A Comparison with Theory of Computation and Estimation of Pitch-Angle Diffusion Coefficients from Simulations in Noisy Reduced Magnetohydrodynamic Turbulence

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Abstract: The transport of energetic charged particles in turbulent magnetic fields is a topic of interest in various astrophysical and laboratory plasma contexts. In order to estimate the mean free path $\lambda_{\|}$ of a particle in the direction parallel to the mean magnetic field, one can use theoretical expressions that include the pitch-angle diffusion coefficient $D_{\mu\mu}$. In this work we evaluate theories for $D_{\mu\mu}$ in the context of the noisy reduced magnetohydrodynamic (NRMHD) model where turbulent fluctuations are absent at large parallel wavenumbers. For most turbulence models, the standard quasilinear theory predicts zero pitch-angle diffusion only for particles with a 90° pitch angle, but for NRMHD a range of pitch angles is affected, leading to infinite $\lambda_{\|}$. We examine two theories that include resonance broadening which yield finite $\lambda_{\|}$ and compare them with test-particle computer simulations in which the parallel mean free path can be readily obtained. We find that both theories are quite accurate in some regions of the parameter space considered, but neither is particularly good when the particle Larmor radius R_{L} becomes much smaller than the quasilinear resonance limit.

Key-Words: Diffusion, Pitch-Angle Scattering, Synthetic Magnetic Fields, Turbulence, Test Particle Simulations

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1 Introduction

The transport of energetic charged particles (e.g., cosmic rays) in turbulent magnetic fields is an important process in various astrophysical and laboratory plasma contexts. A quantity of interest is the parallel (to the mean magnetic field direction) diffusion coefficient, $\kappa_{\|}$, which can be expressed in terms of the parallel mean free path $\lambda_{\|}$ (e.g. [1]):

$$\lambda_{\|} = \frac{3v}{8} \int_{-1}^{1} \frac{(1-\mu^2)^2}{D_{\mu\mu}(\mu)} d\mu, \tag{1}$$

where v is the particle velocity and μ is the cosine of its pitch angle with respect to the mean magnetic field direction. This is expected to be valid when the particle pitch-angle distribution function, $f(\mu, t)$, is almost isotropic. The mean free path is related to the diffusion coefficient via $\lambda_{\|} = 3\kappa_{\|}/v$. Parallel diffusion theory is dominated by the quasilinear theory [3] and its various extensions (e.g. [12, 11]), which focus on obtaining expressions for the pitch-angle diffusion coefficient $D_{\mu\mu}(\mu)$.

For ease of reading, Table 1 gives definitions and assumed values for variables and parameters used in this paper.

Test particle computer simulations are a useful tool in the study of charged particle transport, and in particular can be used to test theoretical expressions for particle diffusion coefficients. For test particle simulations, the parallel diffusion coefficient can be ob-

Table 1: Definitions of variables, parameters and values. Lengths, velocities and magnetic field strengths are given in dimensionless units as noted in the text.

Parameter	Symbol	Value
Parallel length scale	$l_{\|}$	1
Perpendicular length scale	$l_{\perp}$	1
Particle velocity	v	1
Pitch-angle cosine	μ	$-1 \leq \mu \leq 1$
Larmor radius	R_L	variable
Gyrofrequency	Ω	variable
Parallel wavenumber	$k_{\|}$	variable
Perpendicular wavenumber	$k_{\perp}$	variable
Mean magnetic field strength	B_0	1
Turbulence strength	b/B_0	variable
Perpendicular bend-over scale	$\hat{\lambda}_{\perp}$	2.4954
Dimensionless rigidity	R $(R_L/\hat{\lambda}_{\perp})$	variable
Magnetic power spectrum	S	variable
Cut-off wavenumber $k_z = k_{\|}$	K	$\pi/2$, $\pi/20$
$\xi = K\hat{\lambda}_{\perp}$	ξ	0.39198 3.9198

tained as shown in equation (2)

$$\kappa_{\parallel} = \lim_{t\to\infty} \frac{\langle(\Delta z)^2\rangle}{2t}, \qquad (2)$$

where $\Delta z(t)$ is the displacement of a particle in the parallel direction over a time t, and the angle brackets denote an ensemble average over particles which can be obtained by simulating the trajectories of a large number of particles in turbulent magnetic fluctuations up to a sufficiently large time where the limit $\langle(\Delta z)^2\rangle \propto t$ becomes constant. This occurs in most contexts where the magnetic field has a well-defined integral length scale. One could apply an analogous expression for the pitch-angle diffusion

$$D_{\mu\mu}(\mu) = \lim_{t\to\infty} \frac{\langle(\Delta\mu)^2\rangle}{2t}, \qquad (3)$$

where $\Delta\mu = \mu(t) - \mu$ is the pitch-angle displacement from its initial value μ. However, this form has a severe difficulty since the pitch-angle μ is bounded, i.e. $\mu \in [-1, 1]$, so that $(\Delta\mu)^2 \leq 4$, and in the infinite limit one obtains $D_{\mu\mu}(\mu) = 0$. In the literature, this problem is remedied by taking the value of this expression at a sufficiently large time (see, e.g. [17]), or simply after one gyroperiod t_{L} of a particle, as suggested by [18]. An alternative approach to obtain $D_{\mu\mu}$ involves following the evolution of the pitch-angle distribution function $f(\mu, t)$ of the test particles and assume that it obeys a pitch-angle diffusion equation (and hence Fick's law), so that $D_{\mu\mu}$ can be calculated via the derivatives of f and the pitch-angle flux (e.g. [4, 16, 2, 7]).

If we have a theoretical expression for $D_{\mu\mu}$, we can then compare it with the corresponding value of $D_{\mu\mu}$ in computer simulations, or we can use Eq. 1 to compare it with the parallel mean free path in the computer simulations obtained via Eq. 2. In this work we study pitch-angle diffusion in the noisy reduced magnetohydrodynamic (NRMHD) model[10], a synthetic model of homogeneous magnetic turbulence. The model is interesting in that it has no magnetic fluctuations at higher parallel wavenumber, $k_z > K$. According to the classical quasilinear scattering theory [3], particles are in resonance with magnetic fluctuations when $k_z = 1/(|\mu|R_{\mathrm{L}})$, where μ is the cosine of the pitch angle and R_{L} is the particle's Larmor radius. This implies that whenever $R_{\mathrm{L}} < 1/K$, there is no resonant scattering, so that the particle's mean free path $\lambda_{\parallel}$ will be infinite. Due to higher order nonlinear effects, $\lambda_{\parallel}$ is actually finite, but can become very large at small R_{L}. This presents a problem for the theory and the standard quasilinear approach, as detailed below, cannot be used. We here consider a variant of the second order quasilinear theory (SOQLT) [11] and a special case of the weakly nonlinear theory (WNLT) [12].

Below we provide details of the NRMHD model and test particle simulations that we compare directly with the SOQLT and WNLT theories in terms of $D_{\mu\mu}$ and $\lambda_{\parallel}$, as discussed above. This work is quite similar to that of [9], but our computational simulations are performed using a discrete magnetic field mesh, rather than a locally-evaluated continuous magnetic field. Here we also use a slightly different parameter regime (stretching into the non-resonant scattering region), evaluate SOQLT in detail and contrast $D_{\mu\mu}$ results. This work is quite similar to [7], where the widely-studied slab and two-component turbulence models were considered.

2 Noisy RMHD Turbulence

The noisy reduced magnetohydrodynamic (NRMHD) model can be applied to different systems such as fusion plasma and the solar corona (for details of the model, see, e.g. [10]). NRMHD is a model that characterizes RMHD turbulence, which comes from a reduced set of MHD equations, that has 2D-like and 3D-like properties [6]. The model was developed by [10], and it has been applied in studies of the field line random walk (related to test-particle transport, [15]) and in some particle transport studies (see, e.g., [9, 13]). The total magnetic field is expressed as

$$\vec{B}(\vec{x}) = B_0\hat{\vec{z}} + \vec{b}(x, y, z), \quad \vec{b} \perp \hat{\vec{z}}, \qquad (4)$$

where $\vec{B}_0 = B_0\vec{\hat{z}}$ is the mean magnetic field, which is taken to be aligned along the $\hat{\vec{z}}$ direction and to be strong compared with a transverse fluctuating magnetic field $\vec{b}(\vec{x})$ with zero mean $\langle\vec{b}(\vec{x})\rangle = \vec{0}$. We assume that the magnetic fields are static and homogeneous, which means that the field does not depend on time and the statistical properties of the magnetic field are invariant under translations. Although magnetic fields will evolve over time, this time can be large compared with the time taken by particles to become diffusive and the variation can be neglected. This can be justified in a variety of physical contexts, especially for highly relativistic particles, but in general one could compare the length and timescales associated with the magnetic field to the particle's velocity magnitude.

The statistically homogeneous fluctuating field is given in terms of wave vectors $\vec{k}$, for which the potential function in k-space can be written as [10]

$$a(\vec{k}) = \begin{cases} C\sqrt{A^{2D}(k_\perp)}e^{i\varphi(\vec{k})} & \text{for } |k_z| \leq K \\ 0 & \text{for } |k_z| > K, \end{cases} \qquad (5)$$

where $\varphi(\vec{k})$ is a random phase, $A^{2D}(k_\perp)$ is the 2D spectrum and K is a cut-off wavenumber in k_z. This has a different random phase for every independent $\vec{k}$

(in 3D), and has a finite extent of the spectrum in k_z. The normalization constant C is chosen to satisfy a condition on the mean energy density in the magnetic field. The properties of NRMHD are controlled by the dimensionless parameters b/B_0 and $K\hat{\lambda}_\perp$, where b is the rms fluctuation strength (in units of magnetic field strength) and the other parameters are as in Table 1. In computer simulations the magnetic field is constructed on a three-dimensional grid of size $N_x \times N_y \times N_z$, first in wavevector space, then applying an inverse FFT to each component. We use $N_x = N_y = 512$ and $N_z = 4096$ grid points. Multiple realisations of the field are constructed by taking different random values of φ. The details of the synthetic field construction can be found in [15].

3 Test Particle Simulations

For each particle in a realization we solve the (dimensionless) Newton-Lorentz equations for the particle trajectory $\vec{x}(t)$ and velocity $\vec{v}(t)$,

$$\frac{d\vec{v}}{dt} = \alpha\vec{v} \times \vec{B}\left[\vec{x}(t)\right], \quad \frac{d\vec{x}}{dt} = \vec{v}, \tag{6}$$

where $\alpha = qB_0 l_\perp/(\gamma m v_0)$, with q the particle charge, γ the Lorentz factor, m the particle rest mass and v_0 the unit of velocity. Note that these equations imply that $|\vec{v}|$ is constant, or in other words, the particle energy is conserved. For simplicity we take $|\vec{v}| = 1$ in units of v_0, $B_0 = 1$ and then vary the parameter α to control the dimensionless Larmor radius $R_\mathrm{L}/l_\perp = \gamma m v_0/(|q|B_0)$.

We take a large number of particles with initial locations uniformly distributed in space over the magnetic field grid. The initial velocity distribution depends on what quantity we wish to evaluate. To calculate the parallel mean free path $\lambda_\parallel$ we take initially an isotropic velocity distribution and then integrate the equations of motion up to a time when Eq. 2 converges for an ensemble of particle trajectories and magnetic field realisations. In the case of evaluating $D_{\mu\mu}$ in simulation, we take a distribution of velocities parallel to the mean field such that the distribution of pitch angle cosines μ has a constant gradient, i.e. $f(\mu, 0) \propto \mu$ on some interval in $[-1, 1]$. The details of obtaining $D_{\mu\mu}$ in simulations by calculating $\partial df/\partial\mu$ and the pitch-angle flux are given in [7]. In the following, we denote this quantity as $D_{\mu\mu}$[FD], and the $\lambda_\parallel$ obtained directly from simulations as $\lambda_\parallel[sim]$.

4 Pitch Angle Diffusion Theories

In the following we make a specific choice for the 2D spectrum $A(k_\perp)$ in order to implement the model in computer simulations and evaluate theoretical expressions. We take (see [5])

$$A(k_\perp) = \frac{A_0}{\left[1 + (\hat{\lambda}_\perp k_\perp)^2\right]^{7/3}}, \tag{7}$$

where $A_0 = 8b^2\hat{\lambda}_\perp^4/9$.

4.1 Quasilinear theory

QLT was originally developed by Jokipii [3]. It is the simplest approach for describing spatial diffusion. In QLT the actual particle trajectory in the NRMHD field is approximated by the trajectory of a particle in the mean field, i.e. it is assumed that the turbulent fluctuations are relatively weak. With this approximation, a particle is taken to rotate in the direction perpendicular to the mean field and to move with constant velocity along the mean field, i.e., a particle's trajectory is assumed to be a helical motion with a constant Larmor radius. We employ this concept to compute the pitch-angle diffusion coefficient $D_{\mu\mu}(\mu)$ for NRMHD turbulence as in the following expression [9]:

$$\begin{aligned} D_{\mu\mu}^{\mathrm{QLT}}(\mu) &= \tfrac{\pi\Omega^2}{2v|\mu|R_\mathrm{L}^2 K B_0^2} \int_0^\infty \sum_{n=-\infty}^{+\infty} k_\perp A(k_\perp) \\ &\quad \times \mathrm{H}\left(K - \left|\tfrac{n\Omega}{v\mu}\right|\right) n^2 J_n^2(W) dk_\perp. \end{aligned} \tag{8}$$

where $W = R_\mathrm{L} k_\perp \sqrt{1-\mu^2}$, H the Heaviside step function and J_n the Bessel function of order n of the first kind. We denote this estimate by $D_{\mu\mu}$[QLT].

4.2 Second-order quasilinear theory

The 90° scattering problem is a well-known problem in diffusion theory (see, e.g., [14]). The SOQLT provided a solution of this problem of finding a diffusion formula for $\mu = 0$, i.e., at a pitch angle of 90°. Here for NRMHD we adopt the simplified form of SOQLT pitch angle diffusion coefficient as in [9],

$$\begin{aligned} D_{\mu\mu}^{\mathrm{SOQLT}}(\mu) = \frac{2\pi\Omega^2}{R_\mathrm{L}^2 K B_0^2} \sum_{n=0}^{+\infty} \int_0^K \int_0^\infty R_n(\mu, k_z) dk_z \\ \times k_\perp^{-2} \frac{A(k_\perp)}{2\pi} n^2 J_n^2(W) dk_\perp, \end{aligned} \tag{9}$$

with resonance function R_n

$$R_n(\mu, k_z) = \frac{\sqrt{\pi}}{v|k_z|}\frac{B_0}{b} \exp\left[-\frac{(\mu R_\mathrm{L} k_z + n)^2 B_0^2}{(R_\mathrm{L} k_z b)^2}\right], \tag{10}$$

with the various other quantities as defined previously. We denote this estimate by $D_{\mu\mu}$[SOQLT].

4.3 Weakly non-linear theory

As shown in [9], the weakly nonlinear pitch-angle diffusion coefficient can be obtained from the theory [12]. The WNLT formula for $D_{\mu\mu}(\mu)$ that we use in our calculations is

$$
\begin{aligned}
D_{\mu\mu}^{\mathrm{WNLT}} &= \frac{8}{9\xi R^4|\mu|}\frac{b^2}{B_0^2}\int_0^\infty \frac{x}{(1+x^2)^{7/3}} \\
&\times \sum_{n=1}^{+\infty} n^2 J_n^2\left(Rx\sqrt{1-\mu^2}\right) \\
&\times \left[\arctan\left(\frac{n+R_{\mathrm{L}}|\mu|K}{R_{\mathrm{L}}D_\perp k_\perp^2/2}\right)\right. \\
&\left. - \arctan\left(\frac{n-R_{\mathrm{L}}|\mu|K}{R_{\mathrm{L}}D_\perp k_\perp^2/2}\right)\right] dx.
\end{aligned} \tag{11}
$$

In (11), we take the form of $D_\perp(\mu)$ to be

$$D_\perp(\mu) = (\nu+1)|\mu|^\nu \kappa_\perp, \tag{12}$$

where $\kappa_\perp$ is the particle perpendicular diffusion coefficient and ν is an arbitrary parameter ($0 \le \nu < 1$). $D_\perp$ is the pitch-angle dependent perpendicular diffusion coefficient, and the form used here was originally proposed by [14, 9] based on 2D field simulation results and might not be the most appropriate form for NRMHD.

In our calculations, we have tested a range of values of ν in the resonant regime $R_L > \frac{|n|}{\mu K}$ and found that $\nu = 0$ usually gave either the best value for the diffusion coefficients or an acceptable value. In this case,

$$D_\perp = \kappa_\perp, \tag{13}$$

which is independent of μ. This model requires an estimate for $\kappa_\perp$. In our WNLT calculations, we take $\kappa_\perp$ from computer simulations.

5 Results

In our numerical calculations for the NMRHD model, we used the values of parameters summarized in Table 1. As shown in Tables 2, 3 and 4, we carried out test-particle simulations for 12 sets of values of b/B_0 and $R_L/l_\perp$ and computed the $\lambda_\|[sim]/l_\perp$ and $\lambda_\|[\mathrm{FD}]/l_\perp$ values. For the same 12 sets of values, we calculated the $\lambda_\|/l_\perp$ values from the SOQLT and WNLT theories.

The values for $\lambda_\|[sim]/l_\perp$ in Table 2 were computed from the test-particle simulation results for $\kappa_\|$ using

$$\lambda_\| = \frac{3}{v}\kappa_{zz}. \tag{14}$$

The values for $\lambda_\|[\mathrm{FD}]/l_\perp$ and $\lambda_\|[\mathrm{SOQLT}]/l_\perp$ were computed using Eq. (1) from values for the pitch angle coefficients $D_{\mu\mu}(\mu)$ computed from the flux and derivative, and SOQLT (Eq. (9)) formulas. The values for $\lambda_\|[\mathrm{WNLT}]/l_\perp$ in Table 4 was computed using Eq. (11) from values for the pitch-angle coefficients $D_{\mu\mu}(\mu)$. The errors in Tables 2, 3 and 4 are relative errors computed from the formula

$$\lambda_\|[\cdot]error = \frac{\left|\lambda_\|[sim]/l_\perp - \lambda_\|[\cdot]/l_\perp\right|}{\lambda_\|[sim]/l_\perp}. \tag{15}$$

Table 2: Comparison of the values of parallel mean free path ($\lambda_\|$) computed from test-particle simulations with estimates $\lambda_\|[\mathrm{FD}]$.

b/B_0	$R_{\mathrm{L}}/l_\perp$	$\lambda_\|[sim]/l_\perp$	$\lambda_\|[\mathrm{FD}]/l_\perp$	$\lambda_\|[\mathrm{FD}]$ error
0.3	0.37	7478.19	7480.52	0.0003
0.3	1.93	560.19	561.27	0.0019
0.3	3.73	935.34	927.22	0.0087
0.3	10.0	7801.61	7837.17	0.0046
0.5	0.2	292.08	297.13	0.0173
0.5	0.5	154.36	154.30	0.0004
0.5	1.93	92.54	92.07	0.0051
0.5	10.0	1652.40	1650.90	0.0009
1.0	0.2	7.91	7.91	0.0006
1.0	0.5	9.23	9.23	0.0003
1.0	1.93	14.13	14.12	0.0005
1.0	10.0	272.59	272.62	0.0001

Table 3: Comparison of the values of parallel mean free path ($\lambda_\|$) computed from test-particle simulations with estimates $\lambda_\|[\mathrm{SOQLT}]$ based on Eq. (9)

b/B_0	$R_{\mathrm{L}}/l_\perp$	$\lambda_\|[\mathrm{SOQLT}]/l_\perp$	$\lambda_\|[\mathrm{SOQLT}]$ error
0.3	0.37	4.90×10^{14}	6.55×10^{10}
0.3	1.93	195.75	0.6506
0.3	3.73	394.68	0.5780
0.3	10.0	2503.04	0.6792
0.5	0.2	5.20×10^{17}	1.78×10^{15}
0.5	0.5	1738.47	10.2624
0.5	1.93	46.82	0.4940
0.5	10.0	837.05	0.4934
1.0	0.2	10208.36	1289.5638
1.0	0.5	5.71	0.3814
1.0	1.93	9.50	0.3277
1.0	10.0	211.14	0.2254

As shown by the small errors in the $\lambda_\|[\mathrm{FD}]$ error column in Table 2, the flux derivative method gives very good agreement with the test-particle simulations. These results show that the flux derivative method, which is based on interpreting $D_{\mu\mu}$ as a Fokker-Planck coefficient, gives very good results in all cases.

The SOQLT has been discussed by Reimer and Shalchi [9] who concluded that the SOQLT was not

Table 4: Comparison of the values of parallel mean free path ($\lambda_\parallel$) computed from test-particle simulations with estimates $\lambda_\parallel$[WNLT] based on Eq. (11)

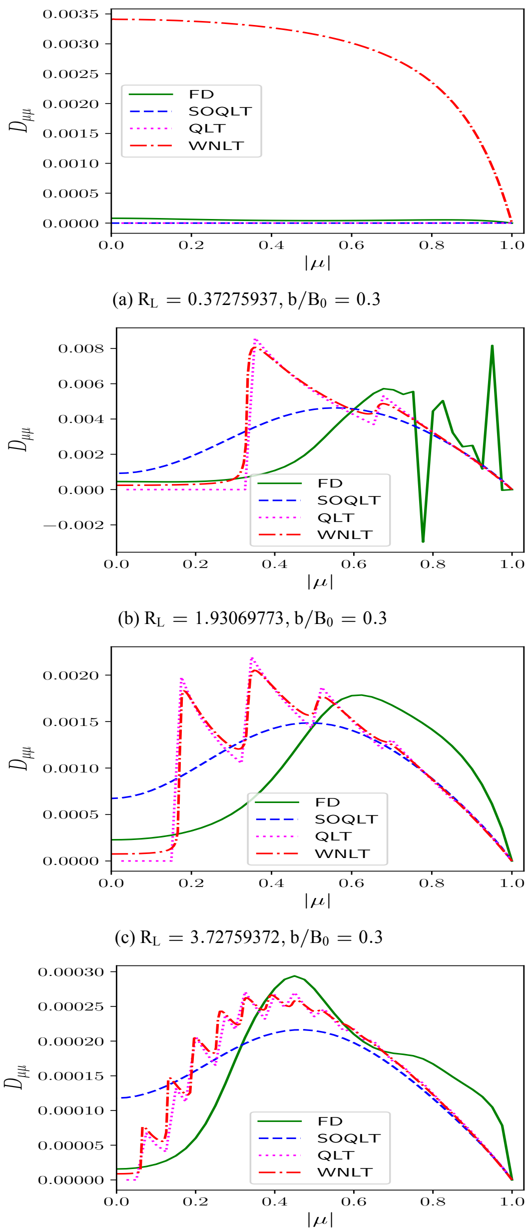
b/B_0	$R_L/l_\perp$	$\lambda_\parallel[\text{WNLT}]/l_\perp$	$\lambda_\parallel$[WNLT] error
0.3	0.37	124.33	0.9834
0.3	1.93	789.43	0.4090
0.3	3.73	1572.02	0.6807
0.3	10.0	7229.86	0.0733
0.5	0.2	13.65	0.9530
0.5	0.5	37.20	0.7590
0.5	1.93	134.82	0.4569
0.5	10.0	1473.36	0.1084
1.0	0.2	2.45	0.6903
1.0	0.5	5.09	0.4485
1.0	1.93	14.04	0.0064
1.0	10.0	215.46	0.2096

useful. They argued that as their theory predicts a dependence $\exp(B_0^2/b^2)$ (see Eq. (10)) it would give very large parallel mean free paths for $b < B_0$. However, their simulation results did not seem to have huge parallel mean free paths and therefore their conclusion that SOQLT was not useful seemed reasonable. In contrast, in the cases shown in Table 2, we usually consider small b/B_0 and our results show very large $\lambda_\parallel$ values when $b/B_0 < 0.5$. Also, our results are actually better than expected for SOQLT as the $\lambda_\parallel$[SOQLT] values are of a similar order of magnitude to the $\lambda_\parallel$[sim] values for all values of b/B_0, except in the small $R_L = 0.2,\ 0.37$ cases. A possible explanation for the results is that $R_L/l_\parallel < 2/\pi$ is the condition for the QLT non-resonant regime. The standard QLT would have $D_{\mu\mu} = 0$ for all μ and so infinite parallel mean free path (see Eq. (1)). In the SOQLT the QLT resonant wavenumber of the particles is broadened and this has the effect of smoothing the QLT result so that it looks more like the simulation result. However, the $R_L = 0.2$ cases are far below the QLT resonant scattering regime and therefore also the SOQLT regime. In this regime, the simulation result is determined by higher-order nonlinear scattering.

The WNLT results in Table 4 are for the special case of $\nu = 0$. In their recent paper on WNLT, Reimer and Shalchi [9] studied the dependence of the diffusion coefficients on ν for $0 \le \nu < 1$ and found that the WNLT results agreed well with the simulation values with suitable choices for ν.

The plots in Figs. 1, 2, 3 show the values of the pitch angle diffusion coefficients calculated from the FD, SOQLT, QLT and WNLT($\nu = 0$) methods for the 12 cases in Tables 2 and 4. In each figure, panels (a)-(l) show plots of $D_{\mu\mu}(\mu)$ values calculated from the diffusion equation integration method $D_{\mu\mu}$[FD] (solid lines) ([2, 7]), the second-order quasilinear theory $D_{\mu\mu}$[SOQLT] (dashed lines), the quasilinear theory $D_{\mu\mu}$[QL] (dotted lines) and the weakly nonlinear theory with $\nu = 0$ $D_{\mu\mu}$[WNLT] (dot-dash lines).

For the QLT non-resonant small R_L cases shown in panels (a) and (e), $D_{\mu\mu}$[WNLT] deviates significantly from $D_{\mu\mu}$[FD].



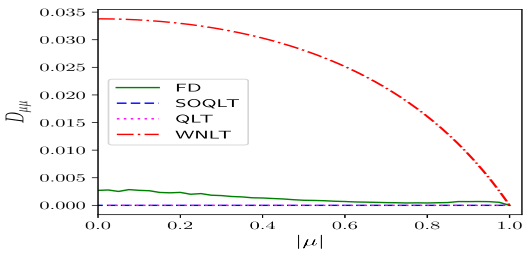
(a) $\text{R}_\text{L} = 0.37275937, \text{b}/\text{B}_0 = 0.3$

(b) $\text{R}_\text{L} = 1.93069773, \text{b}/\text{B}_0 = 0.3$

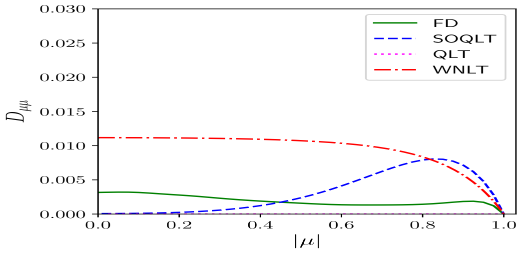
(c) $\text{R}_\text{L} = 3.72759372, \text{b}/\text{B}_0 = 0.3$

(d) $\text{R}_\text{L} = 10.0, \text{b}/\text{B}_0 = 0.3$

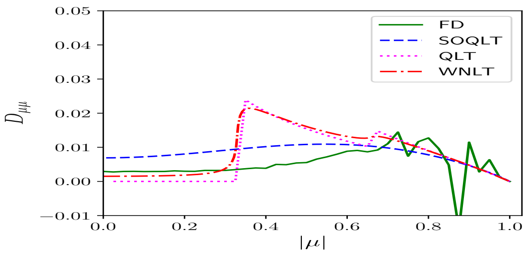
Figure 1: Plots of pitch-angle diffusion coefficients in case $b/B_0 = 0.3$. $D_{\mu\mu}$[FD] (solid lines), $D_{\mu\mu}$[SOQLT] (dashed lines), $D_{\mu\mu}$[QLT] (dotted lines) and $D_{\mu\mu}[\text{WNLT}(\nu = 0)]$ (dash-dot lines) (see Tables 2, 4)



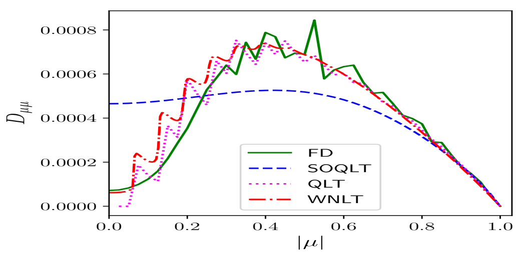
(e) $R_L = 0.2, b/B_0 = 0.5$



(f) $R_L = 0.5, b/B_0 = 0.5$

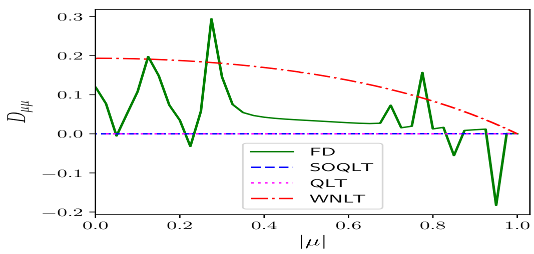


(g) $R_L = 1.93069773, b/B_0 = 0.5$

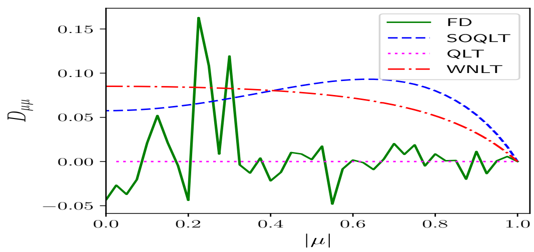


(h) $R_L = 10.0, b/B_0 = 0.5$

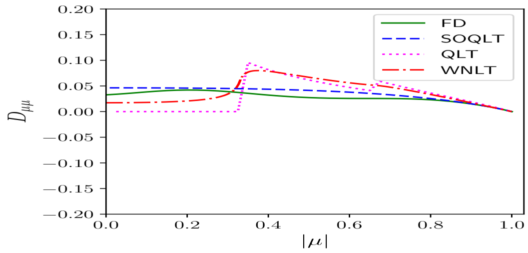
Figure 2: Plots of pitch-angle diffusion coefficients in case $b/B_0 = 0.5$. $D_{\mu\mu}$[FD] (solid lines), $D_{\mu\mu}$[SOQLT] (dashed lines), $D_{\mu\mu}$[QLT] (dotted lines) and $D_{\mu\mu}[\text{WNLT}(\nu = 0)]$ (dash-dot lines) (see Tables 2, 4)



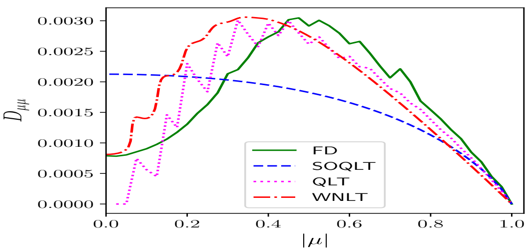
(i) $R_L = 0.2, b/B_0 = 1.0$



(j) $R_L = 0.5, b/B_0 = 1.0$



(k) $R_L = 1.93069773, b/B_0 = 1.0$



((l) $R_L = 10.0, b/B_0 = 1.0$

Figure 3: Plots of pitch-angle diffusion coefficients in case $b/B_0 = 1.0$. $D_{\mu\mu}$[FD] (solid lines), $D_{\mu\mu}$[SOQLT] (dashed lines), $D_{\mu\mu}$[QLT] (dotted lines) and $D_{\mu\mu}[\text{WNLT}(\nu = 0)]$ (dash-dot lines) (see Tables 2, 4)

6 Discussion

In this work we have compared different theories for calculating the pitch-angle diffusion coefficient, $D_{\mu\mu}(\mu)$, with test-particle simulation results for NRMHD turbulence and assessed their ability to predict the parallel mean free path $\lambda_{\|}$ using Eq. (1). As was seen in previous work [7], we find that $D_{\mu\mu}$[FD] (from test-particle simulations) provides $\lambda_{\|}$ in this way that is very close to the value obtained via Eq. (2) in simulations, as shown in Table 2 (i.e. the differences between the third and fourth columns are very small). On this basis, we take $D_{\mu\mu}$[FD] as reference value that theories should try to replicate.

The weakly non-linear theory (WNLT), as in Eq. 11, was found to give $\lambda_{\|}$ closest to simulations when taking a $\mu-$independent $D_{\perp}$, with $\nu = 0$. This does not seem to be consistent with $D_\perp$ used in theory [12] or found in previous simulations [8]. Further study into an appropriate form of $D_\perp$ could help us to understand this discrepancy. WNLT only seems to work well for $R_{\rm L}/l_\perp \lesssim 2/\pi \approx 0.64$, which is the quasilinear limit of parallel resonant scattering. Below this limit, $D_{\mu\mu}$[WNLT] seems too large when compared with $D_{\mu\mu}$[FD], especially near $\mu = 0$, as seen in Fig. 2(a) and Fig. 2(b), so the parallel mean free path is underestimated. When $R_{\rm L}$ is large, as in Figs. 1(d), 2(d) and 3(d), the WNLT theory shifts the QLT value up near $\mu = 0$ to a value close to the FD result, so does very well with predicting $\lambda_\parallel$.

The second-order quasilinear theory (SOQLT), calculated here in detail for the first time for NRMHD turbulence, gives $\lambda_\parallel$ to within a factor of a few for large $R_{\rm L}$ and is still quite accurate for $R_{\rm L} = 0.5$ when $b/B_0 = 1$ (where we might expect the theory to work). In other cases, particularly at small $R_{\rm L}$, the results are unacceptably large, as shown in Table 4. Here we have adopted the same form of SOQLT as in [9], who concluded that the SOQLT was not useful. They argued that as their theory predicts a dependence $\exp(B_0^2/b^2)$ (see Eq. (10)) it would give very large parallel mean free paths for $b < B_0$. However, their simulation results did not seem to have huge parallel mean free paths and therefore their conclusion that SOQLT was not useful seemed reasonable. It is possible an alternative, and more complicated, variant of the theory from [11] could be applied to give greater accuracy.

In summary, for the parameters used here, at least one of the theories for $D_{\mu\mu}$ will provide a $\lambda_\parallel$ value within a factor of a few of the simulation value, provided $R_{\rm L}$ is not too much smaller that $1/K$, the quasilinear resonance limit. Although the original QLT of Eq. 8 provides $D_{\mu\mu}$ of similar shape to the simulations, it always yields an infinite parallel mean free path due to the troublesome resonance gap that NMRHD turbulence presents. The two other theories presented here broaden the resonance and close up the gap, but not in a way that is broadly consistent with the test-particle simulations. Thus NRMHD turbulence remains a challenge for parallel diffusion theories. Since WNLT looks promising, a detailed examination of $D_\perp$ in computer simulations may provide clues on how to proceed.

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Contribution of individual authors to the creation of a scientific article

Chanidaporn Pleumpreedaporn performed test particle simulations, evaluated theories and analysed results.
Andrew P. Snodin contributed to writing and testing the numerical tools, analysis of results and writing the text.
Elvin J. Moore assisted with the writing and the preparation of the final version.

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