$\theta(\star)$-quasi continuity for multifunctions

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Abstract: Our purpose is to introduce the concepts of upper and lower $\theta(\star)$-quasi continuous multifunctions. Several characterizations of upper and lower $\theta(\star)$-quasi continuous multifunctions are investigated.

Key–Words: upper $\theta(\star)$-quasi continuous multifunction, lower $\theta(\star)$-quasi continuous multifunction

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1 Introduction

The notion of continuity is an important concept in general topology as well as all branches of mathematics. This concept has been extended to the setting multifunctions and has been generalized by weaker forms of open sets such as α-open sets [15], semi-open sets [13], preopen sets [14], β-open sets [2] or semi-preopen sets [4]. Levine [13] introduced the concept of semi-continuity in topological spaces. Arya and Bhamini [5] introduced and studied the concept of θ-semi-continuity as a generalization of semi-continuity. Noiri [18] and Jafari and Noiri [10] have further investigated several characterizations of θ-semi-continuity. Popa and Noiri [19] introduced and studied the notions of upper and lower θ-quasi continuous multifunctions. The present authors [16] obtained new characterizations of upper and lower θ-quasi continuous multifunctions. Topological ideals have played an important role in topology. The concept of ideal topological spaces was introduced and studied by Kuratowski [12] and Vaidyanathaswamy [20]. Janković and Hamlett [11] introduced the notion of $\mathscr{I}$-open sets in ideal topologial spaces. Abd El-Monsef et al. [1] further investigated $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions. Later, several authors studied ideal topological spaces giving several convenient definitions. Some authors obtained decompositions of continuity. For instance, Açikgöz et al. [3] introduced and investigated the notions of weakly-$\mathscr{I}$-continuous and weak*-$\mathscr{I}$-continuous functions in ideal topological spaces. Hatir and Noiri [9] introduced the notions of semi-$\mathscr{I}$-open sets, α-$\mathscr{I}$-open sets and β-$\mathscr{I}$-open sets via idealization and using these sets obtained new decompositions of continuity. In [7], the author introduced and investigated the concepts of almost quasi $\star$-continuous multifunctions and weakly quasi $\star$-continuous multifunctions. The purpose of the present paper is to introduce the notions of upper and lower $\theta(\star)$-quasi continuous multifunctions. Moreover, several characterizations of $\theta(\star)$-quasi continuous multifunctions are discussed.

2 Preliminaries

Throughout the present paper, spaces (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space (X,τ). The closure of A and the interior of A are denoted by Cl(A) and Int(A), respectively. An ideal $\mathscr{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathscr{I}$ and $B \subseteq A$ imply $B \in \mathscr{I}$; (2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply $A \cup B \in \mathscr{I}$. A topological space (X,τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X,\tau,\mathscr{I})$. For an ideal topological space $(X,\tau,\mathscr{I})$ and a subset A of X, $A^\star(\mathscr{I})$ is defined as follows:

$$A^\star(\mathscr{I}) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every open neighbourhood } U \text{ of } x\}.$$

In case there is no chance for confusion, $A^\star(\mathscr{I})$ is simply written as $A^\star$. In [12], $A^\star$ is called the local function of A with respect to $\mathscr{I}$ and τ. Observe additionally that Cl$^\star(A) = A^\star \cup A$ defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I})$ finer than τ, generated by the base

$$\mathscr{B}(\mathscr{I},\tau) = \{U - I \mid U \in \tau \text{ and } I \in \mathscr{I}\}.$$

However, $\mathscr{B}(\mathscr{I},\tau)$ is not always a topology [20]. A subset A is said to be $\star$-closed [11] if $A^\star \subseteq A$. The interior of a subset A in $(X, \tau^\star(\mathscr{I}))$ is denoted by $\text{Int}^\star(A)$.

A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ called *R-$\mathscr{I}^\star$-open* (resp. *$\mathscr{I}^\star$-preopen*) [8] if $A = \text{Int}^\star(\text{Cl}^\star(A))$ (resp. $A \subseteq \text{Int}^\star(\text{Cl}^\star(A))$). The complement of a R-$\mathscr{I}^\star$-open (resp. $\mathscr{I}^\star$-preopen) set is called *R-$\mathscr{I}^\star$-closed* (resp. *$\mathscr{I}^\star$-preclosed*). A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is called *semi-$\mathscr{I}^\star$-open* (resp. *semi-$\mathscr{I}^\star$-preopen*) [7] if $A \subseteq \text{Cl}^\star(\text{Int}^\star(A))$ (resp. $A \subseteq \text{Cl}^\star(\text{Int}^\star(\text{Cl}^\star(A)))$). The complement of a semi-$\mathscr{I}^\star$-open (resp. semi-$\mathscr{I}^\star$-preopen) set is called *semi-$\mathscr{I}^\star$-closed* (resp. *semi-$\mathscr{I}^\star$-preclosed*). For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the intersection of all semi-$\mathscr{I}^\star$-closed sets containing A is called the *semi-$\mathscr{I}^\star$-closure* [7] of A and is denoted by $s\text{Cl}_{\mathscr{I}^\star}(A)$. The union of all semi-$\mathscr{I}^\star$-open sets contained in A is called the *semi-$\mathscr{I}^\star$-interior* [7] of A and is denoted by $s\text{Int}_{\mathscr{I}^\star}(A)$.

Lemma 1. [7] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, $x \in sCl_{\mathscr{I}^\star}(A)$ if and only if*

$$U \cap A \neq \emptyset$$

for every semi-$\mathscr{I}^\star$-open set U containing x.

Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. A point x in an ideal topological space $(X, \tau, \mathscr{I})$ is called a *$\star_\theta$-cluster point* [7] of A if $\text{Cl}^\star(U) \cap A \neq \emptyset$ for every $\star$-open set U of X containing x. The set of all $\star_\theta$-cluster points of A is called the *$\star_\theta$-closure* [7] of A and is denoted by $\star_\theta\text{Cl}(A)$. A subset B of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *$\star_\theta$-closed* if $\star_\theta\text{Cl}(B) = B$. The complement of a $\star_\theta$-closed set is said to be *$\star_\theta$-open*.

Lemma 2. [7] *For a subset A of an ideal topological space $(X, \tau, \mathscr{I})$, the following properties hold:*

(1) If A is $\star$-open in X, then $Cl^\star(A) = \star_\theta Cl(A)$.

(2) $\star_\theta Cl(A)$ is $\star$-closed in X.

The family of all semi-$\mathscr{I}^\star$-open sets of an ideal topological space $(X, \tau, \mathscr{I})$ is denoted by $S_{\mathscr{I}^\star}O(X)$. Let A be a subset of an ideal topological space $(X, \tau, \mathscr{I})$. The *semi-$\theta(\star)$-closure* of A, $\star_\theta s\text{Cl}(A)$ and the *semi-$\theta(\star)$-interior* of A, $\star_\theta s\text{Int}(A)$ are defined as follows:

$$\star_\theta s\text{Cl}(A) = \{x \in X \mid A \cap s\text{Cl}_{\mathscr{I}^\star}(U) \neq \emptyset \text{ for every } U \in S_{\mathscr{I}^\star}O(X, x)\},$$

$$\star_\theta s\text{Int}(A) = \{x \in X \mid s\text{Cl}_{\mathscr{I}^\star}(U) \subseteq A \text{ for some } U \in S_{\mathscr{I}^\star}O(X, x)\}$$

where

$$S_{\mathscr{I}^\star}O(X, x) = \{U \mid x \in U \text{ and } U \in S_{\mathscr{I}^\star}O(X)\}.$$

A subset A of an ideal topological space $(X, \tau, \mathscr{I})$ is called *semi-$\theta(\star)$-closed* if $A = \star_\theta s\text{Cl}(A)$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [6] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X \mid F(x) \subseteq B\}$$

and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x \in A} F(x)$. Then F is said to be surjection if $F(X) = Y$, or equivalent, if for each $y \in Y$ there exists $x \in X$ such that $y \in F(x)$ and F is called injection if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

3 Characterizations of upper and lower $\theta(\star)$-quasi continuous multifunctions

In this section, we introduce the notions of upper and lower $\theta(\star)$-quasi continuous multifunctions. Moreover, some characterizations of upper and lower $\theta(\star)$-quasi continuous multifunctions are discussed.

Definition 3. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be:*

(1) upper $\theta(\star)$-quasi continuous if, for each $x \in X$ and each $\star$-open set V of Y containing $F(x)$, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $F(sCl_{\mathscr{I}^\star}(U)) \subseteq Cl^\star(V)$;

(2) lower $\theta(\star)$-quasi continuous if, for $x \in X$ and each $\star$- open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $F(z) \cap Cl^\star(V) \neq \emptyset$ for every $z \in sCl_{\mathscr{I}^\star}(U)$.

Example 4. *Let $X = \{1, 2, 3\}$ with a topology $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{3\}\}$. Let $Y = \{a, b, c\}$ with a topology $\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$ and an ideal $\mathscr{J} = \{\emptyset, \{c\}\}$. A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is defined as follows: $F(1) = \{a\}$, $F(2) = \{c\}$ and $F(3) = \{a, b\}$. Then, F is upper $\theta(\star)$-quasi continuous.*

Example 5. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, \{c\}, X\}$ and an ideal $\mathscr{I} = \{\emptyset, \{c\}\}$. Let $Y = \{-1, 0, 1, 2\}$ with a topology $\sigma = \{\emptyset, \{-1, 0\}, \{1, 2\}, Y\}$ and an ideal $\mathscr{J} = \{\emptyset\}$. Define a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ as follows: $F(a) = \{-1\}$, $F(b) = \{0\}$ and $F(c) = \{1, 2\}$. Then, F is lower $\theta(\star)$-quasi continuous.*

The following theorem gives some characterizations of upper $\theta(\star)$-quasi continuous multifunctions.

Theorem 6. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is upper $\theta(\star)$-quasi continuous;

(2) $\star_\theta s\mathrm{Cl}(F^-(\mathrm{Int}^\star(\star_\theta \mathrm{Cl}(B)))) \subseteq F^-(\star_\theta \mathrm{Cl}(B))$ for every subset B of Y;

(3) $\star_\theta s\mathrm{Cl}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \subseteq F^-(\mathrm{Cl}^\star(V))$ for every $\star$-open set V of Y;

(4) $\star_\theta s\mathrm{Cl}(F^-(\mathrm{Int}^\star(K))) \subseteq F^-(K)$ for every R-$\mathscr{J}^\star$-closed set K of Y;

(5) $F^+(V) \subseteq \star_\theta s\mathrm{Int}(F^+(\mathrm{Cl}^\star(V)))$ for every $\star$-open set V of Y;

(6) $\star_\theta s\mathrm{Cl}(F^-(\mathrm{Int}^\star(K))) \subseteq F^-(K)$ for every $\star$-closed set K of Y;

(7) $\star_\theta s\mathrm{Cl}(F^-(V)) \subseteq F^-(\mathrm{Cl}^\star(V))$ for every $\star$-open set V of Y.

Proof. $(1) \Rightarrow (2)$: Let B be any subset of Y. Suppose that $x \notin F^-(\star_\theta \mathrm{Cl}(B))$. Then, $x \in X - F^-(\star_\theta \mathrm{Cl}(B))$ and $F(x) \subseteq Y - \star_\theta \mathrm{Cl}(B)$. Since $\star_\theta \mathrm{Cl}(B)$ is $\star$-closed in Y, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that

$$F(s\mathrm{Cl}_{\mathscr{I}^\star}(U)) \subseteq \mathrm{Cl}^\star(Y - \star_\theta \mathrm{Cl}(B)) = Y - \mathrm{Int}^\star(\star_\theta \mathrm{Cl}(B)).$$

Thus, $F(s\mathrm{Cl}_{\mathscr{I}^\star}(U)) \cap \mathrm{Int}^\star(\star_\theta \mathrm{Cl}(B)) = \emptyset$ and hence $s\mathrm{Cl}_{\mathscr{I}^\star}(U) \cap F^-(\mathrm{Int}^\star(\star_\theta \mathrm{Cl}(B))) = \emptyset$. This shows that $x \notin \star_\theta s\mathrm{Cl}(F^-(\mathrm{Int}^\star(\star_\theta \mathrm{Cl}(B))))$. Consequently, we obtain $\star_\theta s\mathrm{Cl}(F^-(\mathrm{Int}^\star(\star_\theta \mathrm{Cl}(B)))) \subseteq F^-(\mathrm{Cl}_{\theta^\star}(B))$.

$(2) \Rightarrow (3)$: This is obvious since $\mathrm{Cl}^\star(V) = \star_\theta \mathrm{Cl}(V)$ for every $\star$-open set V of Y.

$(3) \Rightarrow (4)$: Let K be any R-$\mathscr{J}^\star$-closed set of Y. By (3), we have

$$\begin{aligned}
&\star_\theta\ s\mathrm{Cl}(F^-(\mathrm{Int}^\star(K))) \\
&= \star_\theta s\mathrm{Cl}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(\mathrm{Int}^\star(K))))) \\
&\subseteq F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(K))) \\
&= F^-(K).
\end{aligned}$$

$(4) \Rightarrow (5)$: Let V be any $\star$-open set of Y. Then, we have

$$\begin{aligned}
X - &\star_\theta s\mathrm{Int}(F^+(\mathrm{Cl}^\star(V))) \\
&= \star_\theta s\mathrm{Cl}(X - F^+(\mathrm{Cl}^\star(V))) \\
&= \star_\theta s\mathrm{Cl}(F^-(Y - \mathrm{Cl}^\star(V))),
\end{aligned}$$

$$\begin{aligned}
Y - \mathrm{Cl}^\star(V) &= \mathrm{Int}^\star(Y - \mathrm{Cl}^\star(V)) \\
&\subseteq \mathrm{Int}^\star(Y - \mathrm{Int}^\star(\mathrm{Cl}^\star(V)))
\end{aligned}$$

and $Y - \mathrm{Int}^\star(\mathrm{Cl}^\star(V))$ is R-$\mathscr{J}^\star$-closed in Y. By (4),

$$\begin{aligned}
&\star_\theta\ s\mathrm{Cl}(F^-(\mathrm{Int}^\star(Y - \mathrm{Int}^\star(\mathrm{Cl}^\star(V))))) \\
&\subseteq F^-(Y - \mathrm{Int}^\star(\mathrm{Cl}^\star(V))) \\
&= X - F^+(\mathrm{Int}^\star(\mathrm{Cl}^\star(V))) \\
&\subseteq X - F^+(V).
\end{aligned}$$

Therefore, $F^+(V) \subseteq \star_\theta s\mathrm{Int}(F^+(\mathrm{Cl}^\star(V)))$.

$(5) \Rightarrow (6)$: Let K be any $\star$-closed set of Y. By (5), we have

$$\begin{aligned}
X - F^-(K) &= F^+(Y - K) \\
&\subseteq \star_\theta s\mathrm{Int}(F^+(\mathrm{Cl}^\star(Y - K))) \\
&= \star_\theta s\mathrm{Int}(F^+(Y - \mathrm{Int}^\star(K))) \\
&= \star_\theta s\mathrm{Int}(X - F^-(\mathrm{Int}^\star(K))) \\
&= X - \star_\theta s\mathrm{Cl}(F^-(\mathrm{Int}^\star(K)))
\end{aligned}$$

and hence $\star_\theta s\mathrm{Cl}(F^-(\mathrm{Int}^\star(K))) \subseteq F^-(K)$.

$(6) \Rightarrow (7)$: Let V be any $\star$-open set of Y. Then, we have $\mathrm{Cl}^\star(V)$ is $\star$-closed and by (6),

$$\begin{aligned}
\star_\theta s\mathrm{Cl}(F^-(V)) &\subseteq \star_\theta s\mathrm{Cl}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \\
&\subseteq F^-(\mathrm{Cl}^\star(V)).
\end{aligned}$$

$(7) \Rightarrow (1)$: Let $x \in X$ and V be any $\star$-open set of Y containing $F(x)$. Then, we have

$$F(x) \cap \mathrm{Cl}^\star(Y - \mathrm{Cl}^\star(V)) = \emptyset$$

and hence $x \notin F^-(\mathrm{Cl}^\star(Y - \mathrm{Cl}^\star(V)))$. It follows from (7) that $x \notin \star_\theta s\mathrm{Cl}(F^-(Y - \mathrm{Cl}^\star(V)))$. Then, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $s\mathrm{Cl}_{\mathscr{I}^\star}(U) \cap F^-(Y - \mathrm{Cl}^\star(V)) = \emptyset$; hence $F(s\mathrm{Cl}_{\mathscr{I}^\star}(U)) \subseteq \mathrm{Cl}^\star(V)$. This shows that F is upper $\theta(\star)$-quasi continuous. □

An ideal topological space $(X, \tau, \mathscr{I})$ is said to be *$\mathscr{I}^\star$-compact* [8] if every cover of X by $\star$-open sets of X has a finite subcover. A subset K of an ideal topological space $(X, \tau, \mathscr{I})$ is said to be *$\mathscr{I}^\star$-compact* [8] if every cover of K by $\star$-open sets has a finite subcover.

Definition 7. *An ideal topological space $(X, \tau, \mathscr{I})$ is called quasi $\mathscr{H}^\star$-closed (resp. $s^\star$-closed) if, for every $\star$-open (resp. semi-$\mathscr{I}^\star$-open) cover $\{V_\alpha \mid \alpha \in \nabla\}$ of X, there exists a finite subset ∇_0 of ∇ such that*

$$X = \cup\{Cl^\star(V_\alpha) \mid \alpha \in \nabla_0\}$$

(resp. $X = \cup\{sCl_{\mathscr{I}^\star}(V_\alpha) \mid \alpha \in \nabla_0\}$).

Theorem 8. *Let $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ be a surjective multifunction and $F(x)$ be $\mathscr{J}^\star$-compact for each $x \in X$. If F is upper $\theta(\star)$-quasi continuous and $(X, \tau, \mathscr{I})$ is $s^\star$-closed, then $(Y, \sigma, \mathscr{J})$ is quasi $\mathscr{H}^\star$-closed.*

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any $\star$-open cover of Y and let $x \in X$. Since $F(x)$ is $\mathscr{J}^\star$-compact, there exists a finite subset $\nabla(x)$ of ∇ such that

$$F(x) \subseteq \cup\{V_\alpha \mid \alpha \in \nabla(x)\}.$$

Put $V(x) = \cup\{V_\alpha \mid \alpha \in \nabla(x)\}$, then $F(x) \subseteq V(x)$ and $V(x)$ is $\star$-open in Y. Since F is upper $\theta(\star)$-quasi continuous, there exists a semi-$\mathscr{I}^\star$-open set $U(x)$ of X containing x such that

$$F(s\mathrm{Cl}_{\mathscr{I}^\star}(U(x))) \subseteq \mathrm{Cl}^\star(V(x)).$$

The family $\{U(x) \mid x \in X\}$ is a semi-$\mathscr{I}^\star$-open cover of X. Since $(X, \tau, \mathscr{I})$ is $s^\star$-closed, there exists a finite number of points, say, $x_1, x_2, ..., x_n \in X$ such that $X = \cup\{s\mathrm{Cl}_{\mathscr{I}^\star}(U(x_i)) \mid i = 1, 2, ..., n\}$. Since F is surjective, we obtain

$$\begin{aligned} Y = F(X) &= \overset{n}{\underset{i=1}{\cup}} F(s\mathrm{Cl}_{\mathscr{I}^\star}(U(x_i))) \\ &\subseteq \overset{n}{\underset{i=1}{\cup}} \mathrm{Cl}^\star(V(x_i)) \\ &= \overset{n}{\underset{i=1}{\cup}} \cup_{\alpha \in \nabla(x_i)} \mathrm{Cl}^\star(V_\alpha). \end{aligned}$$

This shows that $(Y, \sigma, \mathscr{J})$ is quasi $\mathscr{H}^\star$-closed. □

Lemma 9. *If $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is lower $\theta(\star)$-quasi continuous, then for each $x \in X$ and each subset B of Y such that $F(x) \cap \star_\theta Int(B) \neq \emptyset$, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $sCl_{\mathscr{I}^\star}(U) \subseteq F^-(B)$.*

Proof. Suppose that $F(x) \cap \star_\theta \mathrm{Int}(B) \neq \emptyset$, there exists a $\star$-open set V of Y such that $V \subseteq \mathrm{Cl}^\star(V) \subseteq B$ and $F(x) \cap V \neq \emptyset$. Since F is lower $\theta(\star)$-quasi continuous, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $F(z) \cap \mathrm{Cl}^\star(V) \neq \emptyset$ for every $z \in s\mathrm{Cl}_{\mathscr{I}^\star}(U)$ and hence $s\mathrm{Cl}_{\mathscr{I}^\star}(U) \subseteq F^-(B)$. □

The following theorem gives some characterizations of lower $\theta(\star)$-quasi continuous multifunctions.

Theorem 10. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is lower $\theta(\star)$-quasi continuous;

(2) $\star_\theta sCl(F^+(B)) \subseteq F^+(\star_\theta Cl(B))$ for every subset B of Y;

(3) $\star_\theta sCl(F^+(V)) \subseteq F^+(Cl^\star(V))$ for every $\star$-open set V of Y;

(4) $F^-(V) \subseteq \star_\theta sInt(F^-(Cl^\star(V)))$ for every $\star$-open set V of Y;

(5) $F(\star_\theta sCl(A)) \subseteq \star_\theta Cl(F(A))$ for every subset A of X;

(6) $\star_\theta sCl(F^+(Int^\star(\star_\theta Cl(B)))) \subseteq F^+(\star_\theta Cl(B))$ for every subset B of Y;

(7) $\star_\theta sCl(F^+(Int^\star(Cl^\star(V)))) \subseteq F^+(Cl^\star(V))$ for every $\star$-open set V of Y;

(8) $\star_\theta sCl(F^+(Int^\star(K))) \subseteq F^+(K)$ for every R-$\mathscr{J}^\star$- closed set K of Y;

(9) $\star_\theta sCl(F^+(Int^\star(K))) \subseteq F^+(K)$ for every $\star$-closed set K of Y.

Proof. (1) ⇒ (2): Let B be any subset of Y. Suppose that $x \notin F^+(\star_\theta \mathrm{Cl}(B))$. Then, we have

$$x \in F^-(Y - \star_\theta \mathrm{Cl}(B)) = F^-(\star_\theta \mathrm{Int}(Y - B)).$$

Since F is lower $\theta(\star)$-continuous, by Lemma 9, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $s\mathrm{Cl}_{\mathscr{I}^\star}(U) \subseteq F^-(Y - B) = X - F^+(B)$. Thus, $s\mathrm{Cl}_{\mathscr{I}^\star}(U) \cap F^+(B) = \emptyset$ and hence

$$x \notin \star_\theta s\mathrm{Cl}(F^+(B)).$$

(2) ⇒ (3): This is obvious sine $\mathrm{Cl}^\star(V) = \star_\theta \mathrm{Cl}(V)$ for every $\star$-open set V of Y.

(3) ⇒ (4): Let V be any $\star$-open set of Y. By (3),

$$\begin{aligned} & X - \star_\theta s\mathrm{Int}(F^-(\mathrm{Cl}^\star(V))) \\ &= \star_\theta s\mathrm{Cl}(X - F^-(\mathrm{Cl}^\star(V))) \\ &= \star_\theta s\mathrm{Cl}(F^+(Y - \mathrm{Cl}^\star(V))) \\ &\subseteq F^+(\mathrm{Cl}^\star(Y - \mathrm{Cl}^\star(V))) \\ &\subseteq F^+(\mathrm{Cl}^\star(Y - V)) \\ &= F^+(Y - V) \\ &= X - F^-(V). \end{aligned}$$

Thus, $F^-(V) \subseteq \star_\theta s\mathrm{Int}(F^-(\mathrm{Cl}^\star(V)))$.

(4) ⇒ (1): Let $x \in X$ and V be any $\star$-open set such that $F(x) \cap V \neq \emptyset$. Then, we have

$$x \in F^-(V) \subseteq \star_\theta s\mathrm{Int}(F^-(\mathrm{Cl}^\star(V))).$$

Therefore, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $s\text{Cl}_{\mathscr{I}^\star}(U) \subseteq F^-(\text{Cl}^\star(V))$; hence $F(z) \cap \text{Cl}^\star(V) \neq \emptyset$ for every $z \in s\text{Cl}_{\mathscr{I}^\star}(U)$. This shows that F is lower $\theta(\star)$-quasi continuous.

(2) $\Rightarrow$ (5): Let A be any subset of X. Replacing B in (2) by $F(A)$, we have

$$\begin{aligned}\star_\theta s\text{Cl}(A) &\subseteq \star_\theta s\text{Cl}(F^+(F(A)))\\ &\subseteq F^+(\star_\theta\text{Cl}(F(A)))\end{aligned}$$

and hence $F(\star_\theta s\text{Cl}(A)) \subseteq \star_\theta\text{Cl}(F(A))$.

(5) $\Rightarrow$ (2): Let B be any subset of Y. Replacing A in (5) by $F^+(B)$, we have $F(\star_\theta s\text{Cl}(F^+(B))) \subseteq \star_\theta\text{Cl}(F(F^+(B))) \subseteq \star_\theta\text{Cl}(B)$. Thus, $\star_\theta s\text{Cl}(F^+(B)) \subseteq F^+(\star_\theta\text{Cl}(B))$.

(3) $\Rightarrow$ (6): Let B be any subset of Y. Put $V = \text{Int}^\star(\star_\theta\text{Cl}(B))$ in (3). Then, since $\star_\theta\text{Cl}(B)$ is $\star$-closed in Y, we have

$$\begin{aligned}&\star_\theta\, s\text{Cl}(F^+(\text{Int}^\star(\star_\theta\text{Cl}(B))))\\ &\subseteq F^+(\text{Cl}^\star(\text{Int}^\star(\star_\theta\text{Cl}(B))))\\ &\subseteq F^+(\star_\theta\text{Cl}(B)).\end{aligned}$$

(6) $\Rightarrow$ (7): This is obvious since $\text{Cl}^\star(V) = \star_\theta\text{Cl}(V)$ for every $\star$-open set V of Y.

(7) $\Rightarrow$ (8): Let K be any R-$\mathscr{J}^\star$-closed set of Y. By (7), we have

$$\begin{aligned}&\star_\theta\, s\text{Cl}(F^+(\text{Int}^\star(K)))\\ &= \star_\theta s\text{Cl}(F^+(\text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(K)))))\\ &\subseteq F^+(\text{Cl}^\star(\text{Int}^\star(K)))\\ &= F^+(K).\end{aligned}$$

(8) $\Rightarrow$ (9): Let K be any $\star$-closed set of Y. Since $\text{Cl}^\star(\text{Int}^\star(K))$ is R-$\mathscr{J}^\star$-closed in Y and by (8),

$$\begin{aligned}&\star_\theta\, s\text{Cl}(F^+(\text{Int}^\star(K)))\\ &= \star_\theta s\text{Cl}(F^+(\text{Int}^\star(\text{Cl}^\star(\text{Int}^\star(K)))))\\ &\subseteq F^+(\text{Cl}^\star(\text{Int}^\star(K)))\\ &\subseteq F^+(K).\end{aligned}$$

(9) $\Rightarrow$ (4): Let V be any $\star$-open set of Y. Then, $Y - V$ is $\star$-closed in Y and by (9),

$$\begin{aligned}\star_\theta s\text{Cl}(F^+(\text{Int}^\star(Y-V))) &\subseteq F^+(Y-V)\\ &= X - F^-(V).\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\star_\theta\, s\text{Cl}(F^+(\text{Int}^\star(Y-V)))\\ &= \star_\theta s\text{Cl}(F^+(Y - \text{Cl}^\star(V)))\\ &= \star_\theta s\text{Cl}(X - F^-(\text{Cl}^\star(V)))\\ &= X - \star_\theta s\text{Int}(F^-(\text{Cl}^\star(V)))\end{aligned}$$

and hence $F^-(V) \subseteq \star_\theta s\text{Int}(F^-(\text{Cl}^\star(V)))$. □

Definition 11. *A function $(X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is said to be $\theta(\star)$-quasi continuous if, for each $x \in X$ and each $\star$-open set V of Y containing $f(x)$, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $f(\star_\theta sCl(U)) \subseteq Cl^\star(V)$.*

Corollary 12. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) f is $\theta(\star)$-quasi continuous;

(2) $\star_\theta sCl(f^{-1}(B)) \subseteq f^{-1}(\star_\theta Cl(B))$ for every subset B of Y;

(3) $\star_\theta sCl(f^{-1}(V)) \subseteq f^{-1}(Cl^\star(V))$ for every $\star$-open set V of Y;

(4) $f^{-1}(V) \subseteq \star_\theta sInt(f^{-1}(Cl^\star(V)))$ for every $\star$-open set V of Y;

(5) $f(\star_\theta sCl(A)) \subseteq \star_\theta Cl(f(A))$ for every subset A of X.

For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, a multifunction $s\text{Cl}_{\mathscr{J}^\star} F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is defined as follows: $(s\text{Cl}_{\mathscr{J}^\star}F)(x) = s\text{Cl}_{\mathscr{J}^\star}(F(x))$ for each $x \in X$.

Lemma 13. *Let $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ be a multifunction. Then, $(sCl_{\mathscr{J}^\star}F)^-(V) = F^-(V)$ for each semi-$\mathscr{J}^\star$-open set V of Y.*

Proof. Suppose that V is any semi-$\mathscr{J}^\star$-open set of Y. Let $x \in (s\text{Cl}_{\mathscr{J}^\star}F)^-(V)$. Then,

$$s\text{Cl}_{\mathscr{J}^\star}(F(x)) \cap V \neq \emptyset$$

and so $F(x) \cap V \neq \emptyset$. Thus, $x \in F^-(V)$ and hence $(s\text{Cl}_{\mathscr{J}^\star}F)^-(V) \subseteq F^-(V)$. On the other had, let $x \in F^-(V)$. Then, we have $\emptyset \neq F(x) \cap V \subseteq s\text{Cl}_{\mathscr{J}^\star}(F(x)) \cap V$ and hence $x \in (s\text{Cl}_{\mathscr{J}^\star}F)^-(V)$. This shows that $F^-(V) \subseteq (s\text{Cl}_{\mathscr{J}^\star}F)^-(V)$. Consequently, we obtain $(s\text{Cl}_{\mathscr{J}^\star}F)^-(V) = F^-(V)$. □

Theorem 14. *A multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is lower $\theta(\star)$-quasi continuous if and only if $sCl_{\mathscr{J}^\star}F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$ is lower $\theta(\star)$-quasi continuous.*

Proof. Suppose that F is lower $\theta(\star)$-quasi continuous. Let $x \in X$ and let V be any $\star$-open sets of Y such that $(s\text{Cl}_{\mathscr{J}^\star}F)(x) \cap V \neq \emptyset$. By Lemma 13, we have $F(x) \cap V \neq \emptyset$. Sine F is lower $\theta(\star)$-quasi continuous, there exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that $F(z) \cap \text{Cl}^\star(V) \neq \emptyset$ for every $z \in s\text{Cl}_{\mathscr{I}^\star}(U)$. Since $\text{Cl}^\star(V)$ is semi-$\mathscr{J}^\star$-open in Y, by Lemma 13, $s\text{Cl}_{\mathscr{I}^\star}(U) \subseteq F^-(\text{Cl}^\star(V)) = (s\text{Cl}_{\mathscr{J}^\star}F)^-(\text{Cl}^\star(V))$ and hence $(s\text{Cl}_{\mathscr{J}^\star}F)(z) \cap \text{Cl}^\star(V) \neq \emptyset$ for every

$z \in s\mathrm{Cl}_{\mathscr{I}^\star}(U)$. This shows that $s\mathrm{Cl}_{\mathscr{I}^\star}F$ is lower $\theta(\star)$-quasi continuous.

Conversely, suppose that $s\mathrm{Cl}_{\mathscr{I}^\star}F$ is lower $\theta(\star)$-quasi continuous. Let $x \in X$ and let V be any $\star$-open set of Y such that $F(x) \cap V \neq \emptyset$. Then, $s\mathrm{Cl}_{\mathscr{I}^\star}(F(x)) \cap V \neq \emptyset$. There exists a semi-$\mathscr{I}^\star$-open set U of X containing x such that

$$(s\mathrm{Cl}_{\mathscr{I}^\star}F)(z) \cap \mathrm{Cl}^\star(V) \neq \emptyset$$

for every $z \in s\mathrm{Cl}_{\mathscr{I}^\star}(U)$. Since $\mathrm{Cl}^\star(V)$ is semi-$\mathscr{J}^\star$-open in Y, by Lemma 13, we have

$$\begin{aligned} s\mathrm{Cl}_{\mathscr{I}^\star}(U) &\subseteq (s\mathrm{Cl}_{\mathscr{I}^\star}F)^-(\mathrm{Cl}^\star(V)) \\ &= F^-(\mathrm{Cl}^\star(V)) \end{aligned}$$

and so $F(z) \cap \mathrm{Cl}^\star(V) \neq \emptyset$ for every $z \in s\mathrm{Cl}_{\mathscr{I}^\star}(U)$. This shows that F is lower $\theta(\star)$-quasi continuous. □

Theorem 15. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is upper $\theta(\star)$-quasi continuous;

(2) $\star_\theta sCl(F^-(Int^\star(Cl^\star(V)))) \subseteq F^-(Cl^\star(V))$ for every semi-$\mathscr{J}^\star$-preopen set V of Y;

(3) $\star_\theta sCl(F^-(Int^\star(Cl^\star(V)))) \subseteq F^-(Cl^\star(V))$ for every semi-$\mathscr{J}^\star$-open set V of Y.

Proof. (1) ⇒ (2): Let V be any semi-$\mathscr{J}^\star$-preopen set of Y. Then, we have $V \subseteq \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))$ and hence $\mathrm{Cl}^\star(V) = \mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))$. Since $\mathrm{Cl}^\star(V)$ is R-$\mathscr{J}^\star$-closed, by Theorem 6, we have $\star_\theta s\mathrm{Cl}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \subseteq F^-(\mathrm{Cl}^\star(V))$.

(2) ⇒ (3): This is obvious since every semi-$\mathscr{J}^\star$-open set is semi-$\mathscr{J}^\star$-preopen.

(3) ⇒ (1): Let V be any $\star$-open set of Y. By (3), $\star_\theta s\mathrm{Cl}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \subseteq F^-(\mathrm{Cl}^\star(V))$ and by Theorem 6, F is upper $\theta(\star)$-quasi continuous. □

Theorem 16. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is lower $\theta(\star)$-quasi continuous;

(2) $\star_\theta sCl(F^+(Int^\star(Cl^\star(V)))) \subseteq F^+(Cl^\star(V))$ for every semi-$\mathscr{J}^\star$-preopen set V of Y;

(3) $\star_\theta sCl(F^+(Int^\star(Cl^\star(V)))) \subseteq F^+(Cl^\star(V))$ for every semi-$\mathscr{J}^\star$-open set V of Y.

Proof. The proof is similar to that of Theorem 15. □

Corollary 17. *For a function $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) f is $\theta(\star)$-quasi continuous;

(2) $\star_\theta sCl(f^{-1}(Int^\star(Cl^\star(V)))) \subseteq f^{-1}(Cl^\star(V))$ for every semi-$\mathscr{J}^\star$-preopen set V of Y;

(3) $\star_\theta sCl(f^{-1}(Int^\star(Cl^\star(V)))) \subseteq f^{-1}(Cl^\star(V))$ for every semi-$\mathscr{J}^\star$-open set V of Y.

Theorem 18. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is upper $\theta(\star)$-quasi continuous;

(2) $\star_\theta sCl(F^-(Int^\star(Cl^\star(V)))) \subseteq F^-(Cl^\star(V))$ for every $\mathscr{J}^\star$-preopen set V of Y;

(3) $\star_\theta sCl(F^-(V)) \subseteq F^-(Cl^\star(V))$ for every $\mathscr{J}^\star$-preopen set V of Y;

(4) $F^+(V) \subseteq \star_\theta sInt(F^+(Cl^\star(V)))$ for every $\mathscr{J}^\star$-preopen set V of Y.

Proof. (1) ⇒ (2): Let V be any $\mathscr{J}^\star$-preopen set of Y. Since $\mathrm{Int}^\star(\mathrm{Cl}^\star(V))$ is $\star$-open in Y, by Theorem 6,

$$\begin{aligned} \star_\theta s\mathrm{Cl}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) &\subseteq F^-(\mathrm{Cl}^\star(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \\ &= F^-(\mathrm{Cl}^\star(V)). \end{aligned}$$

(2) ⇒ (3): Let V be any $\mathscr{J}^\star$-preopen set of Y. Then, we have $V \subseteq \mathrm{Int}^\star(\mathrm{Cl}^\star(V))$ and by (2),

$$\begin{aligned} \star_\theta s\mathrm{Cl}(F^-(V)) &\subseteq \star_\theta s\mathrm{Cl}(F^-(\mathrm{Int}^\star(\mathrm{Cl}^\star(V)))) \\ &\subseteq F^-(\mathrm{Cl}^\star(V)). \end{aligned}$$

(3) ⇒ (4): Let V be any $\mathscr{J}^\star$-preopen set of Y. By (3), we have

$$\begin{aligned} &X - \star_\theta s\mathrm{Int}(F^+(\mathrm{Cl}^\star(V))) \\ &= \star_\theta s\mathrm{Cl}(X - F^+(\mathrm{Cl}^\star(V))) \\ &= \star_\theta s\mathrm{Cl}(F^-(Y - \mathrm{Cl}^\star(V))) \\ &\subseteq F^-(\mathrm{Cl}^\star(Y - \mathrm{Cl}^\star(V))) \\ &= X - F^+(\mathrm{Int}^\star(\mathrm{Cl}^\star(V))) \\ &\subseteq X - F^+(V) \end{aligned}$$

and hence $F^+(V) \subseteq \star_\theta s\mathrm{Int}(F^+(\mathrm{Cl}^\star(V)))$.

(4) ⇒ (1): Let V be any $\star$-open set of Y. Then, we have V is $\mathscr{J}^\star$-preopen and by (4),

$$F^+(V) \subseteq \star_\theta s\mathrm{Int}(F^+(\mathrm{Cl}^\star(V))).$$

Thus, F is upper $\theta(\star)$-quasi continuous by Theorem 6. □

Theorem 19. *For a multifunction $F : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$, the following properties are equivalent:*

(1) F is lower $\theta(\star)$-quasi continuous;

(2) $\star_\theta sCl(F^+(Int^\star(Cl^\star(V)))) \subseteq F^+(Cl^\star(V))$ *for every* $\mathscr{J}^\star$*-preopen set* V *of* Y*;*

(3) $\star_\theta sCl(F^+(V)) \subseteq F^+(Cl^\star(V))$ *for every* $\mathscr{J}^\star$*-preopen set* V *of* Y*;*

(4) $F^-(V) \subseteq \star_\theta sInt(F^-(Cl^\star(V)))$ *for every* $\mathscr{J}^\star$*-preopen set* V *of* Y.

Proof. The proof is similar to that of Theorem 18. □

Corollary 20. *For a function* $f : (X, \tau, \mathscr{I}) \to (Y, \sigma, \mathscr{J})$*, the following properties are equivalent:*

(1) f *is lower* $\theta(\star)$*-quasi continuous;*

(2) $\star_\theta sCl(f^{-1}(Int^\star(Cl^\star(V)))) \subseteq f^{-1}(Cl^\star(V))$ *for every* $\mathscr{J}^\star$*-preopen set* V *of* Y*;*

(3) $\star_\theta sCl(f^{-1}(V)) \subseteq f^{-1}(Cl^\star(V))$ *for every* $\mathscr{J}^\star$*-preopen set* V *of* Y*;*

(4) $f^{-1}(V) \subseteq \star_\theta sInt(f^{-1}(Cl^\star(V)))$ *for every* $\mathscr{J}^\star$*-preopen set* V *of* Y.

4 Conclusion

Topology as a field of mathematics is concerned with all questions directly or indirectly related to continuity. Generalization of continuity is one of the main research topics in general topology. This paper is dealing with the concepts of upper and lower $\theta(\star)$-quasi continuous multifunctions. Moreover, some characterizations of upper and lower $\theta(\star)$-quasi continuous multifunctions are obtained. The ideas and results of this paper may motivate further research.

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