Jumping unbounded nonlinearities and ALP condition

PETR TOMICZEK
Department of Mathematics
University of West Bohemia
Univerzitní 2732/8, 301 00 Pilsen
CZECH REPUBLIC
https://www.zcu.cz/en/Employees/person.html?personId=17020

Abstract: We investigate the existence of solutions to the nonlinear problem

$$u''(x) + \lambda_+ u^+(x) - \lambda_- u^-(x) + g(x, u(x)) = f(x)\,, \quad x \in (0, 2\pi)\,,$$
$$u(0) = u(2\pi)\,, \quad u'(0) = u'(2\pi)\,,$$

where the point $[\lambda_+, \lambda_-]$ is a point of the Fučík spectrum $\Sigma = \bigcup_{m=0}^{\infty} \Sigma_m$. We denote φ_m any nontrivial solution to our problem with $g = f = 0$ corresponding to $\lambda_+, \lambda_- \in \Sigma_m$. We assume that $g(x, s) = \gamma(x, s)s + h(x, s)$ and the nonlinearity g satisfies ALP type condition

Key-Words: Second order ODE, periodic, resonance, jumping nonlinearities, Dancer-Fucik spectrum, ALP condition, saddle point theorem.

Received: May 26, 2021. Revised: February 23, 2022. Accepted: March 24, 2022. Published: April 21, 2022.

1 Introduction

The aim of this article is to provide new existence result for the periodic problem with unbounded jumping nonlinearities

$$u''(x) + \lambda_+ u^+(x) - \lambda_- u^-(x) + g(x, u(x)) = f(x),$$
$$x \in (0, 2\pi)\,, \quad u(0) = u(2\pi)\,, \quad u'(0) = u'(2\pi)\,, \tag{1}$$

where nonlinearity $g\colon [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory's function, $f \in L^1(0, 2\pi)$, $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$.

To prove the existence results for nonjumping problems ($\lambda_+ = \lambda_-$) authors formulated several conditions. In 1969, a paper by Landesman and Leach [1] for a periodic problem opened the way towards what today is usually called the Landesman-Lazer condition, introduced one year later in [2] for a semilinear problem.

We can also study the periodic problems with friction $u''(x) + r(x)u'(x) + g(x, u(x)) = f(x)$ in [3] or for positive solutions see [4]. One of the latest results in this regard is [5]. The singular periodic problem is investigate in [6] by lower and upper solution. The authors of [7] use phase-plane analysis to prove the existence of a periodic solution to a nonlinear impact oscillator. The reader is referred to [8], [9] for the problem with impulsive differential equations.

A significant alternative to the Landesman-Lazer condition was proposed by Ahmad, Lazer and Paul [10] (ALP condition) in 1976, but for the bounded nonlinearity g. The ALP condition generalizes (see [11]) the classical Landesman-Lazer condition and also the potential Landesman-Lazer condition (see [12]). Therefore to relax the boundedness of g is a problem which attracted several authors' attention (see [13]). In [14] with $f \equiv 0$, the nonlinearity g is allowed to be unbounded and satisfies $|g(x, s)| \le q(x)|s|^\alpha + h(x)$, where $0 \le \alpha < 1$, $q, h \in L^2(0, 2\pi)$ with assumption $\lim_{|s|\to\infty} \int_0^{2\pi} G(x, s)\, dx/|s|^{2\alpha} = \infty$, where $G(x, s) = \int_0^s g(x, t)\, dt$.

The existence results for jumping problems ($\lambda_+ \ne \lambda_-$) with bounded nonlinearities g are investigated in [15], [16], with sublinear nonlinearities in [17]. In this article we obtain a solution to (1) for g with linear growth.

For $g \equiv 0$ and $f \equiv 0$ problem (1) becomes

$$u''(x) + \lambda_+ u^+(x) - \lambda_- u^-(x) = 0,\ x \in (0, 2\pi), \tag{2}$$
$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi)\,.$$

It is well known (see [18]) that problem (2) has nontrivial solutions only when the pairs (λ_+, λ_-) lies in the set of points made up of the curves

$$\Sigma_0 = \{\, [\lambda_+, \lambda_-] \in \mathbb{R}^2 \mid \lambda_+ \lambda_- = 0 \,\}\,,$$
$$\Sigma_m = \{\, [\lambda_+, \lambda_-] \in \mathbb{R}^2 \mid m \left(\tfrac{1}{\sqrt{\lambda_+}} + \tfrac{1}{\sqrt{\lambda_-}} \right) = 2 \,\}\,,$$

where $m \in \mathbb{N}$. The set $\Sigma = \bigcup_{m=0}^{\infty} \Sigma_m$ is called the Fučík spectrum.

Using the Landesman-Lazer type conditions authors usually suppose that g satisfies the linear growth restriction $|g(x,s)| \leq q(x)|s| + h(x)$ and there are functions $a, A \in L^1(0,2\pi)$, constants $r, R \in \mathbb{R}$ such that $g(x,s) \geq A(x)$ for a.e. $x \in [0,2\pi]$ and all $s \geq R$ and $g(x,s) \leq a(x)$ for a.e. $x \in [0,2\pi]$ and all $s \leq r$ (see [19]). These conditions imply our assumptions (see also [20]), that is the function g can be decomposed as

$$g(x,s) = \gamma(x,s)s + h(x,s)\,, \tag{3}$$

where

$$0 \leq \gamma(x,s) \leq q_1(x)\,, \quad |h(x,s)| \leq q_2(x) \tag{4}$$

for a.e. $x \in (0,2\pi)$, for all $s \in \mathbb{R}$, with some $q_1, q_2 \in L^1(0,2\pi)$. Moreover $\lambda_+ \geq \lambda_-$, $[\lambda_+, \lambda_-] \in \Sigma_m$, $m \in \mathbb{N}$ and there exists $\varepsilon > 0$ such that

$$\begin{aligned} \limsup_{s\to+\infty} \tfrac{g(x,s)}{s} &\leq (m+1)^2 - \lambda_+ - \varepsilon\,, \\ \limsup_{s\to-\infty} \tfrac{g(x,s)}{s} &\leq (m+1)^2 - \lambda_- - \varepsilon\,. \end{aligned} \tag{5}$$

We denote φ_m any nontrivial solution to (2) corresponding to $[\lambda_+, \lambda_-] \in \Sigma_m$. We shall suppose the following ALP type conditions

$$\lim_{|s|\to\infty} \int_0^{2\pi} [G(x, s\,\varphi_m(x)) - f(x)\, s\,\varphi_m(x)]\, dx = +\infty \tag{6}$$

and

$$\liminf_{|s|\to\infty} \int_0^{2\pi} [H(x, s\,\varphi_m(x)) - f(x)\, s\,\varphi_m(x)]\, dx \geq c_1 \tag{7}$$

with some constant c_1, where $H(x,s) = \int_0^s h(x,t)\, dt$.

If the nonlinearity g is L^1-bounded (as in [10]) then clearly (6) implies (7). We obtain for example the existence result to the equation (1) with the nonlinearity $g(x,s) = s/(1+s^2) + f(x)$ or $g(x,s) = [(m+1)^2 - \lambda_+ - \varepsilon]\,|\sin s|\, s + f(x)$ if $\lambda_+ \geq \lambda_-$.

2 Preliminaries

We shall use the Lebesgue space $L^p(0,2\pi)$ with the norm $\|u\|_p$. We denote by H the Sobolev space 2π-periodic absolutely continuous functions $u : \mathbb{R} \to \mathbb{R}$ such that $u' \in L^2(0,2\pi)$ endowed with the norm $\|u\| = \left(\int_0^{2\pi} u^2\, dx + \int_0^{2\pi} (u')^2\, dx\right)^{1/2}$.

By a solution to (1) we mean a function u in $W^{2,1}(0,2\pi)$ such that the equation (1) is satisfied a.e. on $(0,2\pi)$ and $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$.

We study (1) by using of variational method. More precisely, we look for critical points of the functional $I : H \to \mathbb{R}$, which is defined by

$$\begin{aligned} I(u) = \frac{1}{2} \int_0^{2\pi} &[(u')^2 - \lambda_+ (u^+)^2 - \lambda_- (u^-)^2]\, dx \\ &- \int_0^{2\pi} [G(x,u) - fu]\, dx\,. \end{aligned} \tag{8}$$

Every critical point $u \in H$ of the functional I satisfies

$$\begin{aligned} &\int_0^{2\pi} [u'v' - (\lambda_+ u^+ - \lambda_- u^-)v]\, dx \\ &- \int_0^{2\pi} [g(x,u)v - fv]\, dx = 0 \quad \text{for all } v \in H\,. \end{aligned}$$

Then u is also a weak solution to (1) and vice versa.

The usual regularity argument for ODE yields immediately (see Fučík [18]) that any weak solution to (1) is also the solution in the sense mentioned above.

We say that I satisfies Palais-Smale condition (PS) if every sequence (u_n) for which I is bounded in H and $I'(u_n) \to 0$ (as $n \to \infty$) contains a convergent subsequence.

To obtain a critical point of the functional I we will use the following variant of Saddle Point Theorem (see [21]), which is proved in Struwe [21, Theorem 8.4].

Theorem 1 *Let V, H^+ be closed subsets in H, $H = V \oplus H^+$ and Q a bounded subset in V with boundary ∂Q. Set $\Gamma = \{h :\, h \in \mathbf{C}(H,H),\, h(u) = u \text{ on } \partial Q\}$. Suppose $I \in C^1(H,\mathbb{R})$ and*

(i) $H^+ \cap \partial Q = \emptyset$,

(ii) $H^+ \cap h(Q) \neq \emptyset$, for every $h \in \Gamma$,

(iii) there are constants μ, ν such that $\mu = \inf_{u\in H^+} I(u) > \sup_{u\in\partial Q} I(u) = \nu$,

(iv) I satisfies Palais-Smale condition.

Then the number

$$\gamma = \inf_{h\in\Gamma} \sup_{u\in Q} I(h(u))$$

defines a critical value $\gamma > \nu$ of I.

We say that H^+ and ∂Q *link* if they satisfy conditions i), ii) of the theorem above.

We use result from [16, section 2] to assert that any nontrivial solution to the boundary-value problem (2) corresponding to $[\lambda_+, \lambda_-] \in \Sigma_m$, $m \in \mathbb{N}$ must be a translate, or phase shift, of a positive multiple of the function $\varphi_m : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi_m(x) = \begin{cases} \sqrt{\lambda_-}\,\sin(\sqrt{\lambda_+}x), \\ \quad x \in [0, \frac{\pi}{\sqrt{\lambda_+}}), \\ -\sqrt{\lambda_+}\,\sin(\sqrt{\lambda_-}(x - \frac{\pi}{\sqrt{\lambda_+}})), \\ \quad x \in [\frac{\pi}{\sqrt{\lambda_+}}, \frac{\pi}{\sqrt{\lambda_+}} + \frac{\pi}{\sqrt{\lambda_-}}), \\ \sqrt{\lambda_-}\,\sin(\sqrt{\lambda_+}(x - \frac{\pi}{\sqrt{\lambda_+}} - \frac{\pi}{\sqrt{\lambda_-}})), \\ \quad x \in [\frac{\pi}{\sqrt{\lambda_+}} + \frac{\pi}{\sqrt{\lambda_-}}, \frac{2\pi}{\sqrt{\lambda_+}} + \frac{\pi}{\sqrt{\lambda_-}}), \\ \quad \vdots \\ -\sqrt{\lambda_+}\,\sin(\sqrt{\lambda_-}(x - (2\pi - \frac{\pi}{\sqrt{\lambda_-}})), \\ \quad x \in [2\pi - \frac{\pi}{\sqrt{\lambda_-}}, 2\pi] \end{cases}$$

after it has been extended to be 2π-periodic over all of $\mathbb{R}$.

We denote $\theta_1 = \pi/(2\sqrt{\lambda_+})$ and

$$\varphi_\theta(x) = \varphi_m(x + \theta_1 - \theta)\,, \quad x \in [0, 2\pi]\,, \tag{9}$$

where $\theta \in [0, 2\pi]$, then $\varphi_\theta(x)$ is a nontrivial solution to the boundary-value problem (2) corresponding to $[\lambda_+, \lambda_-] \in \Sigma_m$, $m \in \mathbb{N}$.

Let H^- be the subspace of H spanned by $1, \sin x, \cos x, \sin 2x, \dots, \sin(m-1)x, \cos(m-1)x$. For $K > 0$, $L > 0$, we define sets

$$\begin{aligned} V &= \{u \in H : u = a\varphi_\theta + w,\ \theta \in [0, 2\pi],\ a \in \mathbb{R}_0^+, \\ &\qquad w \in H^-\}\,, \\ Q &= \{u \in V : 0 \le a \le K,\ \|w\| \le L\}\,. \end{aligned} \tag{10}$$

Let H^+ be the subspace of H spanned by $\sin(m+1)x, \cos(m+1)x, \sin(m+2)x, \cos(m+2)x, \dots$.

Next, we verify the assumptions (i) of Theorem 1 and assumption $H = V \oplus H^+$.

Lemma 1 *It holds*

$$H^+ \cap \partial Q = \emptyset\,. \tag{11}$$

Proof *We suppose for contradiction that there is $u \in \partial Q \cap H^+$. We denote $\langle \cdot, \cdot \rangle$ the inner product in $L^2(0, 2\pi)$. Then*
$0 \overset{u \in H^+}{=} \langle u, \sin mx \rangle \overset{u \in \partial Q}{=} \langle K\varphi_\theta + w, \sin mx \rangle \overset{w \in H^-}{=} K\langle \varphi_\theta, \sin mx \rangle \overset{K \ge 0}{=} \langle \varphi_\theta, \sin mx \rangle\,.$

Similarly $\langle \varphi_\theta, \cos mx \rangle = 0$. It is easy to see that $\langle \varphi_\theta, \sin mx \rangle = 0$ (see figure 1) only for $\theta = k\pi/m$, $k \in \mathbb{Z}$. But $\langle \varphi_{k\pi/m}, \cos mx \rangle \ne 0$ a contradiction.

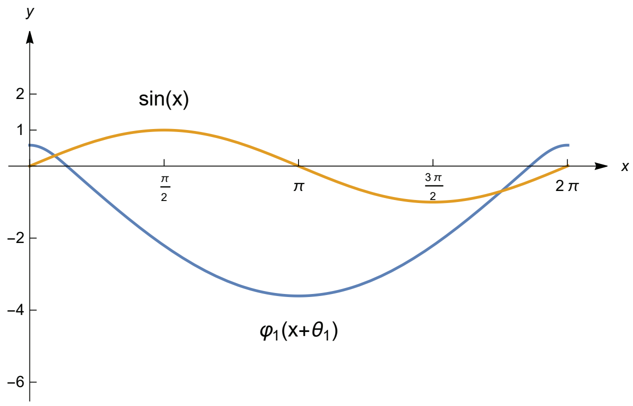

Figure 1: Solution $\varphi_\theta(x) = \varphi_1(x + \theta_1 - \theta)$ to (2) for $\theta = 0$

Lemma 2 *It holds*

$$H = V \oplus H^+\,. \tag{12}$$

Proof *To prove this lemma, we first need to show that an arbitrary element u of H can be expressed in the form*

$$u = v + h\,, \tag{13}$$

where $v \in V$ and $h \in \mathbb{H}^+$. To establish (13), we observe that every $u \in H$ can be written in the form

$$u(x) = \overline{u}(x) + a_m \cos mx + b_m \sin mx + \widetilde{u}(x)\,, \tag{14}$$

for all $x \in [0, 2\pi]$, and some constants a_m, b_m, where $\overline{u} \in H^-$ and $\widetilde{u} \in H^+$. We want to show that we can also write u in the form

$$u(x) = \overline{u}_1(x) + \varrho\varphi_\theta(x) + \widetilde{u}_1(x)\,, \tag{15}$$

for some constants $\varrho > 0$ and $\theta \in [0, 2\pi]$, where $\overline{u}_1 \in H^-$ and $\widetilde{u}_1 \in H^+$. Taking inner products with $\cos mx$ and $\sin mx$ in (14) and (15) gives rise to the system

$$\begin{aligned} \varrho\langle \varphi_\theta, \cos mx \rangle &= \pi a_m \\ \varrho\langle \varphi_\theta, \sin mx \rangle &= \pi b_m\,. \end{aligned} \tag{16}$$

We denote $p(\theta) = \langle \varphi_\theta, \sin mx \rangle$ then $p(0) = 0$ (see figure 1) and

$$\begin{aligned} p(\theta) &= \int_0^{2\pi} \varphi_m(x + \theta_1 - \theta) \sin mx\, dx \\ &= \left\{ y = x + \theta_1 - \theta \right\} \\ &= \int_{\theta_1 - \theta}^{2\pi + \theta_1 - \theta} \varphi_m(y) \sin(m(y - \theta_1 + \theta))\, dy \\ &= \int_0^{2\pi} \varphi_m(y) \sin(m(y - \theta_1 + \theta))\, dy\,, \end{aligned} \tag{17}$$

since the integrated functions are 2π-periodic. Hence function p satisfies $p''(\theta) = -m^2 p(\theta)$, thus $p(\theta) = c \sin m\theta$, $c > 0$.

Therefore we can rewrite (16) to the system

$$\begin{aligned} \varrho\, c \cos m\theta &= \pi a_m \\ \varrho\, c \sin m\theta &= \pi b_m \,. \end{aligned} \tag{18}$$

Hence, the system in (16) is solvable for any a_m and b_m in $\mathbb{R}$ and there exist $\varrho_m \geq 0$ and $\theta_m \in [0, (2\pi)/m]$ such that

$$\varrho_m \varphi_{\theta_m}(x) = h_1(x) + a_m \cos mx + b_m \sin mx + h_2(x),$$
for all $x \in [0, 2\pi]$, (19)

where $h_1 \in H^-$ and $h_2 \in H^+$.

Next, solve for $a_m \cos mx + b_m \sin mx$ in (19) and substitute into the expansion for u in (14) to obtain the representation in (15), where $\overline{u}_1 = \overline{u} - \overline{h}$ and $\widetilde{u}_1 = \widetilde{u} - \widetilde{h}$. We have therefore proved that $H = V + H^+$. To complete the proof of (12), we need to show that $V \cap H^+ = \{0\}$. We can repeat the steps from the proof of lemma 1. For $u \in V \cap H^+$ we obtain:

$$0 \overset{u \in H^+}{=} \langle u, \sin mx \rangle \overset{u \in V}{=} \langle a\varphi_\theta + w, \sin mx \rangle \overset{w \in H^-}{=} a \langle \varphi_\theta, \sin mx \rangle$$

and similarly $a\langle \varphi_\theta, \cos mx \rangle = 0$. Hence $a = 0, u = 0$ and $V \cap H^+ = \{0\}$, the proof is complete. We have proved that H is spanned by V and H^+.

We denote the first integral in the functional I by $J(u) = \int_0^{2\pi} [(u')^2 - \lambda_+ (u^+)^2 - \lambda_- (u^-)^2]\, dx$. and formulate the following lemma, which is proved in [12, Lemma 2.2].

Lemma 3 *Let φ be a solution to (2) with $[\lambda_+, \lambda_-] \in \Sigma_m, m \in \mathbb{N}$, $\lambda_+ \geq \lambda_-$. We put $u = a\varphi + w$, $a \geq 0$, $w \in H$. Then it holds*

$$\int_0^{2\pi} [(w')^2 - \lambda_+ w^2]\, dx \leq J(u) \leq \int_0^{2\pi} [(w')^2 - \lambda_- w^2]\, dx. \tag{20}$$

We will also use the following nonexistence of particular nontrivial solution to a BVP like (1) (see [22, Theorem 8, remarks 2]).

Lemma 4 *Let $\gamma_\pm$ be two maps in $L^\infty(0, 2\pi)$. There exists $m \in \mathbb{N}$, two points $[\lambda_{+,m}, \lambda_{-,m}] \in \Sigma_m$, $[\lambda_{+,m+1}, \lambda_{-,m+1}] \in \Sigma_{m+1}$ such that on $[0, 2\pi]$*

$$\lambda_{\pm,m} \lneqq \gamma_\pm(x) \lneqq \lambda_{\pm,m+1} \tag{21}$$

($\lambda_{\pm,m} \neq \gamma_\pm(x)$ and also $\gamma_\pm(x) \neq \lambda_{\pm,m+1}$ on a set of positive measure), then the problem

$$\begin{gathered} u''(x) + \gamma_+(x) u^+(x) - \gamma_-(x) u_-(x) = 0\,, \\ u(0) = u(2\pi)\,, \quad u'(0) = u'(2\pi) \end{gathered} \tag{22}$$

has only the trivial solution $u(x) \equiv 0$.

3 Main result

Theorem 2 *Let $[\lambda_+, \lambda_-] \in \Sigma_m$, $m \in \mathbb{N}$, $\lambda_+ \geq \lambda_-$. Under the assumptions (3), (4),(5), (6) and (7) Problem (1) has at least one solution in H.*

We shall prove that the functional I defined by (8) satisfies the assumptions in Theorem 1 (Saddle Point Theorem).

i) We infer from Lemmas 1, 2 that $H = V \oplus H^+$ and $\partial Q \cap H^+ = \emptyset$.

ii) The proof of the assumption $H^+ \cap h(Q) \neq \emptyset$ $\forall h \in \Gamma$ is similar to the proof in [13, example 8.2].

Let $\pi \colon H \to V$ be the continuous projection of H onto V. We have to show that $0 \in \pi(h(Q))$. For $t \in [0, 1]$, $u \in Q$ we define $h_t(u) = t\pi(h(u)) + (1-t)u$. Function h_t defines a homotopy of $h_0 = id$ with $h_1 = \pi \circ h$. Moreover, $h_t|\partial Q = \mathrm{id}$ for all $t \in [0, 1]$. Hence the topological degree $\deg(h_t, Q, 0)$ is well-defined and by homotopy invariance we have $\deg(\pi \circ h, Q, 0) = \deg(id, Q, 0) = 1$. Hence $0 \in \pi(h(Q))$, as was to be shown.

iii) Firstly, we note that by (4), (5), we get

$$\begin{aligned} 0 &\leq \liminf_{|s| \to \infty} \frac{g(x, s)}{s}\,, \\ 0 &\leq \liminf_{|s| \to \infty} \frac{G(x, s)}{s^2} \\ &\leq \limsup_{s \to \pm\infty} \frac{G(x, s)}{s^2} \leq \frac{(m+1)^2 - \lambda_\pm - \varepsilon}{2} \end{aligned} \tag{23}$$

for a.e. $x \in [0, 2\pi]$. Now we estimate the functional I on the space H^+, we prove that

$$\lim_{\|u\| \to \infty} I(u) = \infty \quad \text{for all } u \in H^+\,. \tag{24}$$

Since $u \in H^+$, we have

$$\int_0^{2\pi} (u')^2\, dx \geq (m+1)^2 \int_0^{2\pi} u^2\, dx\,. \tag{25}$$

The definition of I, (23), and (25) yield

$$\liminf_{\|u\|\to\infty}\frac{I(u)}{\|u\|^2}=\liminf_{\|u\|\to\infty}\frac{1}{\|u\|^2}\Big[\frac{1}{2}\int_0^{2\pi}[(u')^2-\lambda_+(u^+)^2 -\lambda_-(u^-)^2]\,dx-\int_0^{2\pi}[G(x,u)-fu]\,dx\Big]$$
$$\geq\liminf_{\|u\|\to\infty}\frac{1}{\|u\|^2}\Big[\frac{1}{2}\int_0^{2\pi}[(m+1)^2\,u^2-\lambda_+(u^+)^2 -\lambda_-(u^-)^2]\,dx-\int_0^{2\pi}\frac{G(x,u)}{u^2}u^2\,dx\Big]$$
$$\geq\liminf_{\|u\|\to\infty}\frac{\varepsilon}{2}\frac{\|u\|_2^2}{\|u\|^2}. \tag{26}$$

If $\liminf_{\|u\|\to\infty}\|u\|_2^2/\|u\|^2=0$ then it follows from the definition of I and (23) that

$$\liminf_{\|u\|\to\infty}\frac{I(u)}{\|u\|^2}=\frac{1}{2}\,. \tag{27}$$

Then (26) and (27) imply $\liminf_{\|u\|\to\infty}I(u)=\infty$. It follows from (24) and the fact that H^+ is compactly embedded in $C[0,2\pi]$ that there exists a real number, μ, such that $I(u)\geq\mu$ for all $u\in H^+$; in fact, we may take μ to be defined by

$$\mu=\inf_{u\in H^+}I(u)\,. \tag{28}$$

We will next show that we can pick $K>0$ and $L>0$ such that $\sup_{u\in\partial Q}I(u)<\mu$, where $Q=\{u\in H: u=a\varphi_\theta+w,\ 0\leq a\leq K, w\in H^-, \|w\|\leq L, \theta\in[0,2\pi]\}$, where φ_θ is given in (9). We argue by contradiction. Suppose that $\sup_{\|u\|\to\infty}I(u)=-\infty$ for $u\in\partial Q$ is not true. Then there is a sequence $(u_n)\subset\partial Q$ such that $\|u_n\|\to\infty$ and a constant c_- satisfying

$$\liminf_{n\to\infty}I(u_n)\geq c_-\,. \tag{29}$$

Due to (23)
$\liminf_{n\to\infty}\int_0^{2\pi}(G(x,u_n)-fu_n)/\|u_n\|^2\,dx\geq 0$.
Hence from the definition of I and (29) we have

$$\liminf_{n\to\infty}\frac{1}{2}\int_0^{2\pi}\frac{(u_n')^2-\lambda_+(u_n^+)^2-\lambda_-(u_n^-)^2}{\|u_n\|^2}\,dx\geq 0\,. \tag{30}$$

We denote $v_n=u_n/\|u_n\|$ and we proceed as in [16, pg.24]. Then,

$$v_n\in\partial B\cap V,\qquad\text{for all } n\in\mathbb{N}\,, \tag{31}$$

where B denotes the closed unit ball in H, and V is as defined in (10) ($V=\{u\in H: u=a\varphi_\theta+w,\ 0\leq a, w\in H^-\}$); so that $\partial B\cap V$ lives in a finite dimensional subspace of H (see [16, Remark 3.4]). We also have, that

$$v_n=a_n\varphi_{\theta_n}+z_n\,, \tag{32}$$

where

$$z_n\in B\cap H^-\,,\quad a_n\in[0,1/r]\,, \tag{33}$$

where $r=\|\varphi_\theta\|$. Using the compactness of $B\cap H^-$ and the closed intervals $[0,1/r]$ and $[0,2\pi]$, we may assume, as a consequence of (32), (33), that

$$v_n\to v_0\quad\text{in } H, \tag{34}$$

where

$$v_0=a_0\varphi_{\theta_0}+z_0,\ a_0\in[0,1/r],\ \theta_0\in[0,2\pi],\ z_0\in B\cap H^-.$$

Therefore, letting $n\to\infty$, using (30) and (34) we obtain

$$\int_0^{2\pi}[(v_0')^2-\lambda_+(v_0^+)^2-\lambda_-(v_0^-)^2]\,dx\geq 0\,. \tag{35}$$

By lemma 3 we have for $v_0\in V,\ v_0=a_0\varphi_{\theta_0}+z_0$

$$\int_0^{2\pi}[(v_0')^2-\lambda_+(v_0^+)^2-\lambda_-(v_0^-)^2]\,dx \leq\int_0^{2\pi}[(z_0')^2-\lambda_-z_0^2]\,dx\,,\quad z_0\in H^-. \tag{36}$$

By (35), (36) we get

$$0\leq\int_0^{2\pi}[(z_0')^2-\lambda_-z_0^2]\,dx\,. \tag{37}$$

We note that $0\leq\liminf_{|s|\to\infty}g(x,s)/s\leq\limsup_{|s|\to\infty}g(x,s)/s$, thus (5) implies $\lambda_+\leq(m+1)^2-\varepsilon$ with some $\varepsilon>0$. Since $1/\sqrt{\lambda_+}+1/\sqrt{\lambda_-}=2/m$ we obtain

$$\frac{1}{\sqrt{\lambda_-}}<\frac{2}{m}-\frac{1}{m+1}=\frac{m+2}{m(m+1)} \Rightarrow\sqrt{\lambda_-}>\frac{m(m+1)}{m+2}>m-1\,. \tag{38}$$

We denote $\delta=\lambda_--(m-1)^2>0$. Therefore by (37) we get

$$0\leq\int_0^{2\pi}[(z_0')^2-((m-1)^2+\delta)\,z_0^2]\,dx\,. \tag{39}$$

We note that for $z_0\in H^-$ it holds

$$\int_0^{2\pi}[(z_0')^2-(m-1)^2z_0^2]\,dx\leq 0\,. \tag{40}$$

Combining (39) with (40) we deduce that $z_0\equiv 0$ and $v_0=a_0\varphi_{\theta_0}$, where $a_0=1/\|\varphi_{\theta_0}\|$ and φ_{θ_0} is a nontrivial solution to the homogeneous boundary-value

problem (2) corresponding to $[\lambda_+, \lambda_-] \in \Sigma_m$, we denote $\varphi_{m_0} = a_0\varphi_{\theta_0}$.
Because of the compact imbedding $H \subset C(0, 2\pi)$ and (34), we have $v_n \to \varphi_{m_0}(x)$ in $C(0, 2\pi)$ and

$$\lim_{n\to\infty} u_n(x) = \begin{cases} +\infty & \text{where } \varphi_{m_0}(x) > 0\,, \\ -\infty & \text{where } \varphi_{m_0}(x) < 0\,. \end{cases} \tag{41}$$

We return to (29) and firstly estimate by lemma 3 using (40) (with $z_0 = w_n \in H^-$) the first integral in $I(u_n)$

$$\begin{aligned} &\int_0^{2\pi} (u_n')^2 - \lambda_+(u_n^+)^2 - \lambda_-(u_n^-)^2\, dx \\ &\le \int_0^{2\pi} [(w_n')^2 - \lambda_- w_n^2]\, dx \\ &= \int_0^{2\pi} [(w_n')^2 + w_n^2 - (\lambda_- + 1)w_n^2]\, dx \\ &= \|w_n\|^2 - ((m-1)^2 + \delta + 1)\|w_n\|_2^2 \\ &\le \|w_n\|^2 - \frac{(m-1)^2 + \delta + 1}{(m-1)^2 + 1}\|w_n\|^2 \\ &= -\frac{\delta}{(m-1)^2 + 1}\|w_n\|^2 \end{aligned} \tag{42}$$

since $\|w_n\|^2 \le ((m-1)^2 + 1)\|w_n\|_2^2$. By (29) and (42) we obtain

$$\liminf_{n\to\infty}\Big(-\frac{\delta}{2((m-1)^2+1)} \,\| w_n\|^2 - \int_0^{2\pi} [G(x, u_n) - f u_n]\, dx\Big) \ge c_-\,.$$

We denote $c_m = \frac{\delta}{2((m-1)^2+1)} > 0$, then equivalently

$$\limsup_{n\to\infty}\Big(c_m\| w_n\|^2 + \int_0^{2\pi} [G(x, u_n) - f u_n]\, dx\Big) \le -c_-. \tag{43}$$

We use the decomposition (3) of $g(x, s) = \gamma(x, s)s + h(x, s)$ and denote $\Gamma(x, s) = \int_0^s \gamma(x, t)\, t\, dt$, we rewrite (43) into

$$\limsup_{n\to\infty}\Big(c_m \,\| w_n\|^2 + \int_0^{2\pi} [\Gamma(x, u_n) + H(x, u_n) - f u_n]\, dx\Big) \le -c_-. \tag{44}$$

By the mean value theorem, (3),(4) and the compact embedding H into $C([0, 2\pi])$ ($\|\cdot\|_{C([0,2\pi])} \le c_2\|\cdot\|$) we obtain

$$\begin{aligned} &\int_0^{2\pi} [H(x, u_n) - H(x, a_n\varphi_{m_0})]\, dx \\ &= \int_0^{2\pi} [h(x, \xi_n(x))\, w_n)]\, dx \le \|q_2\|_1 c_2\|w_n\|\,, \end{aligned} \tag{45}$$

where $\xi_n(x) \in (a_n\varphi_{m_0}(x), u_n(x))$.
Similarly $\int_0^{2\pi} f w_n \le \|f\|_1 c_2\|w_n\|$. Therefore by (44), (45) we get $\limsup_{n\to\infty}\Big(c_m\|w_n\|^2 - (\|f\|_1 + \|q_2\|_1)c_2\|w_n\| + \int_0^{2\pi}[\Gamma(x, u_n) + H(x, a_n\varphi_{m_0}) - f a_n\varphi_{m_0}]\, dx\Big) \le -c_-$ and consequently there exists a constant c_3 such that

$$\limsup_{n\to\infty} \int_0^{2\pi} [\Gamma(x, u_n) + H(x, a_n\varphi_{m_0}) - f a_n\varphi_{m_0}]\, dx \le c_3\,. \tag{46}$$

For a.e. $x \in (0, 2\pi)$ function $\Gamma(x, s)$ is nonincreasing for $s < 0$; $\Gamma(x, 0) = 0$ and $\Gamma(x, s)$ is nondecreasing for $s > 0$. Hence we get

$$\lim_{n\to\infty} \int_0^{2\pi} \Gamma(x, u_n)\, dx = \lim_{n\to\infty} \int_0^{2\pi} \Gamma(x, a_n\varphi_{m_0})\, dx \tag{47}$$

since $\lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} a_n\varphi_{m_0}(x) = +\infty$ for $x \in (0, 2\pi)$ such that $\varphi_{m_0}(x) > 0$, and $\lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} a_n\varphi_{m_0} = -\infty$ for $x \in (0, 2\pi)$ such that $\varphi_{m_0}(x) < 0$. We rewrite condition (6) in the following form

$$\lim_{n\to\infty} \int_0^{2\pi} [\Gamma(x, a_n\varphi_{m_0}(x)) + H(x, a_n\varphi_{m_0}(x)) - f a_n\varphi_{m_0}(x)]\, dx = \infty\,. \tag{48}$$

If the limit in (47) is finite we obtain a contradiction to (46), (48). If the limit in (47) is infinite we obtain a contradiction to (46) and assumption (7). Hence $\sup_{\|u\|\to\infty} I(u) = -\infty$ for $u \in \partial Q$ and we have showed that we can pick $K > 0$ and $L > 0$ such that

$$\mu = \inf_{u\in H^+} I(u) > \sup_{u\in\partial Q} I(u) = \nu\,.$$

iv) For Assumption (iv) of theorem 1, we show that functional I satisfies the Palais-Smale condition.

For contradiction we suppose that the sequence (u_n) is unbounded and there exists a constant c_4 such that

$$\Big|\frac{1}{2}\int_0^{2\pi}(u_n')^2-\lambda_+(u_n^+)^2-\lambda_-(u_n^-)^2\,dx \\ -\int_0^{2\pi}[G(x,u_n)-fu_n]\,dx\Big|\le c_4 \tag{49}$$

and

$$\lim_{n\to\infty}\|I'(u_n)\|=0\,. \tag{50}$$

Let (w_k) be an arbitrary sequence bounded in H. It follows from (50) and the Schwarz inequality

$$\Big|\lim_{\substack{n\to\infty\\k\to\infty}}\int_0^{2\pi}[u_n'w_k'-(\lambda_+u_n^+-\lambda_-u_n^-)w_k]\,dx \\ -\int_0^{2\pi}[g(x,u_n)w_k-fw_k]\,dx\Big| \\ =|\lim_{\substack{n\to\infty\\k\to\infty}}\langle I'(u_n),w_k\rangle|\le\lim_{\substack{n\to\infty\\k\to\infty}}\|I'(u_n)\|\cdot\|w_k\|=0\,. \tag{51}$$

Since $\int_0^{2\pi}[(f/\|u_n\|)w_k]\,dx\to 0$ we obtain by (51)

$$\lim_{\substack{n\to\infty\\m\to\infty\\k\to\infty}}\Bigg(\int_0^{2\pi}\Bigg[\Big(\frac{u_n'}{\|u_n\|}-\frac{u_m'}{\|u_m\|}\Big)w_k' \\ -\Big(\lambda_+\Big(\frac{u_n^+}{\|u_n\|}-\frac{u_m^+}{\|u_m\|}\Big)-\lambda_-\Big(\frac{u_n^-}{\|u_n\|}-\frac{u_m^-}{\|u_m\|}\Big)\Big)w_k\Bigg]dx \\ -\int_0^{2\pi}\Bigg[\Big(\frac{g(x,u_n)}{\|u_n\|}-\frac{g(x,u_m)}{\|u_m\|}\Big)w_k\Bigg]\,dx\Bigg)=0\,. \tag{52}$$

We put $v_n=u_n/\|u_n\|$ and $w_k=v_n-v_m$ in (52), we conclude

$$\lim_{\substack{n\to\infty\\m\to\infty}}\Bigg(\int_0^{2\pi}(v_n'-v_m')^2\,dx \\ -\int_0^{2\pi}\Big[(\lambda_+(v_n^+-v_m^+)-\lambda_-(v_n^--v_m^-))(v_n-v_m)\Big]dx \\ -\int_0^{2\pi}\Big[\Big(\frac{g(x,u_n)}{\|u_n\|}-\frac{g(x,u_m)}{\|u_m\|}\Big)(v_n-v_m)\Big]dx\Bigg)=0. \tag{53}$$

Due to compact imbedding $H\subset L^2(0,2\pi), C([0,2\pi])$ there is $v_0\in H$ such that (up to subsequence) $v_n\rightharpoonup v_0$ weakly in H, $v_n\to v_0$ strongly in $L^2(0,2\pi)$, $C([0,2\pi])$. Due to assumption (3), (4) the sequence $(g(x,u_n)/\|u_n\|)$ is L^1-bounded, thus (53) implies $v_n\to v_0$ strongly in H.

It follows from assumptions (3), (4), (5) (up to subsequence) that

$$\frac{g(x,u_n)}{\|u_n\|}=\frac{\gamma(x,u_n)\,u_n}{\|u_n\|}+\frac{h(x,u_n)}{\|u_n\|} \\ \rightharpoonup\gamma_0^+(x)v_0^+-\gamma_0^-(x)v_0^-\quad\text{in } L^1(0,2\pi), \tag{54}$$

where $0\le\gamma_0^+(x)\le(m+1)^2-\lambda_+-\varepsilon$, $0\le\gamma_0^-(x)\le(m+1)^2-\lambda_--\varepsilon$ for a.e. $x\in(0,2\pi)$, since the sequence $\gamma_n(x):=\gamma(x,u_n(x))$ is both bounded and equi-integrable in $L^1(0,2\pi)$ (see Dunford, Schwarz [24]). We get from (51) and (54)

$$\int_0^{2\pi}[v_0'w'-((\lambda_++\gamma_0^+)v_0^+ \\ -(\lambda_-+\gamma_0^-)v_0^-)w]\,dx=0\quad\text{for all } w\in H. \tag{55}$$

It follows from (54), (55) and from the usual regularity argument for ordinary differential equations (see Fučík [18]) that v_0 is a solution with norm $\|v_0\|=1$ to the periodic BVP

$$v_0''-(\lambda_++\gamma_0^+)v_0^++(\lambda_-+\gamma_0^-)v_0^-=0 \\ x\in(0,2\pi),\quad v_0(0)=v_0(2\pi),\quad v_0'(0)=v_0'(2\pi), \tag{56}$$

where by (38)

$$m^2\le\lambda_+\le\lambda_++\gamma_0^+(x)\le(m+1)^2-\varepsilon, \\ (m-1)^2<(m-1)^2+\delta=\lambda_- \\ \le\lambda_-+\gamma_0^-(x)\le(m+1)^2-\varepsilon \tag{57}$$

for a.e. $x\in(0,2\pi)$. Therefore using lemma 4 with $[\lambda_+,\lambda_-]\in\Sigma_m$, $[(m+1)^2,(m+1)^2]\in\Sigma_{m+1}$ equation (56) and inequalities (57) we obtain

$$\gamma(x,u_n(x))\to\gamma_0(x)=0\quad\text{for a.e } x\in(0,2\pi) \\ \text{and}\quad v_n(x)\to v_0(x)=\frac{\varphi_m(x)}{\|\varphi_m\|}\,, \tag{58}$$

where φ_m is a solution to (2) with $[\lambda_+,\lambda_-]\in\Sigma_m$.

Now we estimate the first integral in (51). We set $u_n=a_n\varphi_m+u_n^\perp$, where $a_n\ge 0$ and $u_n^\perp\in H^-\oplus H^+$. We remark that $u=u^+-u^-$ and using (21) in the first integral in (51) we denote

$$I_w\equiv\int_0^{2\pi}[(a_n\varphi_m+u_n^\perp)'w_k' \\ -(\lambda_+u_n^+-\lambda_-u_n^-)w_k]\,dx$$

and we obtain

$$\begin{aligned} I_w = & \int_0^{2\pi} [(a_n\varphi_m + u_n^\perp)' w_k' \\ & -(\lambda_+ u_n^+ - \lambda_- u_n^-) w_k]\, dx \\ = & \int_0^{2\pi} [a_n \varphi_m' w_k' + (u_n^\perp)' w_k' \\ & -((\lambda_+ - \lambda_-) u_n^+ + \lambda_- u_n)\, w_k]\, dx \\ = & \int_0^{2\pi} [a_n(\lambda_+ \varphi_m^+ - \lambda_- \varphi_m^-) w_k + (u_n^\perp)' w_k' \\ & -((\lambda_+ - \lambda_-) u_n^+ + \lambda_- u_n)\, w_k]\, dx \\ = & \int_0^{2\pi} \{a_n[(\lambda_+ - \lambda_-)\varphi_m^+ + \lambda_- \varphi_m]\, w_k \\ & +(u_n^\perp)' w_k' - [(\lambda_+ - \lambda_-)(a_n\varphi_m + u_n^\perp)^+ \\ & +\lambda_-(a_n\varphi_m + u_n^\perp)]\, w_k\}\, dx \\ = & \int_0^{2\pi} [(\lambda_+ - \lambda_-)(a_n\varphi_m^+ - (a_n\varphi_m + u_n^\perp)^+) w_k \\ & +(u_n^\perp)' w_k' - \lambda_- u_n^\perp w_k]\, dx\,. \end{aligned} \tag{59}$$

Similarly

$$\begin{aligned} I_w = & \int_0^{2\pi} [(\lambda_+ - \lambda_-)(a_n\varphi_m^- - (a_n\varphi_m + u_n^\perp)^-) w_k \\ & +(u_n^\perp)' w_k' - \lambda_+ u_n^\perp w_k]\, dx\,. \end{aligned} \tag{60}$$

We add (59) and (60), thus

$$\begin{aligned} 2I_w = & \int_0^{2\pi} [(\lambda_+ - \lambda_-)(|a_n\varphi_m| - |a_n\varphi_m + u_n^\perp|) w_k \\ & + 2(u_n^\perp)' w_k' - (\lambda_+ + \lambda_-) u_n^\perp w_k]\, dx\,. \end{aligned} \tag{61}$$

We set $u_n^\perp = \overline{u}_n + \widetilde{u}_n$ where $\overline{u}_n \in H^-$, $\widetilde{u}_n \in H^+$ and we put $w_k = \overline{u}_n - \widetilde{u}_n + a_n\varphi_m,\ a_n \geq 0,\ (k = n)$ in (61), we get

$$\begin{aligned} 2I_n \equiv & \int_0^{2\pi} ([\lambda_+ - \lambda_-)(|a_n\varphi_m| - |a_n\varphi_m + \overline{u}_n + \widetilde{u}_n|) \\ & \cdot(\overline{u}_n - \widetilde{u}_n) + 2(\overline{u}_n')^2 - 2(\widetilde{u}_n')^2 \\ & -(\lambda_+ + \lambda_-)(\overline{u}_n^2 - \widetilde{u}_n^2)]\, dx \\ & + \int_0^{2\pi} [(\lambda_+ - \lambda_-)(|a_n\varphi_m| - |a_n\varphi_m + u_n^\perp|)\, a_n\varphi_m \\ & +2(u_n^\perp)' a_n \varphi_m' - (\lambda_+ + \lambda_-) u_n^\perp a_n \varphi_m]\, dx \end{aligned} \tag{62}$$

Hence using $|x| - |y| \leq |x - y|$ and (21) we obtain

$$\begin{aligned} 2I_n \leq & \int_0^{2\pi} [\, (\lambda_+ - \lambda_-)\, |\overline{u}_n + \widetilde{u}_n|\, |\overline{u}_n - \widetilde{u}_n| \\ & + 2(\overline{u}_n')^2 - 2(\widetilde{u}_n')^2 \\ & -(\lambda_+ + \lambda_-)((\overline{u}_n)^2 - (\widetilde{u}_n)^2)\,]\, dx \\ & + \int_0^{2\pi} [(\lambda_+ - \lambda_-)(|a_n\varphi_m| - |a_n\varphi_m + u_n^\perp|) a_n\varphi_m \\ & +2a_n(\lambda_+ \varphi_m^+ u_n^\perp - \lambda_- \varphi_m^- u_n^\perp) \\ & -(\lambda_+ + \lambda_-) u_n^\perp a_n \varphi_m\,]\, dx \\ = & \int_0^{2\pi} [\, (\lambda_+ - \lambda_-)\, |\overline{u}_n^2 - \widetilde{u}_n^2| + 2(\overline{u}_n')^2 \\ & -(\lambda_+ + \lambda_-)(\overline{u}_n)^2 - 2(\widetilde{u}_n')^2 \\ & +(\lambda_+ + \lambda_-)(\widetilde{u}_n)^2\,]\, dx \\ & + \int_0^{2\pi} [(\lambda_+ - \lambda_-)(|a_n\varphi_m| - |a_n\varphi_m + u_n^\perp|) a_n\varphi_m \\ & +u_n^\perp(\lambda_+ - \lambda_-)\, |a_n\varphi_m|\,]\, dx\,. \end{aligned} \tag{63}$$

Inequality $|a^2 - b^2| \leq a^2 + b^2$ and (63) yield

$$\begin{aligned} 2I_n \leq & \, 2\Big(\int_0^{2\pi} [\, (\overline{u}_n')^2 - \lambda_-(\overline{u}_n)^2\,]\, dx \\ & + \int_0^{2\pi} [\, -(\widetilde{u}_n')^2 + \lambda_+(\widetilde{u}_n)^2\,]\, dx\Big) \\ & +(\lambda_+ - \lambda_-) \int_0^{2\pi} [(|a_n\varphi_m| - |a_n\varphi_m + u_n^\perp|) a_n\varphi_m \\ & +u_n^\perp\, |a_n\varphi_m|]\, dx \\ \leq & \, 2\Big(\int_0^{2\pi} [\, (\overline{u}_n')^2 - \lambda_-(\overline{u}_n)^2\,]\, dx \\ & + \int_0^{2\pi} [\, -(\widetilde{u}_n')^2 + \lambda_+(\widetilde{u}_n)^2\,]\, dx\Big) \\ & + 2\,(\lambda_+ - \lambda_-) \int_{M_n} (u_n^\perp)^2\, dx\,, \end{aligned} \tag{64}$$

where $M_n = \{x \in [0, 2\pi] : \varphi_m(\varphi_m + u_n^\perp/a_n) < 0\}$. The last inequality in (64) follows from the following estimates

$$\begin{aligned} & (|a_n\varphi_m| - |a_n\varphi_m + u_n^\perp|) a_n\varphi_m + u_n^\perp |a_n\varphi_m| \\ & = \begin{cases} 0 \quad (\text{if } a_n\varphi_m(a_n\varphi_m + u_n^\perp) > 0) & x \notin M_n \\ \operatorname{sign}(\varphi_m)\, 2\,(a_n\varphi_m + u_n^\perp) a_n\varphi_m & x \in M_n \end{cases} \\ & \leq\ 2(u_n^\perp)^2 \end{aligned}$$

since $a_n\varphi_m < 0$ and $a_n\varphi_m + u_n^\perp > 0$ imply $u_n^\perp > a_n\varphi_m + u_n^\perp$, $u_n^\perp > -a_n\varphi_m > 0$ and therefore $-(a_n\varphi_m + u_n^\perp)\, a_n\varphi_m \le (u_n^\perp)^2$.
We use $|x| - |y| \ge -|x-y|$ in (62) obtain similarly

$$I_n \ge \int_0^{2\pi} [\,(\overline{u}_n')^2 - \lambda_+(\overline{u}_n)^2 - (\widetilde{u}_n')^2 + \lambda_-(\widetilde{u}_n)^2\,]\,dx - (\lambda_+ - \lambda_-)\int_{M_n}(u_n^\perp)^2\,dx\,. \tag{65}$$

Using $\|\cdot\|_{C([0,2\pi])} \le c_2\|\cdot\|$ we get

$$\int_{M_n}(u_n^\perp)^2\,dx \le \mu(M_n)\,c_2\|u_n^\perp\|^2 \ \text{ and } \ \mu(M_n)\to 0\,. \tag{66}$$

Since by (58) we have

$$\frac{u_n}{\|u_n\|} = \frac{(\varphi_m + u_n^\perp/a_n)}{\|\varphi_m + u_n^\perp/a_n\|} \to \frac{\varphi_m}{\|\varphi_m\|} \ \text{ and } \ \frac{u_n^\perp}{a_n} \rightrightarrows 0\,.$$

We write $u_n = \overline{u}_n + a_n\varphi_m + \widetilde{u}_n$, $\overline{u}_n \in H^-$, $\widetilde{u}_n \in H^+$. We put $w_k = (\overline{u}_n + a_n\varphi_m - \widetilde{u}_n)/(a_n\|u_n^\perp\|^{\frac{1}{2}})$ in (51) then using (64) we obtain

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{a_n\|u_n^\perp\|^{\frac{1}{2}}}\Big\{&\int_0^{2\pi}[\,(\overline{u}_n')^2 - \lambda_-(\overline{u}_n)^2\,]\,dx\\ &+\int_0^{2\pi}[\,-(\widetilde{u}_n')^2 + \lambda_+(\widetilde{u}_n)^2\,]\,dx\\ &+(\lambda_+ - \lambda_-)\int_{M_n}(u_n^\perp)^2\,dx + \int_0^{2\pi}[\gamma(x,u_n)(\widetilde{u}_n)^2]\,dx\\ &-\int_0^{2\pi}[\gamma(x,u_n)(\,\overline{u}_n + a_n\varphi_m)^2\\ &+(h(x,u_n) - f)\,(\,\overline{u}_n + a_n\varphi_m - \widetilde{u}_n)]\,dx\Big\} \ge 0\,.\end{aligned} \tag{67}$$

We note that it holds $\|\overline{u}_n\|^2 \le ((m-1)^2+1)\|\overline{u}_n\|_2^2$, $\|\widetilde{u}_n\|^2 \ge ((m+1)^2+1)\|\widetilde{u}_n\|_2^2$ and using (66) we get

$$\begin{aligned}&\int_0^{2\pi}[(\overline{u}_n')^2 - \lambda_-(\overline{u}_n)^2]\,dx + \int_0^{2\pi}[-(\widetilde{u}_n')^2 + \lambda_+(\widetilde{u}_n)^2]\,dx\\ &+(\lambda_+ - \lambda_-)\int_{M_n}(u_n^\perp)^2\,dx + \int_0^{2\pi}[\gamma(x,u_n)(\widetilde{u}_n)^2]\,dx\\ &= \|\overline{u}_n\|^2 - (\lambda_- + 1)\|\overline{u}_n\|_2^2 - \|\widetilde{u}_n\|^2 + (\lambda_+ + 1)\|\widetilde{u}_n\|_2^2\\ &+(\lambda_+ - \lambda_-)\int_{M_n}(u_n^\perp)^2\,dx + \int_0^{2\pi}[\gamma(x,u_n)(\widetilde{u}_n)^2]\,dx\\ &\le \frac{(m-1)^2 - \lambda_-}{(m-1)^2+1}\|\overline{u}_n\|^2 + \frac{\lambda_+ - (m+1)^2}{(m+1)^2+1}\|\widetilde{u}_n\|^2\\ &+(\lambda_+ - \lambda_-)\mu(M_n)c_2\|u_n^\perp\|^2\\ &+\int_0^{2\pi}\gamma(x,u_n)\,dx\,c_2\|\widetilde{u}_n\|^2.\end{aligned}$$

Hence and from (57), (58) and (66) it follows

$$\begin{aligned}&\int_0^{2\pi}[\,(\overline{u}_n')^2 - \lambda_-(\overline{u}_n)^2\,]\,dx + \int_0^{2\pi}[\,-(\widetilde{u}_n')^2\\ &+\lambda_+(\widetilde{u}_n)^2\,]\,dx + (\lambda_+ - \lambda_-)\int_{M_n}(u_n^\perp)^2\,dx\\ &+\int_0^{2\pi}[\gamma(x,u_n)(\widetilde{u}_n)^2]\,dx\\ &\le \frac{-\delta/2}{(m-1)^2+1}\|\overline{u}_n\|^2 + \frac{-\varepsilon/2}{(m+1)^2+1}\|\widetilde{u}_n\|^2\\ &\le -\varrho\|u_n^\perp\|^2\end{aligned} \tag{68}$$

with some $\varrho > 0$. Therefore (67) and (68) imply

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{a_n\|u_n^\perp\|^{\frac{1}{2}}}\Big\{&-\int_0^{2\pi}[\gamma(x,u_n)(\,\overline{u}_n + a_n\varphi_m)^2\\ &+(\,h(x,u_n) - f\,)\,(\,\overline{u}_n + a_n\varphi_m - \widetilde{u}_n\,)]\,dx\Big\} \ge 0\,.\end{aligned} \tag{69}$$

Consequently

$$\begin{aligned}&\liminf_{n\to\infty}\int_0^{2\pi}\Big[\frac{h(x,u_n) - f)}{\|u_n^\perp\|^{\frac{1}{2}}}((\overline{u}_n - \widetilde{u}_n)/a_n + \varphi_m)\Big]\,dx\\ &\ge \liminf_{n\to\infty}\int_0^{2\pi}\Big[\frac{\gamma(x,u_n)}{\|u_n^\perp\|^{\frac{1}{2}}}a_n(\,\overline{u}_n/a_n + \varphi_m)^2\,dx\Big] \ge 0.\end{aligned} \tag{70}$$

Now we put $w_k = (\,\overline{u}_n - \widetilde{u}_n)/(\|u_n^\perp\|^2)$ in (51) to obtain

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{\|u_n^\perp\|^2}\Big\{&\int_0^{2\pi}[\,(\overline{u}_n')^2 - \lambda_-(\overline{u}_n)^2\,]\,dx\\ &+\int_0^{2\pi}[-(\widetilde{u}_n')^2 + \lambda_+(\widetilde{u}_n)^2\,]\,dx\\ &+(\lambda_+ - \lambda_-)\int_{M_n}(u_n^\perp)^2\,dx\\ &+\int_0^{2\pi}[\gamma(x,u_n)((\widetilde{u}_n)^2 - (\overline{u}_n)^2)]\,dx\\ &-\int_0^{2\pi}[(\gamma(x,u_n)a_n\varphi_m + h(x,u_n) - f)\\ &\cdot(\overline{u}_n - \widetilde{u}_n)]\,dx\Big\} \ge 0.\end{aligned} \tag{71}$$

We suppose for contradiction that the sequence $(u_n^\perp)$ is unbounded then due to (68) and (71) there exists $\varrho > 0$ such that

$$-\varrho + \liminf_{n\to\infty}\Big\{-\int_0^{2\pi}\Big[\frac{\gamma(x,u_n)}{\|u_n^\perp\|^{\frac{1}{2}}}a_n\varphi_m\frac{\overline{u}_n - \widetilde{u}_n}{\|u_n^\perp\|^{\frac{3}{2}}}\Big]\,dx\Big\} \ge 0 \tag{72}$$

or equivalently

$$-\varrho \ge \limsup_{n\to\infty} \int_0^{2\pi} \Big[\frac{\gamma(x,u_n)}{\|u_n^\perp\|^{\frac12}}\, a_n\varphi_m\, \frac{\overline{u}_n - \widetilde{u}_n}{\|u_n^\perp\|^{\frac32}}\Big]\, dx\,. \tag{73}$$

We note that $u_n^\perp/a_n \rightrightarrows 0$ and we get by (70) (for $\|u_n^\perp\| \to \infty$)

$$\lim_{n\to\infty} \int_0^{2\pi} \frac{\gamma(x,u_n)}{\|u_n^\perp\|^{\frac12}} a_n\varphi_m^2\, dx = 0\,. \tag{74}$$

We denote $S_n = \Big\{x \in [0,2\pi] \mid |\varphi_m(x)| \le (\overline{u}_n(x) - \widetilde{u}_n(x))/(\|u_n^\perp\|^{3/2})\Big\}$ then $\lim_{n\to\infty} \mu(S_n) = 0$ and

$$\begin{aligned}&\int_{[0,2\pi]\setminus S_n} \Big[\frac{\gamma(x,u_n)}{\|u_n^\perp\|^{\frac12}}\, a_n\, \Big|\varphi_m\, \frac{\overline{u}_n - \widetilde{u}_n}{\|u_n^\perp\|^{\frac32}}\Big|\Big]\, dx \\ &\le \int_{[0,2\pi]\setminus S_n} \frac{\gamma(x,u_n)}{\|u_n^\perp\|^{\frac12}} a_n\varphi_m^2\,.\end{aligned} \tag{75}$$

By (51) (with $w_k = (\overline{u}_n - \widetilde{u}_n)/\|u_n^\perp\|^2$), (65) we obtain

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{\|u_n^\perp\|^2}\Big\{\int_0^{2\pi} [\,(\overline{u}_n')^2 - \lambda_+(\overline{u}_n)^2\,]\, dx \\ &+ \int_0^{2\pi} [\,-(\widetilde{u}_n')^2 + \lambda_-(\widetilde{u}_n)^2\,]\, dx \\ &- (\lambda_+ - \lambda_-) \int_{M_n} (u_n^\perp)^2\, dx \\ &+ \int_0^{2\pi} [\gamma(x,u_n)((\widetilde{u}_n)^2 - (\overline{u}_n)^2)]\, dx \\ &- \int_0^{2\pi} [\gamma(x,u_n)a_n\varphi_m + (h(x,u_n) - f) \\ &\cdot(\overline{u}_n - \widetilde{u}_n)]\, dx\Big\} \le 0\,.\end{aligned}$$

Hence there exists a constant c_5 such that $\liminf_{n\to\infty} \frac{1}{\|u_n^\perp\|^2} \int_0^{2\pi} [\gamma(x,u_n)a_n\varphi_m(\overline{u}_n - \widetilde{u}_n)]\, dx \ge c_5$. Thus $\limsup_{n\to\infty} \int_{S_n} \Big[(\gamma(x,u_n)/\|u_n^\perp\|^{1/2})\, a_n\varphi_m\, (\overline{u}_n - \widetilde{u}_n)/(\|u_n^\perp\|^{3/2})\Big]\, dx \ge 0$ since $\mu(S_n) \to 0$. Hence and by (74), (75) we get

$$\limsup_{n\to\infty} \int_0^{2\pi} \Big[\frac{\gamma(x,u_n)}{\|u_n^\perp\|^{\frac12}}\, a_n\varphi_m\, \frac{\overline{u}_n - \widetilde{u}_n}{\|u_n^\perp\|^{\frac32}}\Big]\, dx \ge 0 \tag{76}$$

a contradiction to (73). This implies that the sequence $(u_n^\perp)$ is bounded. We use (20) from Lemma 3 with $w = u_n^\perp$ and we obtain

$$\begin{aligned}&\int_0^{2\pi} [((u_n^\perp)')^2 - \lambda_+(u_n^\perp)^2]\, dx \\ &\le J(u_n) \le \int_0^{2\pi} [((u_n^\perp)')^2 - \lambda_-(u_n^\perp)^2]\, dx\end{aligned} \tag{77}$$

where $J(u_n) = \int_0^{2\pi} \big[(u_n')^2 - \lambda_+ u_n^2 - \lambda_- u_n^2\big]\, dx$. Hence boundedness of $(u_n^\perp)$ implies with (49) that there exists a constant c_6 such that

$$\Big|\int_0^{2\pi} [G(x,u_n) - fu_n]\, dx\Big| \le c_6 \quad \text{for all } n \in \mathbb{N}\,. \tag{78}$$

We again use the decomposition $G(x,s) = \Gamma(x,s) + H(x,s)$ to rewrite (78) into

$$\Big|\int_0^{2\pi} [\Gamma(x,u_n) + H(x,u_n) - f(u_n^\perp + a_n\varphi_m)\, dx\Big| \le c_6 \quad \text{for all } n \in \mathbb{N}\,. \tag{79}$$

We use (45) boundedness of $(u_n^\perp)$ and (79) to obtain a constant c_7 such that

$$\Big|\int_0^{2\pi} [\Gamma(x,u_n) + H(x,a_n\varphi_m) - fa_n\varphi_m\, dx\Big| \le c_7 \quad \text{for all } n \in \mathbb{N}\,. \tag{80}$$

Using (47) and (80) we obtain a contradiction to assumptions (6) (see (48)), (7), hence sequence (u_n) is bounded. Then there exists $u_0 \in H$ such that $u_n \rightharpoonup u_0$ in H, $u_n \to u_0$ in $L^2(0,2\pi)$, $C(0,2\pi)$ (taking a subsequence if it is necessary). It follows from equality (39) that

$$\begin{aligned}&\lim_{\substack{n\to\infty\\ m\to\infty\\ k\to\infty}} \Big\{\int_0^{2\pi} [(u_n - u_m)'w_k' \\ &-(\lambda_+(u_n^+ - u_m^+) - \lambda_-(u_n^- - u_m^-))w_k]\, dx \\ &- \int_0^{2\pi} [g(x,u_n) - g(x,u_m)]w_k\, dx\Big\} = 0\,.\end{aligned} \tag{81}$$

The nonlinearity g is the Carathéodory's function, thus strong convergence $u_n \to u_0$ in $C(0,2\pi)$ imply

$$\lim_{\substack{n\to\infty\\ m\to\infty}} \int_0^{2\pi} [g(x,u_n) - g(x,u_m)](u_n - u_m)\, dx = 0\,. \tag{82}$$

If we set $w_k = u_n$, $w_k = u_m$ in (81) and subtract these equalities, then by (82) we obtain

$$\begin{aligned}&\lim_{\substack{n\to\infty\\ m\to\infty}} \int_0^{2\pi} [(u_n' - u_m')^2 - (\lambda_+(u_n^+ - u_m^+) \\ &-\lambda_-(u_n^- - u_m^-))(u_n - u_m)]\, dx = 0\,.\end{aligned} \tag{83}$$

Hence the strong convergence $u_n \to u_0$ in $L^2(0,2\pi)$ implies the strong convergence $u_n \to u_0$ in H. This shows that J satisfies Palais-Smale condition and the proof of Theorem 2 is complete.

ACKNOWLEDGMENTS

This work was supported by the project LO1506 of the Czech Ministry of Education, Youth and Sports.

References:

[1] A. C. Lazer and D. E. Leach, *Bounded perturbations of forced harmonic oscillators at resonance*, Ann. Mat. Pura Appl. **82** (1969), 49–68.

[2] E. Landesman and A. C. Lazer, *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech. **19** (1970), 609–623.

[3] H. Chen and L. Yi, *Rate of decay of stable periodic solutions of Duffing equations*, Journal of Differential Equations **236** (2007), 493–503.

[4] R. Hakl, P. J. Torres, and M. Zamora, *Periodic solutions of singular second order differential equations: Upper and lower functions*, Nonlinear Analysis **74** (2011), no. 18, 7078–7093.

[5] P.Tomiczek, *Duffing Equation with Nonlinearities Between Eigenvalues*, in *Nonlinear analysis and boundary value problems, NABVP 2018, Santiago de Compostela, Spain, September 4-7*, (Springer Proceedings in Mathematics & Statistics,) 2019, pp. 199–209, DOI: 10.1007/978-3-030-26987-6_13.

[6] I. Rachůnková and V. Polášek, *Singular periodic problem for nonlinear ordinary differential equations with ϕ-laplacian*, Electronic Journal of Differential Equations **2006** (2006), no. 27, 1–12.

[7] A. Fonda and A. Sfecci, *Periodic bouncing solutions for nonlinear impact oscillators*, Advanced Nonlinear Studies **13** (2016), no. 1, https://doi.org/10.1515/ans-2013-0110.

[8] P. Drábek and M. Langerová, *On the second order periodic problem at resonance with impulses*, J. Math. Anal. Appl. **428** (2015), 1339–1353.

[9] H. Chen J. Sun and L. Yang, *The existence and multiplicity of solutions for an impulsive differential equation with two parameters via a variational method*, Nonlinear Analysis **73** (2010), 440–449.

[10] S. Ahmad, A. C. Lazer, and J. L. Paul, *Elementary critical point theory and perturbations of elliptic boundary value problems at resonance*, Indiana Univ. Math. J. **25** (1976), no. 10, 933–944.

[11] A. Fonda and M. Garrione, *Nonlinear resonance: a comparison between Landesman-Lazer and Ahmad-Lazer-Paul conditions*, Advanced Nonlinear Studies **11** (2011), 391–404.

[12] P. Tomiczek, *Potential Landesman-Lazer type conditions and the Fučík spectrum*, Electron. J. Dff. Eqns. **2005** (2005), no. 94, 1–12.

[13] Z. Q. Han, 2π*-periodic solutions to ordinary differential systems at resonance*, Acta Mathematica Sinica. Chinese Series **43** (2000), no. 4, 639–644.

[14] Z. Q. Han, 2π*-periodic solutions to N-dimmensional systems of Duffings type (I)*, D. Guo, Ed., Nonlinear Analysis and Its Applications, Beijing Scientific & Technical Publishers, Beijing, China (1994), 182–191.

[15] D. Bonheure and Ch. Fabry, *A variational approach to resonance for asymmetric oscillators*, Communications on pure and applied analysis **6** (2007), 163–181.

[16] D.A. Bliss, J. Buerger, and A.J. Rumbos, *Periodic boundary and the Dancer-Fucik spectrum under conditions of resonance.*, Electronic Journal of Differential Equations **2011** (2011), no. 112, 1–34.

[17] Ch. Wang, *Multiplicity of periodic solutions of Duffing equations with jumping nonlinearities*, Acta Mathematicae Applicatae Sinica, English Series **18** (2002), no. 3, 513–522.

[18] S. Fučík, *Solvability of nonlinear equations and boundary value problems*, D.Reidel Publ. Company, Holland, 1980.

[19] P. Drábek, *Landesman-Lazer type condition and nonlinearities with linear growth*, Czechoslovak Mathematical Journal **40** (1990), no. 1, 70–86.

[20] R. Iannacci and M. N. Nkashama, *Unbounded perturbations of forced second order ordinary differential equations at resonance*, J. Differential Equations **69** (1987), 289–309.

[21] P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Reg. Conf. Ser. in Math. no 65, Amer. Math. Soc. Providence, RI., 1986.

[22] M. Struwe, *Variational methods*, (Springer, Berlin, 1996.)

[23] P. Habets and G. Metzen, *Existence of periodic solutions of Duffing equations*, Journal of Differential Equations **78** (1989), 1–32.

[24] N. Dunford and J.T. Schwartz, *Linear Operators. Part I*, (Interscience Publ., New York, 1958.)