On the Metacyclicity of the Hilbert 2-Class Field Tower of Imaginary

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Abstract: Let d be a square-free integer <0. In this paper, we will determine all imaginary quadratic number fields $\mathbb{Q}(\sqrt{\mathrm{d}})$ that have a metacyclic Hilbert 2-class field tower. Finally, we will numerically validate our theoretical results.

Key-Words: Keywords: class field tower; class group; imaginary quadratic number field; metacyclic group.

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1 Introduction

Let K be a number field. The maximal unramified abelian 2-extension $K_2^{(1)}$ of K, is called the Hilbert 2-class field of K. We recall that by the class field theory we have $Gal(K_2^{(1)}/K) = Cl_2(K)$, the 2-Sylow subgroup of the class group of K denoted $Cl(K)$. $Cl_2(K)$ is called the 2-class group of K. For a nonnegative integer n, let $K_2^{(n)}$ be defined inductively as $K_2^{(0)} = K$ and $K_2^{(n+1)} = \left(K_2^{(n)}\right)_2^{(1)}$; then

$$K \subset K_2^{(1)} \subset K_2^{(2)} \subset ... \subset K_2^{(n)} \subset ...$$

is called the Hilbert 2-class field tower of K. If n is the minimal integer such that $K_2^{(n)} = K_2^{(n+1)}$, then this tower is called to be finite of length n. If there is no such n, then the tower is called to be infinite. We denote $K_2^{(\infty)} = \underset{i\in\mathbb{N}}{\cup} K_2^{(i)}$. We recall that $K_2^{(\infty)}/K$ is a Galois extension and the tower of K is finite iff $K_2^{(\infty)}/K$ is of finite degree.

The finiteness of the Hilbert 2-class field tower of an imaginary quadratic number field K is still a problem of uncontrollable behavior for some values of $rank(Cl_2(K))$. It's well known, that if $rank(Cl_2(K)) \geq 5$, then, the tower is infinite [3]. For the case where $rank(Cl_2(K)) = 4$, there is no known imaginary quadratic field with finite tower, and according to Martinet's conjecture the tower is infinite [6]. If $rank(Cl_2(K)) = 2$ or 3, the tower may be finite or infinite ([5], [6]), and there is no known procedure for deciding if the tower is finite or not. Let $p_1 = 73$, $p_2 = 373$. The class number of $Q(\sqrt{p_1.p_2})$ is 16. Then, according to [7, Proposition 3.3], the field $K = Q(\sqrt{-p_1.p_2.p})$ has infinite Hilbert 2- class field tower for all prime p satisfying the conditions $p \equiv -1 \mod (4)$ and $\left(\frac{73.373}{p}\right) = -1$. This give an infinite family of imaginary quadratic fields K with $rank(Cl_2(K)) = 2$ and infinite tower. In this paper we give, in theorems 1, 2 and 3, infinite family of imaginary quadratic number fields K having finite Hilbert 2-class field tower and satisfying $rank(Cl_2(K)) = 2$. More precisely, we give the list of all imaginary quadratic number fields that have a metacyclic Hilbert 2-class field tower.

Note that a group G is said to be metacyclic if there is a normal subgroup N of G such that N and G/N are cyclic. For such a group, if we denote $N =< a >$ and $G/N =< bN >$, then $G =< a, b >$ and, thus, G is generated by 2 elements. Let K be a number field and denote $G_2 = Gal(K_2^{(\infty)}/K)$. The Hilbert 2-class field tower of K is said to be metacyclic if G_2 is metacyclic. Note that in this case, the tower terminate at most at the second steep.

2 Notations and useful results

2.1 Notations

- Let k be a number field.
  - $k^{(1)}$ denote the Hilbert class field of k which is the maximal unramified abelian extension of k.
  - O_k is the ring of integers of k.
  - E_k is the unit group of O_k.
- Let K/k be an extension of number fields.
  - $ram(K/k)$ is the number of all places of k that ramify in K.

  - If K/k is a quadratic extension, then $e(K/k)$ is the rank of the elementary 2-group $E_k/\big(E_k \cap N_{K/k}(K^\times)\big)$ where $K^\times = \{\alpha \in K/\ \alpha \neq 0\}$.
    Note that an elementary 2-group G can be seen as a $\mathbb{F}_2$-vector space and its dimension is said the 2-rank of G which we denote $rank_2(G)$.
  - Let $\beta \in k$ and $\mathfrak{p}$ is a place of k. $\left(\frac{\beta,K/k}{\mathfrak{p}}\right)$ is the norm residue symbol of β in k relatively to $\mathfrak{p}$, we may denote it also $\left(\frac{\beta,K}{\mathfrak{p}}\right)$ or $\left(\frac{\beta,\alpha}{\mathfrak{p}}\right)$ if $K = k(\sqrt{\alpha})$ is a quadratic extension of k.

- If m is a square-free positive integer, then ε_m denotes the fundamental unit of $\mathbb{Q}(\sqrt{m})$.

2.2 Preliminary results

Lemma 1. Let K/k be a quadratic extension of number fields. We have

$$rank_2(Cl(K)) \geq ram(K/k) - 1 - e(K/k).$$

If the 2-class group of k is trivial, then the preceding inequality becomes an equality

Proof. For the inequality, see [4] and for the equality, see [1]. □

Lemma 2. Let K be a number field. If $G_2 = Gal(K_2^{(\infty)}/K)$ is metacyclic nonabelian, then $rank_2(Cl(K)) = 2$.

Proof. see [2] □

Remarks. Let K be a quadratic number field.

- If $G_2 = Gal(K_2^{(\infty)}/K)$ is metacyclic, then $rank_2(Cl(K)) \leq 2$. If $rank_2(Cl(K)) = 1$, then $K_2^{(1)} = K_2^{(2)}$, and the Hilbert 2-class field tower of K is cyclic. We deduce that the important case to study is when $rank_2(Cl(K)) = 2$.
- If $G_2 = Gal(K_2^{(\infty)}/K)$ is metacyclic nonabelian, then $rank_2(Cl(K)) = 2$, and K has three quadratic extensions L_1, L_2 and L_3 contained in $K^{(1)}$.

Lemma 3. Let K be a number field such that $rank_2(Cl(K)) = 2$ and denote L_1, L_2 and L_3 the three quadratic extensions of K contained in $K^{(1)}$. If we denote $G = Gal(K_2^{(\infty)}/K)$ and $C_i = Cl_2(L_i)$ for $i = 1, 2, 3$, Then G is metacyclic if and only if $rank(C_i) \leq 2$ for $i = 1, 2, 3$.

Proof. see [2] □

Lemma 4. Let $m \leq -2$ and $d \geq 2$ be two integers, $k = \mathbb{Q}(\sqrt{m})$ and $K = k(\sqrt{d})$. Suppose that 2 splits in k and ramifies in $\mathbb{Q}(\sqrt{d})$. Then if $\mathcal{P}$ is a prime ideal of k that divides 2, then

$$\left(\frac{-1,K/k}{\mathcal{P}}\right) = \left(\frac{-1}{d}\right) \text{ if } d \text{ is odd and}$$
$$\left(\frac{-1,K/k}{\mathcal{P}}\right) = \left(\frac{-1}{c}\right) \text{ if } d = 2c.$$

Proof. Let $Gal(K/k) =< \sigma >$. We have $2O_k = \mathcal{P}\overline{\mathcal{P}}$ with $\overline{\mathcal{P}} = \sigma(\mathcal{P})$. We have

$$\begin{aligned}\left(\frac{-1,K/k}{\mathcal{P}}\right) &= \left(\frac{-1,d}{\mathcal{P}}\right)\\ &= \left(\frac{d,-1}{\mathcal{P}}\right)\end{aligned}$$

The conductor of the extension $\mathbb{Q}(i)/\mathbb{Q}$ is $4\mathbb{Z}p_\infty$ where p_∞ is the infinite prime of $\mathbb{Q}$, then $4O_k = \mathcal{P}^2\overline{\mathcal{P}}^2$ is an admissible modulus for the extension $k(i)/k$.

- Suppose that $d \equiv -1\ mod(4)$. Thus

$$\left(\frac{d,-1}{\mathcal{P}}\right) = \left(\frac{-1,-1}{\mathcal{P}}\right).$$

Let $b \in k$ such that $b \equiv -1\ mod(\mathcal{P}^2)$ and $b \equiv 1\ mod(\overline{\mathcal{P}}^2)$. Then

$$\begin{aligned}\left(\frac{-1,-1}{\mathcal{P}}\right) &= \left(\frac{k(i)/k}{bO_k}\right)\\ &= \left(\frac{\mathbb{Q}(i)/\mathbb{Q}}{N_{k/\mathbb{Q}}(b)}\right)\\ &= \left(\frac{-1}{N_{k/\mathbb{Q}}(b)}\right)\end{aligned}$$

We have $b \equiv 1\ mod(\overline{\mathcal{P}}^2)$ then $\sigma(b) \equiv 1\ mod(\mathcal{P}^2)$. We deduce that $N_{k/Q}(b) \equiv -1\ mod(4)$. Then

$$\begin{aligned}\left(\frac{d,-1}{\mathcal{P}}\right) &= \left(\frac{-1,-1}{\mathcal{P}}\right)\\ &= \left(\frac{-1}{N_{k/\mathbb{Q}}(b)}\right)\\ &= -1\\ &= \left(\frac{-1}{d}\right)\end{aligned}$$

If $d \equiv 1\ mod(4)$, Then $d \equiv 1\ mod(\mathcal{P}^2)$, and, then

$$\left(\frac{d,-1}{\mathcal{P}}\right) = \left(\frac{1,-1}{\mathcal{P}}\right) = 1 = \left(\frac{-1}{d}\right)$$

- Suppose that $d = 2c$ where $c \in \mathbb{N}^*$. We have

$$\begin{aligned}\left(\frac{d,-1}{\mathcal{P}}\right) &= \left(\frac{2c,-1}{\mathcal{P}}\right)\\ &= \left(\frac{2,-1}{\mathcal{P}}\right)\left(\frac{c,-1}{\mathcal{P}}\right)\\ &= \left(\frac{-1,2}{\mathcal{P}}\right)\left(\frac{c,-1}{\mathcal{P}}\right)\\ &= \left(\frac{c,-1}{\mathcal{P}}\right)\\ &= \left(\frac{-1}{c}\right)\end{aligned}$$

□

3 Imaginary quadratic fields with metacyclic Hilbert 2-class field tower

Let $d \in \mathbb{N}^*$ be a positive square-free integer and $K = \mathbb{Q}(\sqrt{-d})$ such that $rank_2(Cl(K)) = 2$. According to the genus theory, d will be of one of the following forms:

$$p_1p_2,\ 2p_1p_2,\ 2p_1q_1,\ p_1p_2q_1,\ 2q_1q_2,\ q_1q_2,\ q_1q_2q_3,$$

where p_1 and p_2 are positive prime integers $\equiv 1 \ mod(4)$ and q_1, q_2 and q_3 are positive prime integers $\equiv 3 \ mod(4)$.

In what follows, we study all these cases in order to see when the Hilbert 2-class field tower of $\mathbb{Q}(\sqrt{-d})$ is metacyclic.

Theorem 1. Let $K = \mathbb{Q}(\sqrt{-d})$ with $d = 2q_1q_2$, q_1q_2 or $q_1q_2q_3$. Then the Hilbert 2-class field tower of K is metacyclic.

Proof. The three quadratic extensions of K contained in $K^{(1)}$ are $L_1 = K(\sqrt{-q_1})$, $L_2 = K(\sqrt{-q_2})$ and $L_3 = K(\sqrt{q_1q_2})$.

Suppose that $d = q_1q_2$. Then $L_1 = k_1(\sqrt{q_2})$, $L_2 = k_2(\sqrt{q_1})$ and $L_3 = k_3(\sqrt{q_1q_2})$ with $k_1 = \mathbb{Q}(\sqrt{-q_1})$, $k_2 = \mathbb{Q}(\sqrt{-q_2})$ and $k_3 = \mathbb{Q}(i)$. Since $Cl_2(k_1)$ is trivial, then by lemma 1, we have

$$rank_2(Cl(L_1)) = ram(L_1/k_1) - 1 - e(L_1/k_1).$$

We have $rank_2(Cl(L_1)) \leq 3$ and $rank_2(Cl(L_1)) = 3$ iff 2 and q_2 splits in k_1 and $e(L/k_1) = 0$. But these conditions cannot be satisfied simultaneously. In fact, We have $E_{k_1} =< -1 >$ if $q_1 \neq 3$ and $E_{k_1} =< \zeta_6 >=< -1, \zeta_6 >$ if not. Then if q_2 splits in k_1 and $\mathcal{Q}$ is a prime ideal of k_1 dividing q_2, then

$$\begin{aligned}\left(\frac{-1, L_1/k_1}{\mathcal{Q}}\right) &= \left(\frac{-1,q_2}{\mathcal{Q}}\right)\\ &= \left(\frac{q_2,-1}{\mathcal{Q}}\right)\\ &= \left(\frac{-1}{q_2}\right)\\ &= -1\end{aligned}$$

Thus $e(L_1/k_1) = 1$. We conclude that $rank_2(Cl(L_1)) \leq 2$. In the same way, $rank_2(Cl(L_2)) \leq 2$. We have $ram(L_3/k_3) = 2$ then $rank_2(Cl(L_3)) \leq 2$. Then K has a metacyclic Hilbert 2-class field tower.

Suppose that $d = q_1q_2q_3$. Then $L_1 = k_1(\sqrt{q_2q_3})$, $L_2 = k_2(\sqrt{q_1q_3})$ and $L_3 = k_3(\sqrt{q_1q_2})$ with $k_1 = \mathbb{Q}(\sqrt{-q_1})$, $k_2 = \mathbb{Q}(\sqrt{-q_2})$ and $k_3 = \mathbb{Q}(\sqrt{-q_3})$. We have $rank_2(Cl(L_1)) = ram(L_1/k_1) - 1 - e(L_1/k_1)$. If q_2 is inert in k_1, then $ram(L_1/k_1) \leq 3$ and thus

$$rank_2(Cl(L_1)) \leq 2.$$

If q_2 splits in k_1 and $\mathcal{Q}$ is a prime ideal of k_1 dividing q_2, then

$$\begin{aligned}\left(\frac{-1, L_1/k_1}{\mathcal{Q}}\right) &= \left(\frac{-1,q_2q_3}{\mathcal{Q}}\right)\\ &= \left(\frac{q_2,-1}{\mathcal{Q}}\right)\\ &= \left(\frac{-1}{q_2}\right)\\ &= -1\end{aligned}$$

Thus $e(L_1/k_1) = 1$, and $rank_2(Cl(L_1)) \leq 2$. We conclude that in all cases we have $rank_2(Cl(L_1)) \leq 2$. In the same way, we have $rank_2(Cl(L_j)) \leq 2$ for $j = 2, 3$. Then K has a metacyclic Hilbert 2-class field tower.

Suppose that $d = 2q_1q_2$. Then $L_1 = k_1(\sqrt{2q_2})$, $L_2 = k_2(\sqrt{2q_1})$ and $L_3 = k_3(\sqrt{-2})$ with $k_1 = \mathbb{Q}(\sqrt{-q_1})$, $k_2 = \mathbb{Q}(\sqrt{-q_2})$ and $k_3 = \mathbb{Q}(\sqrt{q_1q_2})$. We have $rank_2(Cl(L_1)) = ram(L_1/k_1) - 1 - e(L_1/k_1)$. If 2 is inert in k_1, then $ram(L_1/k_1) \leq 3$ and thus $rank_2(Cl(L_1)) \leq 2$. If 2 splits in k_1 and $\mathcal{Q}$ is a prime ideal of k_1 dividing 2, then, by lemma 4,

$$\left(\frac{-1, L_1/k_1}{\mathcal{Q}}\right) = \left(\frac{-1}{q_2}\right) = -1.$$

Thus $e(L_1/k_1) = 1$, and $rank_2(Cl(L_1)) \leq 2$. We conclude that in all cases, we have

$$rank_2(Cl(L_1)) \leq 2.$$

In the same way, $rank_2(Cl(L_2)) \leq 2$.
We have $ram(L_3/k_3) \leq 4$ and if $\mathcal{P}$ is an infinite place of k_3 then $\left(\frac{-1, L_3/k_3}{\mathcal{P}}\right) = -1$, thus $e(L_3/k_3) \geq 1$. We conclude that

$$rank_2(Cl(L_3)) \leq 2.$$

Thus, using lemma 3, K has a metacyclic Hilbert 2-class field tower. □

Theorem 2. Let $K = \mathbb{Q}(\sqrt{-d})$ with $d = p_1p_2$ or $d = 2p_1p_2$. Then the Hilbert 2-class field tower of K is metacyclic iff, after a permutation of the p_i's, we have one of the two following conditions:

(C1) $p_1 \equiv p_2 \equiv 5 \ mod(8)$

(C2) $p_1 \equiv 1 \ mod(8)$, $p_2 \equiv 5 \ mod(8)$ and $\left(\frac{p_2}{p_1}\right) = -1$

Proof. The three unramified quadratic extensions of K are $L_1 = K(\sqrt{p_1})$, $L_2 = K(\sqrt{p_2})$ and $L_3 = K(\sqrt{p_1p_2})$.

- Suppose that $d = p_1p_2$. Then $L_1 = k_1(\sqrt{-p_2})$, $L_2 = k_2(\sqrt{-p_1})$ and $L_3 = k_3(\sqrt{p_1p_2})$ with $k_1 = \mathbb{Q}(\sqrt{p_1})$, $k_2 = \mathbb{Q}(\sqrt{p_2})$ and $k_3 = \mathbb{Q}(i)$. We have

$$\begin{cases} rank_2(Cl(L_1)) = ram(L_1/k_1) - 1 - e(L_1/k_1) \\ \text{and} \\ E_{k_1} =< -1, \varepsilon_{p_1} > \end{cases}$$

Let $\mathcal{P}_\infty$ be one of the two infinite places of k_1. We have

$$\left(\frac{-1, L_1/k_1}{\mathcal{P}_\infty}\right) = \left(\frac{-\varepsilon_{p_1}, L_1/k_1}{\mathcal{P}_\infty}\right) = -1$$

On the other hand, ε_{p_1} can not be a norm in L_1/k_1. In fact, if there is an element $v \in L_1$ such that $N_{L_1/k_1}(v) = \varepsilon_{p_1}$, then we will have $N_{L_1/\mathbb{Q}}(v) = N_{k_1/\mathbb{Q}}(\varepsilon_{p_1}) = -1$. This is impossible since $\left(\frac{-1, L_1/\mathbb{Q}}{\mathcal{P}_\infty}\right) = -1$. We deduce that $e(L_1/k_1) = 2$, and then $rank_2(Cl(L_1)) = ram(L_1/k_1) - 3$. The places of k_1 that ramify in L_1 are exactly the 2 infinite places, the place(s) above 2 and the place(s) above p_2. Thus $ram(L_1/k_1) = 4, 5$ or 6 and $rank_2(Cl(L_1)) = 1, 2$ or 3. In addition, $rank_2(Cl(L_1)) = 3$ iff $p_1 \equiv 1 \ mod(8)$ and $\left(\frac{p_2}{p_1}\right) = 1$.

In the same way, we have $rank_2(Cl(L_2)) \leq 3$ and $rank_2(Cl(L_2)) = 3$ iff $p_2 \equiv 1 \ mod(8)$ and $\left(\frac{p_2}{p_1}\right) = 1$.

For the 2-rank of L_3, we have $ram(L_3/k_3) = 4$. Then $rank_2(Cl(L_3)) = 3 - e(L_3/K_3)$. Let $\mathcal{P}$ be a prime ideal of k_3 dividing p_1, for example. We have

$$\begin{aligned} \left(\frac{i, L_3/k_3}{\mathcal{P}}\right) &= \left(\frac{i, p_1p_2}{\mathcal{P}}\right) \\ &= \left(\frac{p_1, i}{\mathcal{P}}\right) \\ &= \left(\frac{k_3(\zeta_8)/k_3}{\mathcal{P}}\right) \\ &= \left(\frac{\mathbb{Q}(\zeta_8)/\mathbb{Q}}{p_1}\right) \end{aligned}$$

Thus $e(L_1/k_1) = 0$ iff $p_1 \equiv p_2 \equiv 1 \ mod(8)$. We conclude that $rank_2(Cl(L_3)) \leq 3$ and $rank_2(Cl(L_3)) = 3$ iff $p_1 \equiv p_2 \equiv 1 \ mod(8)$.

- Suppose that $d = 2p_1p_2$. We have $L_1 = k_1(\sqrt{-2p_2})$, $L_2 = k_2(\sqrt{-2p_1})$ and $L_3 = k_3(\sqrt{p_1p_2})$ with $k_1 = \mathbb{Q}(\sqrt{p_1})$, $k_2 = \mathbb{Q}(\sqrt{p_2})$ and $k_3 = \mathbb{Q}(\sqrt{-2})$.
As in the previous case, We have, for $j = 1, 2$, $e(L_j/k_j) = 2$, and then $rank_2(Cl(L_j)) = ram(L_j/k_j) - 3 \leq 3$. In addition, $rank_2(Cl(L_j)) = 3$ iff $p_j \equiv 1 \ mod(8)$ and $\left(\frac{p_2}{p_1}\right) = 1$.
For the 2-rank of L_3, we have $ram(L_3/k_3) \leq 4$. Then $rank_2(Cl(L_3)) \leq 3$ with equality iff $p_1 \equiv p_2 \equiv 1 \ mod(8)$ and $e(L_1/k_1) = 0$. If $p_1 \equiv p_2 \equiv 1 \ mod(8)$ and $\mathcal{P}$ is a prime ideal of k_3 dividing p_1, for example, then We have

$$\begin{aligned} \left(\frac{-1, L_3/k_3}{\mathcal{P}}\right) &= \left(\frac{-1, p_1p_2}{\mathcal{P}}\right) \\ &= \left(\frac{p_1, -1}{\mathcal{P}}\right) \\ &= \left(\frac{-1}{p_1}\right) \\ &= 1 \end{aligned}$$

Thus $e(L_1/k_1) = 0$. We conclude that $rank_2(Cl(L_3)) \leq 3$ and $rank_2(Cl(L_3)) = 3$ iff $p_1 \equiv p_2 \equiv 1 \ mod(8)$.

The above two cases can be summarized by:
The tower of k is not metacyclic iff $\Big(p_1 \equiv 1 \ mod(8)$ and $\left(\frac{p_1}{p_1}\right) = 1\Big)$ or $\Big(p_2 \equiv 1 \ mod(8)$ and $\left(\frac{p_2}{p_1}\right) = 1\Big)$ or $p_1 \equiv p_2 \equiv 1 \ mod(8)$. The theorem will be proved by discussing according first to (p_1 mod (8), p_2 mod (8)) and then $\left(\frac{p_2}{p_1}\right)$. □

Theorem 3. Let $K = \mathbb{Q}(\sqrt{-d})$ with $d = r_1p_2q_1$ and $r_1 = 2$ or $r_1 = p_1$ is a positive prime integer $\equiv 1 \ mod(4)$. Then the Hilbert 2-class field tower of K is metacyclic except in the following cases:

(C1) $\left(\frac{r_1}{p_2}\right) = \left(\frac{r_1}{q_1}\right) = 1$

(C2) $\left(\frac{r_1}{p_2}\right) = \left(\frac{q_1}{p_2}\right) = 1$

(C3) $\left(\frac{r_1}{q_1}\right) = \left(\frac{p_2}{q_1}\right) = 1$

Proof. The three quadratic extensions of K contained in $K^{(1)}$ are $L_1 = K(\sqrt{r_1})$, $L_2 = K(\sqrt{p_2})$ and $L_3 = K(\sqrt{r_1p_2})$.

- Suppose that $d = p_1p_2q_1$. Then $L_1 = k_1(\sqrt{-p_2q_1})$, $L_2 = k_2(\sqrt{-p_1q_1})$ and $L_3 =$

$k_3(\sqrt{p_1p_2})$ with $k_1 = \mathbb{Q}(\sqrt{p_1})$, $k_2 = \mathbb{Q}(\sqrt{p_2})$ and $k_3 = \mathbb{Q}(\sqrt{-q_1})$. We have $rank_2(Cl(L_j)) = ram(L_j/k_j) - 1 - e(L_j/k_j)$, for $j = 1, 2$ and 3. As in the cases in the previous theorem, we have $e(L_1/k_1) = 2$, and then $rank_2(Cl(L_1)) = ram(L_1/k_1) - 3$. Since $ram(L_1/k_1) \leq 6$, we have $rank_2(Cl(L_1)) \leq 3$ and $rank_2(Cl(L_1)) = 3$ iff $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{q_1}\right) = 1$.
In the same way, we have $rank_2(Cl(L_2)) \leq 3$ and $rank_2(Cl(L_2)) = 3$ iff $\left(\frac{p_2}{p_1}\right) = \left(\frac{p_2}{q_1}\right) = 1$.
If $\left(\left(\frac{q_1}{p_1}\right), \left(\frac{q_1}{p_2}\right)\right) \neq (1,1)$, then $ram(L_3/k_3) \leq 3$ and thus $rank_2(Cl(L_3)) \leq 2$.
Suppose that $\left(\frac{q_1}{p_1}\right) = \left(\frac{q_1}{p_2}\right) = 1$. Then $ram(L_3/k_3) = 4$. let $\mathcal{P}$ be a prime ideal of k_3 lying over p_j with $j = 1$ or 2. We have

$$\begin{aligned}\left(\frac{-1, L_3/k_3}{\mathcal{P}}\right) &= \left(\frac{-1, p_1p_2}{\mathcal{P}}\right)\\ &= \left(\frac{p_j, -1}{\mathcal{P}}\right)\\ &= \left(\frac{-1}{p_j}\right)\\ &= 1\end{aligned}$$

If $q_1 \neq 3$, then $e(L_3/k_3) = 0$. If $q = 3$, then $E_{k_3} =< \zeta_6 >$ and

$$\begin{aligned}\left(\frac{\zeta_6, L_3/k_3}{\mathcal{P}}\right) &= \left(\frac{\zeta_6^3, L_3/k_3}{\mathcal{P}}\right)\\ &= \left(\frac{-1, L_3/k_3}{\mathcal{P}}\right)\\ &= \left(\frac{-1, p_1p_2}{\mathcal{P}}\right)\\ &= \left(\frac{p_j, -1}{\mathcal{P}}\right)\\ &= \left(\frac{-1}{p_j}\right)\\ &= 1\end{aligned}$$

Then $e(L_3/k_3) = 0$. In the two cases, we have $rank_2(Cl(L_3)) = 3$. We conclude that $rank_2(Cl(L_3)) = 3$ iff $\left(\frac{q_1}{p_1}\right) = \left(\frac{q_1}{p_2}\right) = 1$.

- Suppose that $d = 2p_2q_1$. Then $L_1 = k_1(\sqrt{-p_2q_1})$, $L_2 = k_2(\sqrt{-2q_1})$ and $L_3 = k_3(\sqrt{2p_2})$ with $k_1 = \mathbb{Q}(\sqrt{2})$, $k_2 = \mathbb{Q}(\sqrt{p_2})$ and $k_3 = \mathbb{Q}(\sqrt{-q_1})$. We have $rank_2(Cl(L_2)) = ram(L_2/k_2) - 1 - e(L_2/k_2)$. Since $e(L_2/k_2) = 2$, $rank_2(Cl(L_2)) = ram(L_2/k_2) - 3 \leq 3$. In addition $rank_2(Cl(L_2)) = 3$ iff $\left(\frac{2}{p_2}\right) = \left(\frac{q_1}{p_2}\right) = 1$. Similarly, we have $rank_2(Cl(L_1)) \leq 3$ and $rank_2(Cl(L_1)) = 3$ iff $\left(\frac{2}{p_2}\right) = \left(\frac{2}{q_1}\right) = 1$.
We have $rank_2(Cl(L_3)) \leq 3$ with equality iff $ram(L_3/k_3) = 4$ and $e(L_3/k_3) = 0$.
Suppose that $ram(L_3/k_3) = 4$ (i.e $\left(\frac{p_2}{q_1}\right) = \left(\frac{2}{q_1}\right) = 1$). Let $\mathcal{P}$ be a prime ideal of k_3 lying over p_2. We have

$$\begin{aligned}\left(\frac{-1, L_3/k_3}{\mathcal{P}}\right) &= \left(\frac{-1, 2p_2}{\mathcal{P}}\right)\\ &= \left(\frac{p_2, -1}{\mathcal{P}}\right)\\ &= 1\end{aligned}$$

Let $\mathcal{Q}$ be a prime ideal of k_3 lying over 2. By lemma 4, we have

$$\left(\frac{-1, L_3/k_3}{\mathcal{Q}}\right) = \left(\frac{-1}{p_2}\right) = 1$$

Then $e(L_3/k_3) = 0$. We deduce that $rank_2(Cl(L_3)) \leq 3$ and $rank_2(Cl(L_2)) = 3$ iff $\left(\frac{p_2}{q_1}\right) = \left(\frac{2}{q_1}\right) = 1$.

□

4 Numerical verification

Our results can be checked. Indeed let k be a quadratic number field verifying $rank(Cl_2(k)) = 2$. Using a computer algebra system, we can compute the rank of the 2-class group of its three quadratic unramified extensions. Then, by lemma 3, we can decide if the tower of the Hilbert 2-class field of k is metacyclic or not. We will do that, as an example, for the case where $k = \mathbb{Q}(\sqrt{-p_1p_2})$.
Let p_1 and p_2 be two positive prime integers $\equiv 1 \mod (4)$ and $k = \mathbb{Q}(\sqrt{-p_1p_2})$. The three quadratic unramified extensions of k are $L_1 = \mathbb{Q}(\sqrt{p_1}, \sqrt{-p_2})$, $L_2 = \mathbb{Q}(\sqrt{p_2}, \sqrt{-p_1})$ and $L_3 = \mathbb{Q}(i, \sqrt{p_1p_2})$. We denote $r_j = rank(Cl_2(L_j))$ for $j = 1, 2, 3$.

Let B_1 and B_2 be respectively the truth values (0 if false and 1 if true) of the proposition " The Hilbert 2-class field tower of k is metacyclic" and the statement " After a permutation of p_1 and p_2, we have (C_1) or (C_2), the two conditions in theorem 2". In the following table, We compute B_1 and B_2. By comparing them, we can see that they are equivalent. This is in agreement with Theorem 2.
The numerical results in this table were obtained using Pari [8].

p_1	p_2	r_1	r_2	r_3	B_1	$p_1 \mod (8)$	$p_2 \mod (8)$	$\left(\frac{p_1}{p_2}\right)$	B_2
5	13	1	1	1	1	5	5	-1	1
5	17	1	2	2	1	5	1	-1	1
5	29	2	2	2	1	5	5	1	1
5	241	2	3	2	0	5	1	1	0
13	17	2	3	2	0	5	1	1	0
13	29	2	2	2	1	5	5	1	1
13	149	1	1	2	1	5	5	-1	1
13	233	2	3	2	0	5	1	1	0
13	241	1	2	2	1	5	1	-1	1
17	41	2	2	3	0	1	1	-1	0
17	53	3	2	2	0	1	5	1	0
17	89	3	3	3	0	1	1	1	0
17	109	2	1	2	1	1	5	-1	1
17	149	3	2	2	0	1	5	1	0
29	37	1	1	2	1	5	5	-1	1
29	89	1	2	2	1	5	1	-1	1
29	149	2	2	2	1	5	5	1	1
29	233	2	3	2	0	5	1	1	0
37	41	2	3	2	0	5	1	1	0
37	53	2	2	2	1	5	5	1	1
37	149	2	2	2	1	5	5	1	1
37	193	1	2	2	1	5	1	-1	1
41	53	2	1	2	1	1	5	-1	1
41	61	3	2	2	0	1	5	1	0
41	89	2	2	3	0	1	1	-1	0
41	113	3	3	3	0	1	1	1	0
53	61	1	1	2	1	5	5	-1	1
53	73	1	2	2	1	5	1	-1	1
53	113	2	3	2	0	5	1	1	0
61	73	2	3	2	0	5	1	1	0
61	89	1	2	2	1	5	1	-1	1

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