points are two light vectors $\overrightarrow{m_0}$ and $\overrightarrow{m_2}$ representing the two points M_0 and M_2: the first component of $\overrightarrow{m_0}$ and $\overrightarrow{m_2}$ equals 1.

Consider the two points $M_0\,(-2;0;0)$ and $M_2\,(2;0;0)$. The weight ω_1 is chosen equal to 2. In order to obtain a weight equals to 1 for O_5, it is sufficient to replace both light vectors $\overrightarrow{m_0}$ and $\overrightarrow{m_2}$ by the new both light vectors

$$\overrightarrow{W_0} = \frac{1}{2}\overrightarrow{m_0} \qquad \text{et} \qquad \overrightarrow{W_2} = -\frac{1}{2}\overrightarrow{m_2}$$

given that $\overrightarrow{O_3M_2} = -\overrightarrow{O_3M_0}$ and $O_3M_0 = 2$, it yields finally

$$\overrightarrow{O_5\sigma\,(t)} = \frac{1}{2\,t\,(1-t)}\left(\frac{1-2\,t}{2}\overrightarrow{e_0} + (1-2\,t)\,\overrightarrow{e_\infty}\right) + \frac{1}{2\,t\,(1-t)}\,\frac{1-2\,t+2\,t^2}{2}\overrightarrow{O_3M_1}$$

The Figure 6 shows some spheres. Let us recall that for $t=0$ (resp. $t=1$), the vector $\overrightarrow{W_0}$ (resp. $\overrightarrow{W_2}$) defines the point M_0 (resp. M_2).

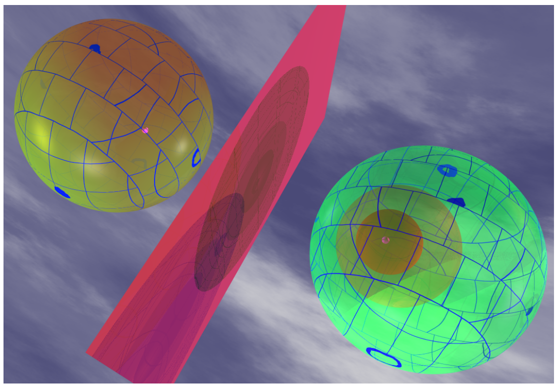


Figure 6: Some spheres of a spheres pencil with limit points modeled by a connected circle arc represented by a BR curve with control mass points.

5.2 Canal surface on the space of sphere Λ^4

The circular cylinders and the circular cones are the well known canal surfaces of degree 2. Moreover, these surfaces are defined by two equivalent families of spheres. The first type uses oriented spheres as the second is based on planes. The surface canal envelopes includes curvature circles for the first type and generatrices for the second type of the quadric surfaces. In the case of a cone, its vertex -a zero radius sphere- splits the cone in two parts. A first half cone is composed of spheres with positive radices. The negative radices are for the second half cone.

A Dupin cyclide can be defined as the image by an inversion mapping of a torus, a circular cylinder or a circular cone. The three last surfaces can be considered as degenerated cyclides [40]. Any Dupin cyclide can be defined as the envelope of a one-parameter family of spheres in two equivalent ways. A Dupin cyclide is determined by the section of Λ^4 with a affine 2-plane which does not contain O_5. If it is a time-like type plane, the Dupin cyclide gets two singular points. Its representation on Λ^4 is a circle drawn as a hyperbola [41, 37].

A circular cone is a degenerated Dupin cyclide [40] including its vertex S and the point at infinity of $\mathcal{E}_3$. These two points are represented in the Lorentz space by two light vectors, $\overrightarrow{m_S}$ and $\overrightarrow{e_\infty}$. The vector $\overrightarrow{e_\infty}$ is a shared vector to all cones representations on Λ^4. Thus a cone is totally determined by its vertex and a sphere. The Algorithm 1 offers a way to model a cone by a BR curve with control mass points coming from its vertex and a sphere.

The assumption $\chi_{\mathrm{S}_1}(S) > 0$, Formula (9), guarantees the point is outside (resp. inside) of the ball defined by the sphere S_1 if its radius is positive (resp. negative). Depending of the orientation of S_1, this bounded set means either the outside or the inside of S_1. The centre of the circle Ω which is the section of Λ^4 by the plane $\mathcal{P}_1$ generated by σ_1, $\overrightarrow{m_S}$ and $\overrightarrow{e_\infty}$, is given by

$$\overrightarrow{\sigma_1\Omega} = \alpha\,\overrightarrow{m_S} + \beta\,\overrightarrow{e_\infty}$$

where α et β must be calculated. The point Ω is the orthogonal projection[3] O_5 on $\mathcal{P}_1$ iff

$$\left\{ \begin{array}{lcl} \overrightarrow{O_5\Omega}\cdot\overrightarrow{m_S} & = & 0 \\ \overrightarrow{O_5\Omega}\cdot\overrightarrow{e_\infty} & = & 0 \end{array} \right. \tag{23}$$

Moreover, α and β are calculated from

$$\overrightarrow{O_5\Omega} = \overrightarrow{O_5\sigma_1} + \alpha\,\overrightarrow{m_S} + \beta\,\overrightarrow{e_\infty}$$

and the System (23) equivalent to the system

$$\left\{ \begin{array}{lcl} \overrightarrow{O_5\sigma_1}\cdot\overrightarrow{m_S} + \beta\,\overrightarrow{e_\infty}\cdot\overrightarrow{m_S} & = & 0 \\ \overrightarrow{O_5\sigma_1}\cdot\overrightarrow{e_\infty} + \alpha\,\overrightarrow{m_S}\cdot\overrightarrow{e_\infty} & = & 0 \end{array} \right.$$

That provides the Formula (21). The BR curve with control mass points $(\overrightarrow{m_0};0)$, $(\Omega;\omega_1)$ and $(\overrightarrow{m_2};0)$, is defined by

[3] for the Lorentz quadratic form

Algorithm 1 Sketch of a circular cone from a point and a sphere.

Input : A point S and a sphere S_1 such that $\chi_{\mathrm{S}_1}(S) > 0$

1. Determine the light vector $\overrightarrow{m_S}$ that represents the vertex S
2. Determine σ_1 that represents the sphere S_1 on Λ^4
3. Determine Ω the centre of the circle that represents the cone by :
$$\overrightarrow{\sigma_1\Omega} = -\frac{\overrightarrow{O_5\sigma_1}\cdot\overrightarrow{e_\infty}}{\overrightarrow{m_S}\cdot\overrightarrow{e_\infty}}\overrightarrow{m_S} - \frac{\overrightarrow{O_5\sigma_1}\cdot\overrightarrow{m_S}}{\overrightarrow{m_S}\cdot\overrightarrow{e_\infty}}\overrightarrow{e_\infty} \tag{21}$$
4. Determine the weight ω_1 root of
$$\omega_1^2 = \frac{1}{2}\frac{\overrightarrow{m_S}\cdot\overrightarrow{e_\infty}}{1-\overrightarrow{O_5\Omega}^2} \tag{22}$$
5. Determine the BR curve γ with mass control points $(\overrightarrow{m_S};0)$, $(\Omega;\omega_1)$ and $(\overrightarrow{e_\infty};0)$ that model a circle arc.

Output : A one parameter set that models a cone itself represented by a BR curve with mass control points $(\overrightarrow{m_S};0)$, $(\Omega;\omega_1)$ and $(\overrightarrow{e_\infty};0)$.

$$\begin{array}{rl} \overrightarrow{O_5\gamma(t)} = & \dfrac{1}{\omega_1 B_1(t)}\left(B_0(t)\,\overrightarrow{m_S} + B_2(t)\,\overrightarrow{e_\infty}\right) \\ + & \dfrac{1}{\omega_1 B_1(t)}\omega_1 B_1(t)\,\overrightarrow{O_5\Omega} \end{array} \tag{24}$$

and for $t=\frac{1}{2}$, it yields

$$\overrightarrow{O_5\gamma\left(\frac{1}{2}\right)} = \frac{2}{\omega_1}\left(\frac{1}{4}\,\overrightarrow{m_S} + \frac{1}{2}\,\omega_1\,\overrightarrow{O_5\Omega} + \frac{1}{4}\,\overrightarrow{e_\infty}\right) \tag{25}$$

reduced into

$$\overrightarrow{O_5\gamma\left(\frac{1}{2}\right)} = \frac{1}{2\,\omega_1}\left(\overrightarrow{m_S} + 2\,\omega_1\,\overrightarrow{O_5\Omega} + \overrightarrow{e_\infty}\right) \tag{26}$$

The point $\gamma\left(\frac{1}{2}\right)$ belongs to Λ^4 iff

$$\overrightarrow{O_5\gamma\left(\frac{1}{2}\right)}^2 = 1$$

iff

$$4\,\omega_1^2 = \left(\overrightarrow{m_S} + 2\,\omega_1\,\overrightarrow{O_5\Omega} + \overrightarrow{e_\infty}\right)^2$$

that is equivalent to

$$4\,\omega_1^2 = 4\,\omega_1^2\,\overrightarrow{O_5\Omega}^2 + 2\overrightarrow{m_S}\centerdot\overrightarrow{e_\infty}$$

that proves ω_1 is a solution of the equation (22). Any solution of ω_1 provides half a cone with vertex S given that $\overrightarrow{\gamma(0)} = \overrightarrow{m_S}$.

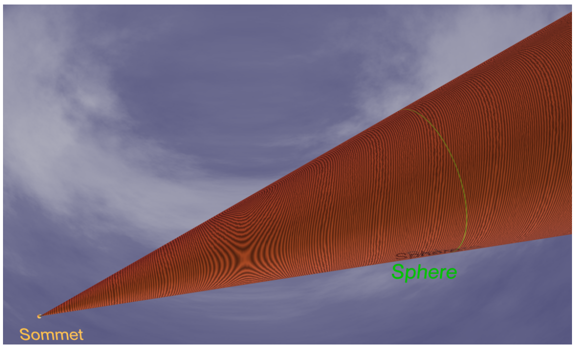


Figure 7: A cone defined by its vertex S and the sphere S_1 shown on Λ^4 by a circle arc modelled by a BR curve with mass control points.

The Figures 7 and 8 show half a cone known by its vertex $S\,(-5;-1;-1)$, its point at infinity and the sphere S_1- green colored- with centre $O_1\,(5;1;2)$ and radius 2. In the Lorentz space, the vertex is represented by the light-like vector $\overrightarrow{m_s}\left(1;-5;-1;-1;\frac{27}{2}\right)$. Keeping in mind that the first coordinates must be equal to 1. The sphere S_1 is represented by the point $\sigma_1\left(\frac{1}{2};\frac{5}{2};\frac{1}{2};1;\frac{13}{2}\right)$. the cone is represented by the circle with centre $\Omega\left(0;5;1;\frac{3}{2};-\frac{55}{2}\right)$. The weight equals $\omega_1 = \frac{\sqrt{218}}{109}$. An inversion maps a spindle or horned Dupin cy-

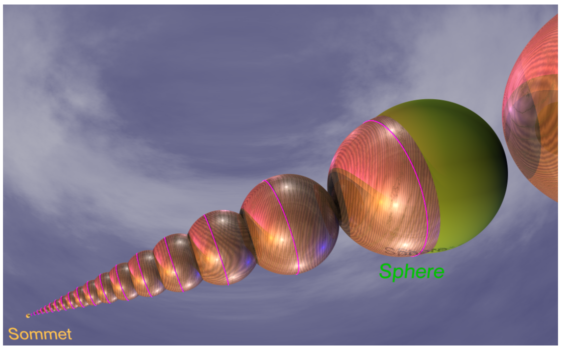


Figure 8: Few spheres and characteristic circles of the cone in the figure 7,obtained for some t values of the BR curve

clide into a horn torus or a circular cone. In case of the pole inversion does not belong to the initial circle, a quartic Dupin cyclide is obtained, otherwise a cubic Dupin cyclide is obtained. In both cases, such a Dupin cyclide is the envelop of a one-parameter family of spheres in two ways. The Dupin cyclide is thus sketched in the sphere space by two circles. One circle is drawn as an ellipse while the second is drawn as an hyperbola [41]. A cubic cyclide is a non bounded surface. It takes two lines of curvature. The following applies the quartic Dupin cyclides bacause they are bounded and the most often used.

The algorithm in this section does not distinguish horned or spindle Dupin cyclide [40], circular cone or horn torus. If the vector $\overrightarrow{e_\infty}$ is replaced by a light-like vector $\overrightarrow{m_\infty}$, the Algorithm 1 provides a spindle or horned Dupin Cyclide. Like the union of the paraboloid with $\overrightarrow{e_\infty}$ is a Alexandroff compactification of $\mathcal{E}_3$, it is possible to send the point represented by $\overrightarrow{m_\infty}$ and then, the Dupin cyclide becomes a circular cone. The reader can find an animation using this link : http://ufrsciencestech.u-bourgogne.fr/~garnier/publications/refigCD4EN2Cone26.gif .

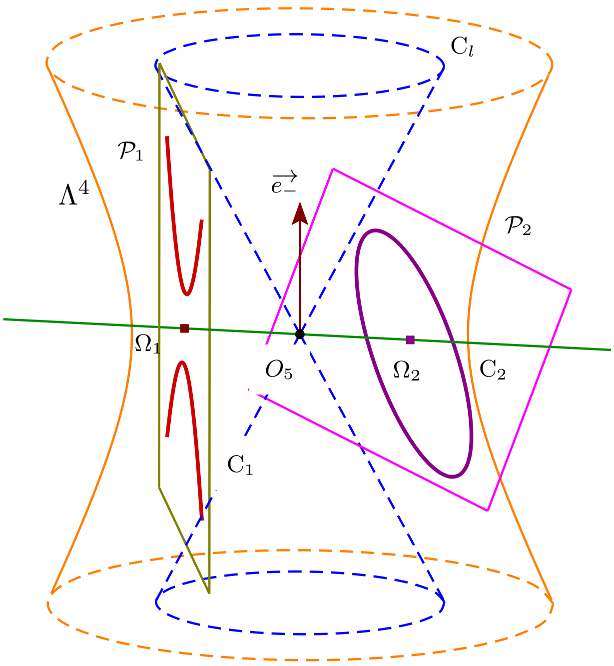


Figure 9: A Dupin cyclide on Λ^4 with two singularities sketched as a connected circle drawn as an hyperbola and a non connected circle drawn as an ellipse.

The Figure 10 shows characteristic circles on a horned Dupin cyclide while changing the light-like vector inside the algorithm 1. The point M_1 and the sphere belong to Figure 7, the vector $\overrightarrow{e_\infty}$ has been substituted by the vector $\overrightarrow{m_2}\,(1;-5;-5;10;75)$ that represents the point

$M_2\,(-5;-5;10)$ and

$$\omega_1 = 137\,\frac{\sqrt{109}}{1526}$$

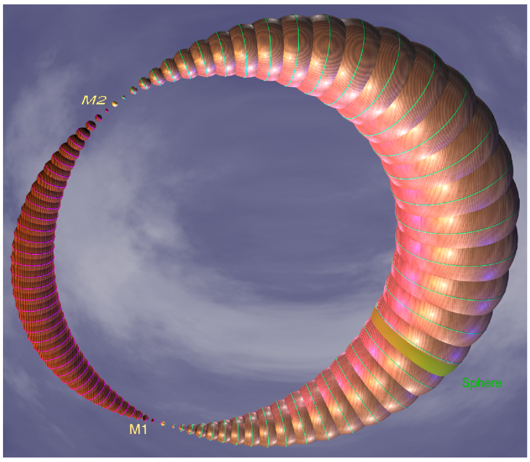


Figure 10: Characteristic circles of a horned Dupin cyclide.

The figure 11 shows the whole cyclide with a non zero external crescent.

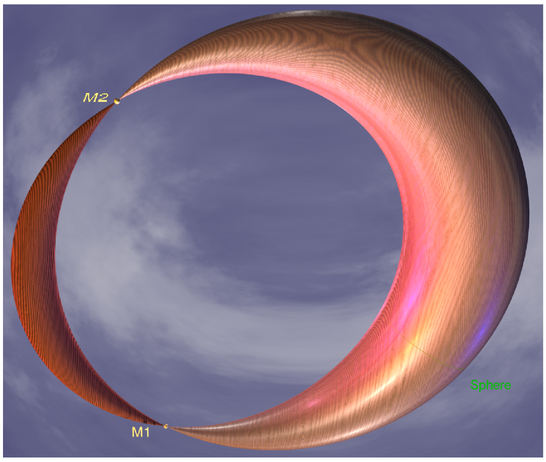


Figure 11: A horned Dupin cyclide.

6 Computation of the foci and the directrices of a conic using and Quételet theorems

A conic is defined as the section of a cone by a plane which does not contain the apex of the cone. The Algorithm 2 determines the foci and the directrices of such a conic, using the geometric characterisation of Dandelin spheres [42, 2].

6.1 Characterisation of the foci and the directrices based on Dandelin spheres

Given a conic defined as the section of a cone and a plane $\mathcal{P}$, there exists one or two spheres called Dandelin spheres both tangent at the cone and at the plane $\mathcal{P}$. A Dandelin sphere is tangent at the plane $\mathcal{P}$ at a point F_i, and tangent at the cone in a circle on a plane $\mathcal{P}_i$. Then, the point F_i is a focus of the conic, and the corresponding directrix is defined as the intersection of the planes $\mathcal{P}$ and $\mathcal{P}_i$, as illustrated in Figure 12.

Each Dandelin sphere defines a focus. An ellipse and a hyperbola have two Dandelin spheres, tangent at the same nappe of the cone for the conic, tangent at the two nappes of the cone for the hyperbola. For a parabola, there exists only one Dandelin sphere, and therefore one focus point and a single directrix. Formally, the second focus is the point at infinity of $\mathcal{E}_3$.

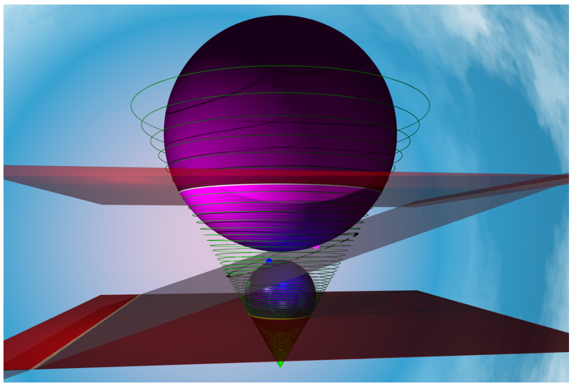


Figure 12: An ellipse and the corresponding Dandelin spheres : the blue points are the foci and the directrices are the yellow lines, intersection of the planes.

6.2 Determining the Dandelin spheres

The Dandelin spheres belong to the family of spheres whose envelop is the cone. First, note that, for an arbitrary vector $\overrightarrow{m}$, its first coordinate equals 1 iff

$$\overrightarrow{m} \cdot \overrightarrow{e_\infty} = -1$$

The Algorithm 2 requires the calculation of only a root of a quadratic equation coming from the Lorentz dot product given that the other equals 1 according to the point at infinity of the intersection of the plane and the cone. The Algorithm 2 determines the Dandelin spheres of a conic defined as the intersection of a cone and a plane. Indeed, each orientation of the plane $\mathcal{P}$ leads to determine one of the two Dandelin spheres.

Algorithm 2 Computing Dandelin spheres from a cone and a plane.

Input : A BR curve $t \mapsto \sigma(t)$ representing a cone on the space of spheres

1. Fix a plane $\mathcal{P}$ that does not contain the apex of the cone.
2. Compute π, modeling the plane $\mathcal{P}$ in Λ^4.
3. Compute π^-, symmetric of sπ relatively to O_5.
4. Compute t_1 in $\overline{\mathbb{R}}$, solution of the equation: :
$$\overrightarrow{O_5\sigma(t)} \centerdot \overrightarrow{O_5\pi} = 1 \tag{27}$$
5. Compute t_{-1} in $\overline{\mathbb{R}}$, solution of the equation:
$$\overrightarrow{O_5\sigma(t)} \centerdot \overrightarrow{O_5\pi} = -1 \tag{28}$$
6. If $t_1 = 1$ or $t_{-1} = 1$ then: // one of the foci is at infinity

then if $t_1 = 1$

then F_{-1} is $\overrightarrow{\sigma(t_1)\,\pi^-}$

else F_1 is $\overrightarrow{\sigma(t_0)\,\pi}$

End if

else F_1 is $\overrightarrow{\sigma(t_0)\,\pi}$

F_{-1} isr $\overrightarrow{\sigma(t_1)\,\pi^-}$

End if

Output: One or two Dandelin sphere(s), and the foci of the conic defined by the cone and the chosen plane.

When the conic is an ellipse, it is sufficient to consider only a nappe of the cone, and therefore the BR curve can be taken on the interval $[0, 1]$. For the hyperplane, the domain $\overline{\mathbb{R}}$ is considered in order to model the two nappes of the cone. Alternatively, we could have considered two different BR curves on $[0, 1]$.

Equations (27) and (28) model the fact that the plane $\mathcal{P}$ and the spheres S_0 and S_1 represented by $\sigma(t_0)$ and $\sigma(t_1)$ are tangent.

As the focus is characterised as the contact point between the plane and one of the spheres, they are modeled by the light vectors $\overrightarrow{\sigma(t_0)\,\pi}$ et $\overrightarrow{\sigma(t_1)\,\pi^-}$.

Note that equations (27) and (28) can be rewritten as linear equations:

Lemma 1 : Simplification of Equations (27) and (28)

Let $\varepsilon \in \{-1; 1\}$, and let $\overrightarrow{N}$ the unit normal vector of the oriented plane $\mathcal{P}$ and P_0 a point on this plane.

The solutions of equations (27) and (28) are:

$$t_\varepsilon = \frac{\overrightarrow{N} \bullet \overrightarrow{P_0S}}{\overrightarrow{N} \bullet \overrightarrow{P_0S} - 2\,\omega_1\, \overrightarrow{O_5\Omega} \centerdot \overrightarrow{N} + 2\omega_1\,\varepsilon}$$

with

$$\overrightarrow{O_5\Omega} \centerdot \overrightarrow{N} = \overrightarrow{O_5\Omega} \centerdot \left(0\,\overrightarrow{e_0} + \overrightarrow{N} + 0\,\overrightarrow{e_\infty}\right)$$

Proof: See Appendix A
■

If one of the solutions t_1 or t_{-1} equals 1, a parabola is obtained and one of the foci is the point at infinity represented in $\mathcal{E}_3$ by the vector $\overrightarrow{\sigma(1)} = \overrightarrow{e_\infty}$. Taking the control point $(\overrightarrow{e_\infty}; 0)$ simplifies (27) and (28) into linear equations instead of quadratic equations.

6.3 Three numerical examples

In the three following examples shown in Figures 12, 13 and 14, the cone $\mathscr{C}$ is characterised by its apex $S = O_3$ and the sphere of centre $O\,(0; 0; 2)$ and radius $\rho = 2$ represented in Λ^4 by σ_O. Numerical results are given in B.

In the following, the computation of the foci and the computation of the directrices are developed.

6.4 Computing the directrices of the conic

Let us consider S_1, one of the Dandelin spheres represented by $\overrightarrow{\sigma_1}$. The characteristic circle corresponding to S_1 is the intersection of the spheres σ_1 and $\dot{\sigma}_1$ where $\dot{\sigma}_1$ is the derivative sphere of σ_1 and it yields

$$\left\{\begin{array}{lcl} \overrightarrow{\sigma_1} & = & \dfrac{1}{\rho_1}\left(\overrightarrow{e_o} + \overrightarrow{\Omega_1} + \dfrac{1}{2}\left(\left\|\overrightarrow{\Omega_1}\right\|^2 - \rho_1^2\right)\overrightarrow{e_\infty}\right) \\ \overrightarrow{\dot{\sigma}_1} & = & \dfrac{1}{\dot{\rho}_1}\left(\overrightarrow{e_o} + \overrightarrow{\dot{\Omega}_1} + \dfrac{1}{2}\left(\left\|\overrightarrow{\dot{\Omega}_1}\right\|^2 - \dot{\rho}_1^{\,2}\right)\overrightarrow{e_\infty}\right) \end{array}\right.$$

where the center of S_1 (resp. $\dot{\mathrm{S}}_1$) is Ω_1 (resp. $\dot{\Omega}_1$) and its radius is ρ_1 (resp. $\dot{\rho}_1$). Note that $\dot{\Omega}_1$ is the cone apex.

These two spheres define a pencil of spheres containing the circle which is the intersection

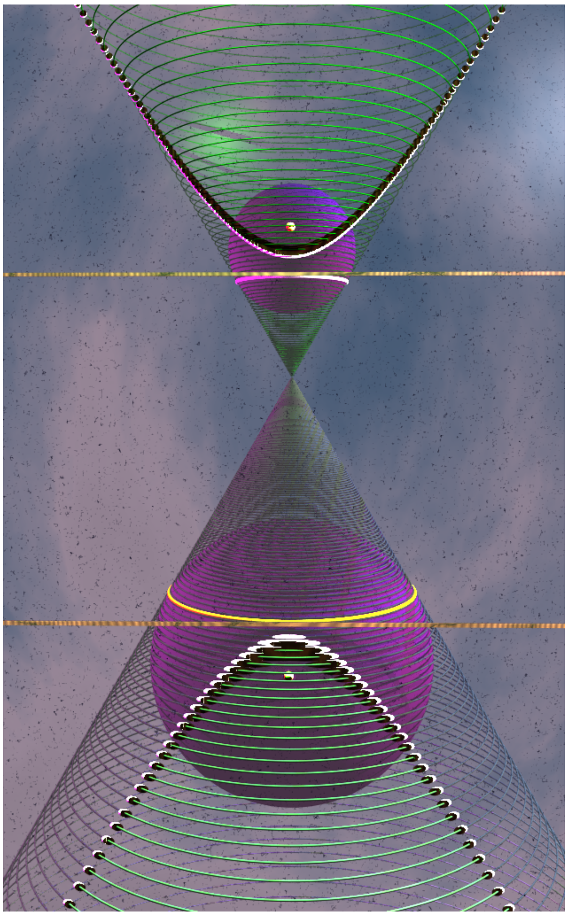


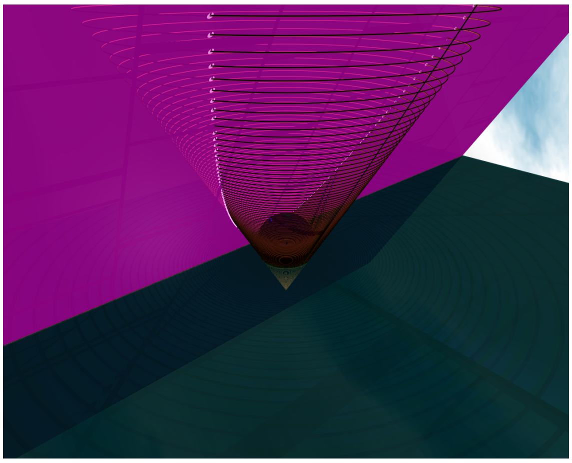
Figure 13: A hyperbola and its two Dandelin spheres. The foci of the hyperbola are the two yellow points. The directrices of the hyperbola are the two golden lines.

between the sphere S_1 and its derivative sphere $\dot{S}_1$. Let $\mathcal{P}_1$, be an oriented plane of this family, represented by $\overrightarrow{\pi_1}$. The vector $\overrightarrow{\pi_1}$ is a linear combination of $\overrightarrow{\sigma_1}$ and $\overrightarrow{\dot{\sigma}_1}$ and its coordinate in $\overrightarrow{e_o}$ vanishes. So

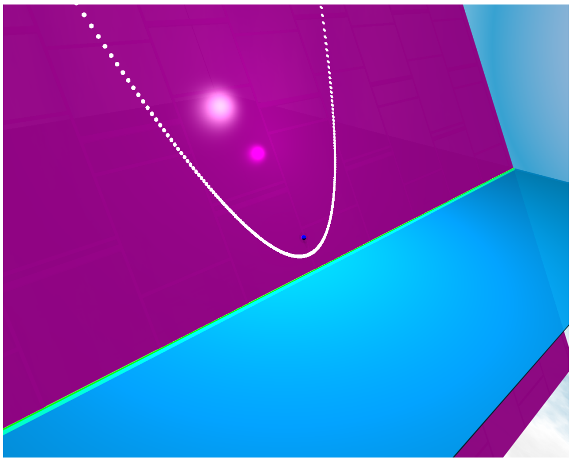
$$\overrightarrow{\pi_1} = \lambda \left(\frac{1}{\dot{\rho}_1} \overrightarrow{\sigma_1} - \frac{1}{\rho_1} \overrightarrow{\dot{\sigma}_1} \right) = \frac{\lambda}{\rho_1 \; \dot{\rho}_1} \overrightarrow{\dot{\Omega}_1 \; \Omega_1} + \frac{\lambda}{\rho_1 \; \dot{\rho}_1} \left(\frac{1}{2} \left(\left\| \overrightarrow{\Omega_1} \right\|^2 - \left\| \overrightarrow{\dot{\Omega}_1} \right\|^2 + \dot{\rho}_1^{\;2} - \rho_1^2 \right) \overrightarrow{e_\infty} \right)$$

where λ is given such that π_1 is on Λ^4, which leads:

$$\left(\frac{\lambda}{\rho_1 \; \dot{\rho}_1} \right)^2 \overrightarrow{\dot{\Omega}_1 \; \Omega_1}^2 = 1.$$



(a)



(b)

Figure 14: Dandelin sphere and a parabola. The foci are the blue point on both figures and the point at infinity. The directrix is the intersection of the two planes. (a): Characteristic circles of the cone nappe, the Dandelin sphere (in red), focus and directrix. (b): Different view of the same elements except the characteristic circles.

The real number λ is chosen equal to

$$\lambda = \frac{\rho_1 \; \dot{\rho}_1}{\sqrt{\overrightarrow{\dot{\Omega}_1 \; \Omega_1}^2}}.$$

without taking care of the plane orientation. Finally

$$\overrightarrow{\pi_1} = \frac{\rho_1 \; \dot{\rho}_1}{\sqrt{\overrightarrow{\dot{\Omega}_1 \; \Omega_1}^2}} \left(\frac{1}{\dot{\rho}_1} \overrightarrow{\sigma_1} - \frac{1}{\rho_1} \overrightarrow{\dot{\sigma}_1} \right)$$

that is,

$$\begin{aligned}\overrightarrow{\pi_1} = \; & \frac{1}{\sqrt{\overrightarrow{\dot{\Omega}_1\,\Omega_1}^2}}\overrightarrow{\dot{\Omega}_1\,\Omega_1} \\ + \; & \frac{1}{2\sqrt{\overrightarrow{\dot{\Omega}_1\,\Omega_1}^2}}\left(\left\|\overrightarrow{\Omega_1}\right\|^2 - \left\|\overrightarrow{\dot{\Omega}_1}\right\|^2\right)\overrightarrow{e_\infty} \\ + \; & \frac{1}{2\sqrt{\overrightarrow{\dot{\Omega}_1\,\Omega_1}^2}}\left(\dot{\rho}_1^{\,2} - \rho_1^2\right)\overrightarrow{e_\infty}\end{aligned}$$

The first directrix line is the intersection of the planes represented by π_1 et π. The second directrix line is obtained in the same way using the second Dandelin sphere (if not in the parabola setting).

7 Conclusion and prospect

This article shows the modelling with one degree of freedom of the families of spheres including two points by a rational quadratic Bézier curve with mass control points. A circle on the space of spheres, represented as a hyperbola leads to three cases: a cone, a horned or spindle Dupin cyclide, or pencils of spheres with limit point. The points in these families define the direction of the asymptote to the circle. Representing such a surface in 3D by a curve in 5D simplifies the computation of principal spheres of a Dupin cyclide, or the construction of a cone from its apex and a sphere. An example shows how this formalism simplifies the computation of Dandelin spheres.

In the space of spheres, this work is going on with the representation of the revolution cylinders and the singly horned Dupin cyclides or one-singularity spindle Dupin cyclides by the use of Bézier curves with mass control points.

Another idea gets up while the use of an inversion that transforms the space of spheres Λ^4 into a hyperplan, sketches a Bézier curve in that hyperplan then taking the same inversion to obtain a Bézier curve on the space of spheres that represents a canal surface. Such a canal surface is considered as a $\mathcal{C}^1$ curve on Λ^4 such that all the derivative vectors are space-like vectors and the geodesic vectors are time-like vectors [43].

A last problem must be tackled. That is to find the conditions for the curve laying on the hyperplane such the resulting curve in the space of spheres satisfies both conditions. This work will be extended to the surfaces with a curvilinear skeleton. Thus, from a finite number of spheres, a continuous skeleton can be obtained by the interpolation of points in the space of spheres easing the work with curves rather surfaces.

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A Proof of Lemma 1

A BR curve models a family of spheres whose envelop is the cone. A sphere of the rational BR curve is defined as

$$\begin{aligned}\overrightarrow{O_5\sigma\left(t\right)} &= \frac{1}{\omega_1 B_1\left(t\right)}\left(B_0\left(t\right)\overrightarrow{m_S}+\omega_1 B_1\left(t\right)\overrightarrow{O_5\Omega}\right)\\ &+ \frac{1}{\omega_1 B_1\left(t\right)}B_2\left(t\right)\overrightarrow{e_\infty}\end{aligned}$$

The plane is represented by:

$$\overrightarrow{O_5\pi}=\overrightarrow{N}+\left(\overrightarrow{N}\bullet\overrightarrow{P_0}\right)\overrightarrow{e_\infty}.$$

Then, equations (27) and (28) can be written:

$$\overrightarrow{O_5\sigma\left(t\right)}\,\boldsymbol{.}\,\overrightarrow{O_5\pi}=\varepsilon \tag{29}$$

In the product $\overrightarrow{O_5\sigma\left(t\right)}\,\boldsymbol{.}\,\overrightarrow{O_5\pi}$, the remaining terms are $\overrightarrow{e_0}\,\boldsymbol{.}\,\overrightarrow{e_\infty}$ and the remaining terms come from the 3D Euclidean scalar product defined on $\overrightarrow{e_1}$, $\overrightarrow{e_2}$ and $\overrightarrow{e_3}$.
Recall that:

$$\overrightarrow{m_S}=\overrightarrow{e_0}+\overrightarrow{S}+\frac{\overrightarrow{S}^2}{2}\overrightarrow{e_\infty}$$

and equation (29) leads to:

$$\begin{aligned}B_0\left(t\right)\left(\overrightarrow{N}\bullet\overrightarrow{P_0}\right)\overrightarrow{e_o}\,\boldsymbol{.}\,\overrightarrow{e_\infty}+B_0\left(t\right)\overrightarrow{S}\bullet\overrightarrow{N}\\ +\omega_1 B_1\left(t\right)\overrightarrow{O_5\Omega}\,\boldsymbol{.}\,\overrightarrow{N}=\varepsilon\,\omega_1 B_1\left(t\right)\end{aligned} \tag{30}$$

where:

$$\overrightarrow{O_5\Omega}\,\boldsymbol{.}\,\overrightarrow{N}=\overrightarrow{O_5\Omega}\,\boldsymbol{.}\left(0\,\overrightarrow{e_0}+\overrightarrow{N}+0\,\overrightarrow{e_\infty}\right)$$

Formula (30) holds iff:

$$\begin{aligned}-\left(1-t\right)\left(\overrightarrow{N}\bullet\overrightarrow{P_0}\right)+\left(1-t\right)\left(\overrightarrow{S}\bullet\overrightarrow{N}\right)+\\ 2\omega_1\overrightarrow{O_5\Omega}\,\boldsymbol{.}\,\overrightarrow{N}=2\,\varepsilon\,\omega_1\,t\end{aligned}$$

and

$$1-t=0$$

Since:

$$\overrightarrow{S}\bullet\overrightarrow{N}-\overrightarrow{N}\bullet\overrightarrow{P_0}=\overrightarrow{N}\bullet\left(\overrightarrow{O_3S}-\overrightarrow{O_3P_0}\right)=\overrightarrow{N}\bullet\overrightarrow{P_0S}$$

the solution of the first equation is:

$$t_\varepsilon=\frac{\overrightarrow{N}\bullet\overrightarrow{P_0S}}{\overrightarrow{N}\bullet\overrightarrow{P_0S}-2\,\omega_1\,\overrightarrow{O_5\Omega}\,\boldsymbol{.}\,\overrightarrow{N}+2\omega_1\,\varepsilon}$$

whereas the solution of the second equation is simply 1, that is, $\overrightarrow{e_\infty}$: this is expected since all planes of $\mathcal{E}_3$ contain the point at infinity of $\mathcal{E}_3$.
■

B Numerical examples

One nappe of the cone is modeled in Λ^4 by the BR curve of control (mass) points $(\overrightarrow{e_o};0)$, $\left(\Omega;\frac{\sqrt{6}}{6}\right)$ and $(\overrightarrow{e_\infty};0)$ where $\Omega\,(0;0;0;2;0)$ is the orthogonal projection of O_5 on the plane defined by σ_O, $\overrightarrow{e_o}$ and $\overrightarrow{e_\infty}$.

B.1 The ellipse case

Figure 12 show the ellipse intersection of the cone $\mathscr{C}$ with the plane containing $P_0\,(0;0;5)$ of unit normal vector $\overrightarrow{N}\left(\frac{\sqrt{10}}{10};0;3\,\frac{\sqrt{10}}{10}\right)$ which is represented in Λ^4 by $\pi\left(0;\frac{\sqrt{10}}{10};0;3\,\frac{\sqrt{10}}{10};15\,\frac{\sqrt{10}}{10}\right)$.

The solutions of equations (27) and (28) are respectively

$$\begin{aligned}t_1 &= \tfrac{-16200\,\sqrt{10}-56070\,\sqrt{6}+21810\,\sqrt{15}+93525}{334129}\\ &\simeq 0.866\end{aligned}$$

and

$$\begin{aligned}t_{-1} &= \tfrac{16200\,\sqrt{10}-56070\,\sqrt{6}-21810\,\sqrt{15}+93525}{334129}\\ &\simeq 0.667\end{aligned}$$

So

$$\sigma(t_1) = \left(\frac{6-\sqrt{10}}{15}; 0; 0; 2; \frac{270-45\sqrt{10}}{52} \right)$$

and

$$\sigma(t_{-1}) = \left(\frac{6+\sqrt{10}}{15}; 0; 0; 2; \frac{270-45\sqrt{10}}{52} \right).$$

The point $\sigma(t_1)$ characterises the sphere of centre

$$O_1 \left(0; 0; \frac{90+15\sqrt{10}}{13} \right)$$

and radius

$$r_1 = \frac{90+15\sqrt{10}}{26}.$$

The corresponding focus is the point

$$F_1 \left(-\frac{15+9\sqrt{10}}{26}; 0; \frac{135+3\sqrt{10}}{26} \right)$$

i.e.

$$F_1 \simeq (-1.672, 0, 5.557)$$

The point $\sigma(t_{-1})$ characterises the sphere of centre

$$O_{-1} \left(0; 0; \frac{90-15\sqrt{10}}{13} \right)$$

and radius

$$r_{-1} = \frac{90-15\sqrt{10}}{26}.$$

The corresponding focus is the point

$$F_{-1} \left(\frac{-15+9\sqrt{10}}{26}; 0; \frac{135-3\sqrt{10}}{26} \right)$$

i.e.

$$F_{-1} \simeq (0.518, 0, 4.827)$$

B.2 The hyperbola case

Figure 13 shows the hyperbola, intersection of $\mathscr{C}$ and the plane containing $P_0\left(\frac{1}{2}; 0; 3\right)$ of unit normal vector $\overrightarrow{N}\left(5\frac{\sqrt{26}}{26}; 0; \frac{\sqrt{26}}{26}\right)$ represented in Λ^4 by $\pi\left(0; 5\frac{\sqrt{26}}{26}; 0; \frac{\sqrt{26}}{26}; \frac{11\sqrt{26}}{52}\right)$.

Solutions of equations (27) and (28) are respectively

$$t_1 = \frac{96\sqrt{26}+196\sqrt{6}-68\sqrt{39}-369}{95} \simeq 1,852$$

and

$$t_{-1} = \frac{-96\sqrt{26}+196\sqrt{6}+68\sqrt{39}-369}{95} \simeq 0,487.$$

So

$$\sigma(t_1) = \left(\frac{4-2\sqrt{26}}{11}; 0; 0; 2; \frac{-6-3\sqrt{26}}{8} \right)$$

and

$$\sigma(t_{-1}) = \left(\frac{4+2\sqrt{26}}{11}; 0; 0; 2; \frac{-6+3\sqrt{26}}{8} \right)$$

The point $\sigma(t_1)$ defines the sphere of centre

$$O_1 \left(0; 0; -\frac{2+\sqrt{26}}{2} \right)$$

and radius

$$r_1 = -\frac{2+\sqrt{26}}{4}.$$

The corresponding focus point is

$$F_1 \left(\frac{65+5\sqrt{26}}{52}; 0; -\frac{39+25\sqrt{26}}{52} \right)$$

i.e.

$$F_1 \simeq (1.740, 0, -3.201)$$

The point $\sigma(t_{-1})$ defines the second sphere of centre

$$O_{-1} \left(0; 0; \frac{-2+\sqrt{26}}{2} \right)$$

and radius

$$r_{-1} = \frac{-2+\sqrt{26}}{4}.$$

The corresponding focus point is

$$F_{-1} \left(\frac{65-5\sqrt{26}}{52}; 0; \frac{-39+25\sqrt{26}}{52} \right)$$

i.e.

$$F_{-1} \simeq (0,760; 0; 1,702)$$

B.3 The parabola case

Figure 14 shows a setting where the conic is a parabola, intersection of $\mathscr{C}$ with the plane containing $P_0(0; 0; 5)$ of normal unit vector $\overrightarrow{N}\left(\frac{\sqrt{3}}{2}; 0; \frac{1}{2}\right)$ which representation in Λ^4 is $\pi\left(0; -\frac{\sqrt{3}}{2}; 0; \frac{1}{2}; \frac{5}{2}\right)$.

Solutions of equations (27) and (28) are respectively

$$t_1 = 1$$

and

$$t_{-1} = \frac{75 - 20\sqrt{6}}{43} \simeq 0,605.$$

So

$$\overrightarrow{\sigma(t_1)} = \overrightarrow{e_\infty}$$

and

$$\sigma(t_{-1}) = \left(\frac{4}{5}; 0; 0; 2; \frac{15}{8}\right).$$

The point $\sigma(t_{-1})$ defines the sphere of centre

$$O_{-1}\left(0; 0; \frac{5}{2}\right)$$

and radius

$$r_{-1} = \frac{5}{4}.$$

Thus, the focus point is

$$F_{-1}\left(\frac{-5\sqrt{3}}{8}; 0; \frac{25}{8}\right) \simeq (-1,083; 0; 3,125)$$