# Modification of Sumudu Decomposition Method for Nonlinear Fractional Volterra Integro-Differential Equations

NONGLUK VIRIYAPONG
Mathematics and Applied Mathematics Research Unit
Department of Mathematics, Faculty of Science, Mahasarakham University
Maha Sarakham, 44150
THAILAND

*Abstract:* In this paper, modification of Sumudu decomposition method is successfully applied to find the approximate solution of nonlinear Volterra integro-differential equation of fractional order. The proposed method is based on the combining of two powerful techniques, Sumudu decomposition method and Daftardar-Gejji and Jafari (DGJ) method. Several illustrative examples are given to demonstrate the validity, reliability, and efficiency of the proposed technique.

*Key-Words:* Volterra integro-differential equation, Caputo fractional derivative, Sumudu transform, Adomian decomposition method, DGJ method

Received: May 23, 2021. Revised: February 21, 2022. Accepted: March 22, 2022. Published: April 18, 2022.

## 1 Introduction

The fractional integro-differential equations (FIDEs) are in general form of integer order integro-differential equations. In this study concerns with the approximate analytical solution of the nonlinear Caputo fractional integro-differential equation of the following form:

$$D^{\alpha}y(t) = p(t)y(t) + g(t) + \int_0^t K(t,\tau)F(y(\tau))d\tau, \quad (1)$$

with the initial condition

$$y^{(i)}(0) = \beta_i; \quad i = 0, 1, 2, \ldots, m-1, \quad (2)$$

where $D^{\alpha}$ is the Caputo fractional differential operator of order $\alpha$, $m-1 < \alpha \leq m$, $f(t) \in L^2([0,1])$, $p(t) \in L^2([0,1])$ and $K(t,\tau) \in L^2([0,1]^2)$ are known functions, $y(t)$ is unknown function.

Such kind of equations are the focus of research due to their pivotal role in the mathematical modeling of many physical problems in several fields of physics, engineering, and economics, such as arising in heat conduction in materials with memory, signal processing and fluid mechanics [1, 2, 3]. However, the fractional integro-differential equations are usually difficult to solve analytically and may not have exact or analytical solutions, so approximate and numerical methods for approximate solutions to integro-differential equation of integer order are extended to solve fractional integro-differential equations.

In recent years, many methods have been developed to solve fractional integro-differential equations, especially nonlinear, which are receiving a lot of attention. For instance, we can mention the following works. Momani and Noor [4] applied the Adomian decomposition method (ADM) to solve fourth-order FIDEs, Mittal and Nigam [5] applied ADM to find the approximate solutions for the FIDEs, Yang and Hou [6] developed and applied the Laplace decomposition method to solve linear and nonlinear FIDEs, Tate and Dinde [7] presented a new modification of Adomian decomposition method for nonlinear Volterra FIDEs, Hamoud and Ghadle [8] applied ADM and modified Laplace Adomian decomposition method to find the approximate solution for Voterra integro-differential equation of fractional order, and Al-Khaled and Yousef [9] applied Sumudu decomposition method to solve the fractional nonlinear Volterra-Fredholm integro-differential equation. In addition, the applications of the homotopy analysis method [10] and CAS wavelets method [11] for solution of fractional differential equations.

This article aims to introduce a new method for solving fractional nonlinear integro-differential equations, called the modified Sumudu decomposition method (MSDM) which is a modification to the Sumudu decomposition method (SDM). The MSDM is a combination of the two powerful techniques, Sumudu transform and the iterative DGJ method which is presented by Daftardar-Gejji and Jafari [12] in which the Adomian polynomials in SDM have been replaced by Daftardar-Jafari polynomials. The Sumudu transform was used to avoid calculation of fractional derivative and integration of some difficult functions while the nonlinear term can be easily handled by the use of the approach given by Daftardar-Gejji [12], and there is no need to calculate the Adomian polynomials as compared with SDM.

## 2 Preliminaries

Some useful definitions and properties of fractional calculus and Sumudu transform, are presented in this section.

### 2.1 Fractional calculus

**Definition 1** [13] *The Riemann-Liouville fractional integral of order $\alpha > 0$, of a real valued function $f(t)$ is defined as:*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

**Definition 2** [13] *The fractional derivative $D^{\alpha}$ of $f(t)$ in the Caputo's sense is defined as*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau,$$

*for $m-1 < \alpha \leq m, m \in \mathbb{N}$.*

### 2.2 Sumudu transform

The Sumudu transform is an integral transform, which was first introduced by Watugala and applied to solve differential equations and control engineering problem [14].

**Definition 3** [14] *The Sumudu transform over the following set of functions*

$$\mathbb{A} = \big\{ f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{|t|/\tau_j} \text{ if } t \in (-1)^j \times [0, \infty) \big\}$$

*is defined for $u \in (-\tau_1, \tau_2)$ as*

$$G(u) = \mathscr{S}[f(t)] = \int_0^{\infty} e^{-t} f(ut) dt = \int_0^{\infty} \frac{1}{u} e^{-t/u} f(t) dt. \quad (3)$$

*Function $f(t)$ in* (3) *is called inverse Sumudu transform of $F(u)$ and is denoted by $f(t) = \mathscr{S}^{-1}[F(u)]$.*

In Belgacem et al. [15], the Sumudu transform was shown to be the duality of the Laplace transform. Hence, one should be able to compete to a great extent in problem-solving. The Sumudu and Laplace transforms exhibit a duality relation expressed as follows:

$$G(u) = \frac{F(\frac{1}{u})}{u}, \quad F(s) = \frac{G(\frac{1}{s})}{s},$$

where $G(u) = \mathscr{S}[f(t)]$ and $F(s) = \mathscr{L}[f(t)]$. For further detail and properties about Sumudu transform can found in [15]. Some basic transform of the functions related to present work are as follow:

1. $\mathscr{S}[1] = 1$
2. $\mathscr{S}[t^n] = n! u^n; \quad n = 1, 2, \ldots$
3. $\mathscr{S}[t^{\alpha}] = \Gamma(\alpha+1) u^{\alpha}; \quad \alpha > 0$
4. $\mathscr{S}\Big[\int_0^t f(\tau) d\tau\Big] = u \mathscr{S}[f(t)]$

**Definition 4** *Let $f(t)$ and $g(t)$ are continuous functions and exponential order, the convolution of $f(t)$ and $g(t)$ is defined as*

$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau$$

**Theorem 1** [15] *Let $f(t)$ and $g(t)$ are continuous function and exponential order. If $\mathscr{S}[f(t)] = F(u)$ and $\mathscr{S}[g(t)] = G(u)$ then*

$$\mathscr{S}[(f * g)(t)] = \mathscr{S}\left[\int_0^t f(\tau) g(t-\tau) d\tau\right] = u F(u) G(u)$$

**Theorem 2** [15] *The Sumudu transform of the Caputo fractional derivative id defined as*

$$\mathscr{S}[D^{\alpha} f(t)] = u^{-\alpha} \mathscr{S}[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0),$$

*for $m-1 < \alpha < m, m \in \mathbb{N}$.*

## 3 Analysis of the method

Firstly, we consider the fractional integro-differential equation of Volterra type. According to modified Sumudu decomposition method, we apply Sumudu transform first on both side of (1), we get

$$\mathscr{S}[D^{\alpha} y(t)] = \mathscr{S}[p(t) y(t)] + \mathscr{S}[g(t)] + \mathscr{S}\Big[\int_0^t K(t,\tau) F(y(\tau)) d\tau\Big]. \quad (4)$$

Using the property of Sumudu transform and simplifying, we can obtain

$$\mathscr{S}[y(t)] = \sum_{k=0}^{m-1} u^k y^{(k)}(0) + u^{\alpha} \mathscr{S}[g(t)] + u^{\alpha} \mathscr{S}[p(t) y(t)] + u^{\alpha} \mathscr{S}\Big[\int_0^t K(t,\tau) F(y(\tau)) d\tau\Big]. \quad (5)$$

Operating the inverse Sumudu transform on both sides of (5), we get

$$y(t) = \mathscr{S}^{-1}\Big[\sum_{k=0}^{m-1} u^k y^{(k)}(0)\Big] + \mathscr{S}^{-1}[u^\alpha \mathscr{S}[g(t)]] + \mathscr{S}^{-1}[u^\alpha \mathscr{S}[p(t)y(t)]] + \mathscr{S}^{-1}\Big[u^\alpha \mathscr{S}\Big[\int_0^t K(t,\tau)F(y(\tau))d\tau\Big]\Big]. \quad (6)$$

Next assume that

$$\left.\begin{aligned} f(t) &= \mathscr{S}^{-1}\Big[\sum_{k=0}^{m-1} u^k y^{(k)}(0)\Big] + \mathscr{S}^{-1}[u^\alpha \mathscr{S}[g(t)]], \\ R(y(t)) &= \mathscr{S}^{-1}[u^\alpha \mathscr{S}[p(t)y(t)]], \\ N(y(t)) &= \mathscr{S}^{-1}\Big[u^\alpha \mathscr{S}\Big[\int_0^t K(t,\tau)F(y(\tau))d\tau\Big]\Big]. \end{aligned}\right\} \quad (7)$$

Thus, equation (6) can be written in the following form

$$y(t) = f(t) + R(y(t)) + N(y(t)), \quad (8)$$

where $f$ is a known function, $R$ and $N$ are given linear and nonlinear operator of $y$, respectively.

The second step in modified Sumudu decomposition method is that we represent solution as in form of infinite series, given as follow:

$$y(t) = \sum_{n=0}^{\infty} y_n, \quad (9)$$

where the term $y_n$ are to be recursively computed. Then we have

$$R\Big(\sum_{n=0}^{\infty} y_n\Big) = \sum_{n=0}^{\infty} R(y_n), \quad (10)$$

The nonlinear operator $N$ is decomposed as

$$N\Big(\sum_{n=0}^{\infty} y_n\Big) = N(y_0) + \sum_{n=1}^{\infty}\Big[N\Big(\sum_{i=0}^{n} y_i\Big) - N\Big(\sum_{i=0}^{n-1} y_i\Big)\Big] \quad (11)$$

Thus, equation (8) is given as

$$\sum_{n=0}^{\infty} y_n = f + \sum_{n=0}^{\infty} R(y_n) + N(y_0) + \sum_{n=1}^{\infty}\Big[N\Big(\sum_{i=0}^{n} y_i\Big) - N\Big(\sum_{i=0}^{n-1} y_i\Big)\Big], \quad (12)$$

then a recurrence relation is defined as follow:

$$y_0 = f, \quad (13)$$

$$y_{n+1} = R(y_n) + N(y_0 + y_1 + \ldots + y_n) - N(y_0 + y_1 + \ldots + y_{n-1}), \quad (14)$$

and we have

$$y_0 + y_1 + \ldots + y_{n+1} = R(y_0 + y_1 + \ldots + y_n) + N(y_0 + y_1 + \ldots + y_n). \quad (15)$$

Thus, we obtain

$$\sum_{n=0}^{\infty} y_n = f + R\Big(\sum_{n=0}^{\infty} y_n\Big) + N\Big(\sum_{n=0}^{\infty} y_n\Big). \quad (16)$$

The $m$-term approximate solution of (8) is given by

$$y = y_0 + y_1 + \ldots + y_{m-1}.$$

However, when $\alpha$ is an integer, the exact solution may be obtained. The $m$-term approximation $y_a = \sum_{n=0}^{m-1} y_n$ can be used to approximate the solution. The zeroth component is very important, the choice of (13) as the initial solution always leads to noise oscillation during the iteration procedure [6]. Moreover, the selection of $y_0$ to contain a minimal number of terms is giving more flexibility to solve complicated nonlinear equations. The modification of MSDM is based on the assumption that the function $f$ that arises from the source term and prescribed initial conditions can be divided into two parts, namely, $f_0$ and $f_1$. Under this assumption, we set

$$f = f_0 + f_1, \quad (17)$$

and on applying a slight variation to the component $y_0$ and $y_1$, the modified recursive relation defined as follows,

$$\left.\begin{aligned} y_0 &= f_1, \\ y_1 &= f_2 + R(y_0) + N(y_0), \\ y_{n+1} &= R(y_n) + N(y_0 + y_1 + \ldots + y_n) \\ &\quad - N(y_0 + y_1 + \ldots + y_{n-1}). \end{aligned}\right\} \quad (18)$$

The slight variation in reducing the number of terms of $y_0$ will help in a reduction of computation and will accelerate the convergence. This slight variation in the component $y_0$ and $y_1$ may provide the exact solution by using two iterations only. It should be noted that the success of this method depends on proper choice of the part $f_0$ and $f_1$.

## 4 Applications

In this section, to demonstrate the applicability and validity of the proposed method, we have applied it to nonlinear Volterra integro-differential equations with fractional order and the result obtained will be compared with the exact solution.

**Example 4.1** *First, we consider the following nonlinear fractional Volterra integro-differential equation :*

$$D^{\alpha}y(t) = 2t - \frac{t^6}{6} + 5\int_0^t (t-\tau)y^2(\tau)d\tau,\ 0 < \alpha \le 1, \tag{19}$$

*with the following initial condition*

$$y(0) = 0, \tag{20}$$

*which has the exact solution in the case of* $\alpha = 1$ *is* $y(t) = t^2$.

To solve this problem by the proposed method, we apply Sumudu transform on both side of (19) and using the inverse Sumudu transform, we have

$$\left.\begin{aligned} f(t) =& \mathscr{S}^{-1}\big[u^{\alpha}\mathscr{S}\big[2t - \frac{t^6}{6}\big]\big],\\ N(y(t)) =& 5\mathscr{S}^{-1}\big[u^{\alpha+1}\mathscr{S}[t]\mathscr{S}[y^2]\big]. \end{aligned}\right\} \tag{21}$$

By using the modified recursive relation (18) and (21), we get

$$\left.\begin{aligned} y_0 =& \mathscr{S}^{-1}\big[u^{\alpha}\mathscr{S}[2t]\big],\\ y_1 =& -\frac{1}{6}\mathscr{S}^{-1}\big[u^{\alpha}\mathscr{S}\big[t^6\big]\big] + 5\mathscr{S}^{-1}\big[u^{\alpha+2}\mathscr{S}[y_0^2]\big],\\ y_{n+1} =& 5\mathscr{S}^{-1}\big[u^{\alpha+2}\mathscr{S}[(y_0+y_1+\ldots+y_n)^2\\ &- (y_0+y_1+\ldots+y_{n-1})^2]\big]. \end{aligned}\right\} \tag{22}$$

By using the recursive relation (22), we get

$$\begin{aligned} y_0 =& \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)},\\ y_1 =& \frac{20\Gamma(2\alpha+3)}{\Gamma^2(\alpha+2)\Gamma(3\alpha+5)}t^{3\alpha+4} - \frac{5!}{\Gamma(\alpha+7)}t^{\alpha+6}\\ y_2 =& \frac{400\Gamma(2\alpha+3)\Gamma(4\alpha+6)}{\Gamma^3(\alpha+2)\Gamma(3\alpha+5)\Gamma(5\alpha+8)}t^{5\alpha+7}\\ &- \frac{20\cdot 5!\Gamma(2\alpha+8)}{\Gamma(\alpha+2)\Gamma(\alpha+7)\Gamma(3\alpha+10)}t^{3\alpha+9}\\ &+ \frac{2000\Gamma^2(2\alpha+3)\Gamma(6\alpha+9)}{\Gamma^4(\alpha+2)\Gamma^2(3\alpha+5)\Gamma(7\alpha+11)}t^{7\alpha+10}\\ &- \frac{200\cdot 5!\Gamma(2\alpha+3)\Gamma(4\alpha+11)}{\Gamma^2(\alpha+2)\Gamma(3\alpha+5)\Gamma(\alpha+7)\Gamma(5\alpha+13)}t^{5\alpha+12}\\ &+ \frac{(\cdot 5!)^2\Gamma(2\alpha+13)}{\Gamma^2(\alpha+7)\Gamma(3\alpha+15)}t^{3\alpha+14}\\ &\vdots \end{aligned}$$

Hence, the 3-term approximate solution of problem (19) and (20) is

$$\begin{aligned} y_a(t) =& y_0 + y_1 + y_2\\ =& \frac{2t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{20\Gamma(2\alpha+3)t^{3\alpha+4}}{\Gamma^2(\alpha+2)\Gamma(3\alpha+5)}\\ &- \frac{5!t^{\alpha+6}}{\Gamma(\alpha+7)} + \frac{400\Gamma(2\alpha+3)\Gamma(4\alpha+6)t^{5\alpha+7}}{\Gamma^3(\alpha+2)\Gamma(3\alpha+5)\Gamma(5\alpha+8)}\\ &- \frac{20\cdot 5!\Gamma(2\alpha+8)t^{3\alpha+9}}{\Gamma(\alpha+2)\Gamma(\alpha+7)\Gamma(3\alpha+10)}\\ &+ \frac{2000\Gamma^2(2\alpha+3)\Gamma(6\alpha+9)t^{7\alpha+10}}{\Gamma^4(\alpha+2)\Gamma^2(3\alpha+5)\Gamma(7\alpha+11)}\\ &- \frac{200\cdot 5!\Gamma(2\alpha+3)\Gamma(4\alpha+11)t^{5\alpha+12}}{\Gamma^2(\alpha+2)\Gamma(3\alpha+5)\Gamma(\alpha+7)\Gamma(5\alpha+13)}\\ &+ \frac{(\cdot 5!)^2\Gamma(2\alpha+13)t^{3\alpha+14}}{\Gamma^2(\alpha+7)\Gamma(3\alpha+15)} \end{aligned} \tag{23}$$

In the case of $\alpha = 1$, we get

$$y_0 = t^2, y_1 = 0,$$

and

$$y_n = 0, n \ge 2.$$

Therefore, $y_a(t) = t^2$, which is the same of exact solution. For $\alpha = 1$, it is the only case that the exact solution is known, and the obtained solution is in good agreement with the exact solutions for only two iterations of MSDM.

Figure 1 shows the behavior of the approximate solution of problem (19) and (20) using the MSDM for different values of $\alpha$.

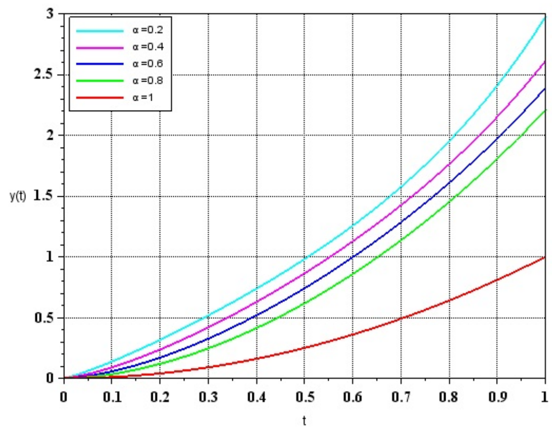


Figure 1: Approximate solutions for Example 4.1 for different values of $\alpha$.

**Example 4.2** *Consider the following nonlinear fractional Volterra integro-differential equation:*

$$D^\alpha y(t) = 1 - t^2 y(t) + 3\int_0^t y^2(\tau)d\tau,\ 0 < \alpha \le 1, \tag{24}$$

*with the following initial condition*

$$y(0) = 0 \tag{25}$$

*which has the exact solution in the case of $\alpha = 1$ is $y(t) = t$.*

To solve this problem by the proposed method, we apply Sumudu transform on both side of (24) and using the inverse Sumudu transform, we have

$$\left.\begin{aligned} f(t) &= \mathscr{S}^{-1}\big[u^\alpha \mathscr{S}[1]\big], \\ R(y(t)) &= -\mathscr{S}^{-1}\big[u^\alpha \mathscr{S}[t^2 y]\big], \\ N(y(t)) &= 3\mathscr{S}^{-1}\big[u^{\alpha+1}\mathscr{S}[y^2]\big]. \end{aligned}\right\} \tag{26}$$

By using the recursive relation (14) and (26), we get

$$\left.\begin{aligned} y_0 &= \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ y_1 &= -\mathscr{S}^{-1}\big[u^\alpha \mathscr{S}[t^2 y_0]\big] + 3\mathscr{S}^{-1}\big[u^{\alpha+1}\mathscr{S}[y_0^2]\big], \\ y_{n+1} &= -\mathscr{S}^{-1}\big[u^\alpha \mathscr{S}[t^2 y_n]\big] \\ &\quad + 3\mathscr{S}^{-1}\big[u^{\alpha+1}\mathscr{S}[(y_0+y_1+\ldots+y_n)^2 \\ &\quad - (y_0+y_1+\ldots+y_{n-1})^2]\big]. \end{aligned}\right\} \tag{27}$$

By using the recursive relation (27), we get

$$\begin{aligned} y_1 &= -\frac{(\alpha+2)t^{2\alpha+2}}{2\Gamma(2\alpha+2)} + \frac{6\Gamma(2\alpha)t^{3\alpha+1}}{\alpha\Gamma^2(\alpha)\Gamma(3\alpha+2)} \\ y_2 &= -\mathscr{S}^{-1}\big[u^\alpha \mathscr{S}[t^2 y_1]\big] + 3\mathscr{S}^{-1}\big[u^{\alpha+1}\mathscr{S}[2y_0y_1 + y_1^2]\big] \\ &= \frac{(\alpha+2)\Gamma(2\alpha+5)}{2\Gamma(2\alpha+2)\Gamma(3\alpha+5)}t^{3\alpha+4} \\ &\quad - \frac{6\Gamma(2\alpha)\Gamma(3\alpha+4)}{\alpha\Gamma^2(\alpha)\Gamma(3\alpha+2)\Gamma(4\alpha+4)}t^{4\alpha+3} \\ &\quad - \frac{3(\alpha+2)\Gamma(3\alpha+3)}{\alpha\Gamma(\alpha)\Gamma(2\alpha+2)\Gamma(4\alpha+4)}t^{4\alpha+3} \\ &\quad + \frac{36\Gamma(2\alpha)\Gamma(4\alpha+2)}{\alpha^2\Gamma^3(\alpha)\Gamma(3\alpha+2)\Gamma(5\alpha+3)}t^{5\alpha+2} \\ &\quad + \frac{36\Gamma(2\alpha)\Gamma(4\alpha+5)}{\alpha^2\Gamma^3(\alpha)\Gamma(3\alpha+2)\Gamma(5\alpha+6)}t^{5\alpha+5} \\ &\quad - \frac{36\Gamma(2\alpha)\Gamma(5\alpha+4)}{\alpha^2\Gamma^3(\alpha)\Gamma(3\alpha+2)\Gamma(6\alpha+5)}t^{6\alpha+4} \\ &\quad + \frac{108\Gamma^2(2\alpha)\Gamma(6\alpha+3)}{\alpha^2\Gamma^4(\alpha)\Gamma^2(3\alpha+2)\Gamma(7\alpha+4)}t^{7\alpha+3} \\ &\quad \vdots \end{aligned}$$

The 3-term approximate solution of (24) is given by

$$y_a(t) = y_0 + y_1 + y_2. \tag{28}$$

For $\alpha = 1$, it is the only case which the exact solution is known. When $\alpha = 1$, then $y_a(t) = t$ which is the same of exact solution. Thus, the approximate solution using only two-steps of MSDM is in excellent agreement with the exact solutions.

The approximate solution of problem (24) and (25) for $\alpha = 0.2, 0.4, 0.6, 0.8$ and $1$ are shown in Figure 2.

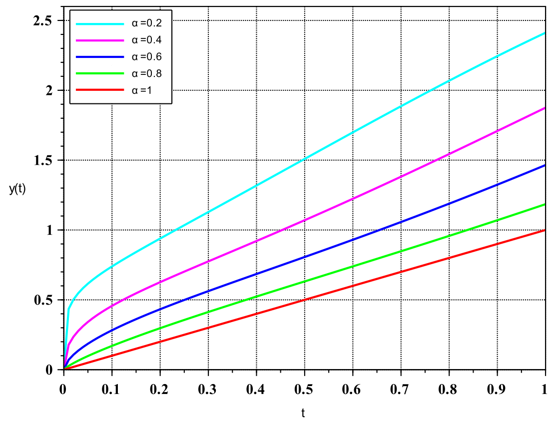


Figure 2: Approximate solutions for Example 4.2 for distinct values of $\alpha$.

**Example 4.3** *Consider the following nonlinear fractional Volterra integro-differential equation* [6]*:*

$$D^\alpha y(t) = 1 + \int_0^t y(\tau)y'(\tau)d\tau, \quad 0 \le t < 1, 0 < \alpha \le 1, \tag{29}$$

*with the initial condition*

$$y(0) = 0. \tag{30}$$

Similar to the previous example, to solve this problem by the proposed method, we apply Sumudu transform on both side of (29) and using the inverse Sumudu transform, we have

$$\left.\begin{aligned} f(t) &= \mathscr{S}^{-1}\big[u^\alpha \mathscr{S}[1]\big], \\ N(y(t)) &= \mathscr{S}^{-1}\big[u^{\alpha+1}\mathscr{S}[yy']\big]. \end{aligned}\right\} \tag{31}$$

By using the recursive relation (14) and (31), we get

$$\left.\begin{aligned} y_0 &= \frac{t^\alpha}{\Gamma(\alpha+1)}, \\ y_1 &= \mathscr{S}^{-1}\big[u^{\alpha+1}\mathscr{S}[y_0 y_0']\big], \\ y_{n+1} &= \mathscr{S}^{-1}\big[u^{\alpha+1}\mathscr{S}[(y_0+\ldots+y_n)(y_0'+\ldots+y_n') \\ &\quad - (y_0+\ldots+y_{n-1})(y_0'+\ldots+y_{n-1}')]\big]. \end{aligned}\right\} \tag{32}$$

By using the recursive relation (32), we get

$$\begin{aligned} y_1 &= \frac{\Gamma(2\alpha)}{3\alpha^2\Gamma^2(\alpha)\Gamma(3\alpha)}t^{3\alpha}, \\ y_2 &= \mathscr{S}^{-1}\big[u^{\alpha+1}\mathscr{S}[(y_0+y_1)(y_0'+y_1')-y_0y_0']\big] \\ &= \mathscr{S}^{-1}\big[u^{\alpha+1}\mathscr{S}[y_0y_1'+y_1y_0'+y_1y_1']\big] \\ &= \frac{4\Gamma(2\alpha)\Gamma(4\alpha)}{15\alpha^3\Gamma^3(\alpha)\Gamma(3\alpha)\Gamma(5\alpha)}t^{5\alpha} \\ &\quad + \frac{\Gamma^2(2\alpha)\Gamma(6\alpha)}{21\alpha^4\Gamma^4(\alpha)\Gamma^2(3\alpha)\Gamma(7\alpha)}t^{7\alpha}, \\ &\vdots \end{aligned}$$

Hence, the 3-term approximate solution of problem (29) and (30) is

$$\begin{aligned} y_a(t) =& y_0+y_1+y_2 \\ =& \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha)}{3\alpha^2\Gamma^2(\alpha)\Gamma(3\alpha)}t^{3\alpha} \\ &+ \frac{4\Gamma(2\alpha)\Gamma(4\alpha)}{15\alpha^3\Gamma^3(\alpha)\Gamma(3\alpha)\Gamma(5\alpha)}t^{5\alpha} \\ &+ \frac{\Gamma^2(2\alpha)\Gamma(6\alpha)}{21\alpha^4\Gamma^4(\alpha)\Gamma^2(3\alpha)\Gamma(7\alpha)}t^{7\alpha}. \end{aligned} \tag{33}$$

In the integer case, $\alpha = 1$, the exact solution is $y(t) = \sqrt{2}\tan(\frac{t}{\sqrt{2}})$ and 3-term approximate solution by MSDM are plotted in Figure 3. It can be found that the obtained approximate solution is close to the exact solution and the result show good agreement with the result of Ref. [6]. The accuracy of the obtained solution can be improved by taking more terms in the series approximate solution.

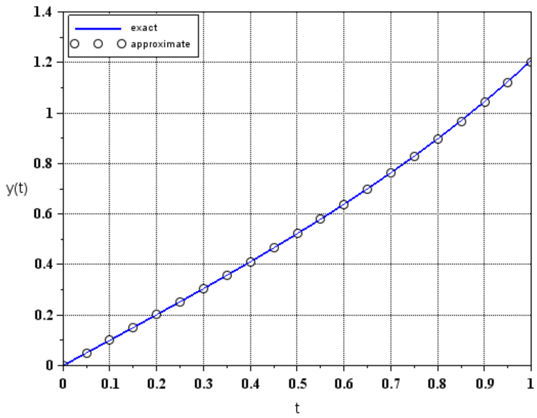


Figure 3: The comparison between the approximate and exact solution of Example 4.3, in the case $\alpha = 1$.

Figure 4 shows the behavior of approximate solution of problem (29) and (30) using the MSDM for different values of $\alpha$.

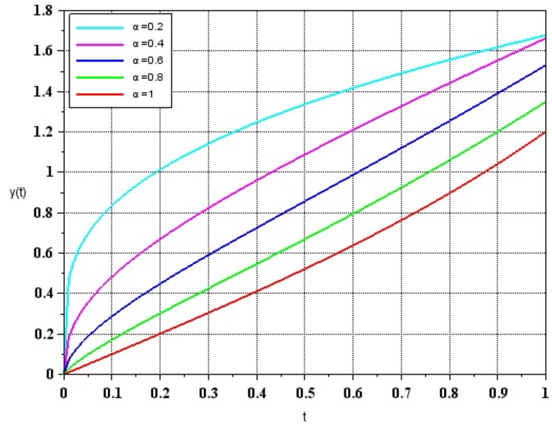


Figure 4: Approximate solutions for Example 4.3 for distinct values of $\alpha$.

**Example 4.4** *Consider the following nonlinear fractional Volterra integro-differential equation* [6]*:*

$$D^{\frac{6}{5}}y(t) = \frac{5t^{\frac{4}{5}}}{2\Gamma(\frac{4}{5})} - \frac{t^9}{252} + \int_0^t (t-\tau)^2 y^3(\tau)d\tau, \tag{34}$$

*with the following initial conditions*

$$y(0) = 0, y'(0) = 0. \tag{35}$$

Applying the proposed algorithm of MSDM, we obtain

$$\left.\begin{aligned} f(t) =& \mathscr{S}^{-1}\big[u^{\frac{6}{5}}\mathscr{S}[\frac{5t^{\frac{4}{5}}}{2\Gamma(\frac{4}{5})} - \frac{t^9}{252}]\big], \\ N(y(t)) =& \mathscr{S}^{-1}\big[u^{\frac{6}{5}+1}\mathscr{S}[t^2]\mathscr{S}[y^3]\big]. \end{aligned}\right\} \tag{36}$$

By using the modified recursive relation (18) and (36), we get

$$\left.\begin{aligned} y_0 =& \mathscr{S}^{-1}\big[u^{\frac{6}{5}}\mathscr{S}[\frac{5t^{\frac{4}{5}}}{2\Gamma(\frac{4}{5})}]\big] = t^2 \\ y_1 =& -\mathscr{S}^{-1}\big[u^{\frac{6}{5}}\mathscr{S}[\frac{t^9}{252}]\big] \\ &+ \mathscr{S}^{-1}\big[u^{\frac{11}{5}}\mathscr{S}[t^2]\mathscr{S}[y_0^3]\big], \\ y_{n+1} =& \mathscr{S}^{-1}\big[u^{\frac{11}{5}}\mathscr{S}[t^2]\mathscr{S}[(y_0+\ldots+y_n)^3 \\ &- (y_0+\ldots+y_{n-1})^3]\big]. \end{aligned}\right\} \tag{37}$$

By using the recursive relation (37), we get

$$\begin{aligned} y_1 &= -\frac{9!}{252}\mathscr{S}^{-1}\left[u^{\frac{51}{5}}\right] + \mathscr{S}^{-1}\left[u^{\frac{11}{5}}\mathscr{S}[t^2]\mathscr{S}[t^6]\right] \\ &= -2\cdot 6!\mathscr{S}^{-1}\left[u^{\frac{51}{5}}\right] + 2\cdot 6!\mathscr{S}^{-1}\left[u^{\frac{51}{5}}\right] \\ &= 0, \\ y_2 &= \mathscr{S}^{-1}\left[u^{\frac{11}{5}}\mathscr{S}[t^2]\mathscr{S}[(y_0+y_1)^3 - y_0^3]\right] \\ &= 0, \\ y_n &= 0, n > 2. \end{aligned}$$

Therefore, the obtain solution is

$$y(t) = \sum_{n=0}^{\infty} y_n = t^2,$$

which is the exact solution.

**Example 4.5** *Consider the following initial value problem of nonlinear fractional Volterra integro-differential equation [4, 11]:*

$$D^\alpha y(t) = 1 + \int_0^t e^{-\tau} y^2(\tau)d\tau, \qquad (38)$$

*for* $0 \le t < 1, 3 < \alpha \le 4$ *and with the following initial conditions*

$$y(0) = 0, y'(0), y''(0), y'''(0) = 1. \qquad (39)$$

The exact solution of this problem for $\alpha = 4$ is $y(t) = e^t$. To solve this problem by the proposed method, we apply Sumudu transform on both side of (38) and using the inverse Sumudu transform, we have

$$\left.\begin{aligned} f(t) &= \mathscr{S}^{-1}\left[\sum_{k=0}^{3} u^k y^{(k)}(0)\right] + \mathscr{S}^{-1}\left[u^\alpha \mathscr{S}[1]\right], \\ N(y(t)) &= \mathscr{S}^{-1}\left[u^{\alpha+1}\mathscr{S}[e^{-t}y^2]\right]. \end{aligned}\right\} \qquad (40)$$

Following the same procedure as the previous example, we take the truncated Taylor expansions for exponential term in (40), that is $e^{-t} \approx 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}$.

By using the recursive relation (14) and (40), the recursive MSDM algorithm is

$$\left.\begin{aligned} y_0 &= 1, \\ y_1 &= t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^\alpha}{\Gamma(\alpha+1)} \\ &\quad + \mathscr{S}^{-1}\left[u^{\alpha+1}\mathscr{S}\left[(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!})y_0^2\right]\right], \\ y_{n+1} &= \mathscr{S}^{-1}\Big[u^{\alpha+1}\mathscr{S}\Big[(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!})\cdot \\ &\quad [(y_0 + \ldots + y_n)^2 - (y_0 + \ldots + y_{n-1})^2]\Big]\Big] \end{aligned}\right\} \qquad (41)$$

By using the recursive relation (41), we get

$$\begin{aligned} y_0 &= 1, \\ y_1 &= t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ &\quad - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{t^{\alpha+4}}{\Gamma(\alpha+5)} \\ &\vdots \end{aligned}$$

Hence, the 2-term approximate solution of problem (38) and (39) is

$$\begin{aligned} y_a(t) &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\ &\quad - \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{t^{\alpha+4}}{\Gamma(\alpha+5)}. \end{aligned} \qquad (42)$$

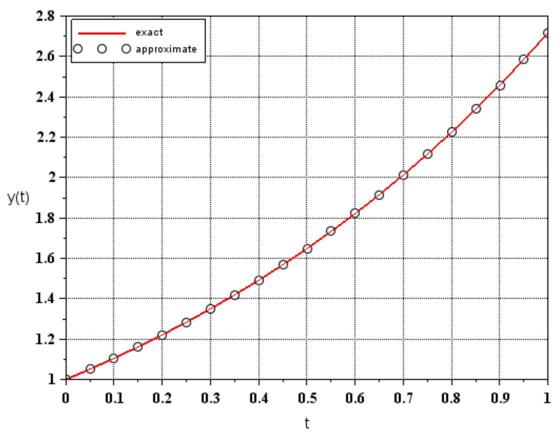


Figure 5: The comparison between the approximate and exact solution of Example 4.5, in the case $\alpha = 4$.

In the integer case, $\alpha = 4$, the exact solution is $y(t) = e^t$ and 2-term approximate solution by MSDM are plotted in Figure 5. It can be found that the obtained solution is close to the exact solution. It is remarkable to note that the accuracy of the obtained solution can be improved by taking more terms in the series approximate solution.

Figure 6 shows the behavior of the approximate solution of this problem using the MSDM for different values of $\alpha$. The numerical results for some $3 < \alpha < 4$, are shown in Table 1 with a comparison to Ref.[4] and [11]. Table 1 presents the MSDM numerical solutions to be good agreement with the numerical solution of ADM in [4] and CAS method in [11].

| $t$ | $\alpha = 3.25$ | | | $\alpha = 3.5$ | | | $\alpha = 3.75$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | ADM | CAS | MSMD | ADM | CAS | MSMD | ADM | CAS | MSMD |
| 0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| 0.1 | 1.10655 | 1.10526 | 1.10524 | 1.10675 | 1.10520 | 1.10519 | 1.10615 | 1.10518 | 1.10518 |
| 0.2 | 1.22393 | 1.22189 | 1.22204 | 1.22432 | 1.22159 | 1.22165 | 1.22323 | 1.22145 | 1.22148 |
| 0.3 | 1.35320 | 1.35231 | 1.35207 | 1.35376 | 1.35099 | 1.35085 | 1.35231 | 1.35027 | 1.35020 |
| 0.3 | 1.49560 | 1.49676 | 1.49735 | 1.49627 | 1.49408 | 1.49443 | 1.49464 | 1.49254 | 1.49276 |
| 0.5 | 1.65255 | 1.66349 | 1.65988 | 1.65327 | 1.65647 | 1.65421 | 1.65162 | 1.65218 | 1.65075 |
| 0.6 | 1.82566 | 1.84380 | 1.84185 | 1.82635 | 1.83337 | 1.83211 | 1.82482 | 1.82670 | 1.82589 |
| 0.7 | 2.01669 | 2.04438 | 2.04552 | 2.01930 | 2.02935 | 2.03024 | 2.01602 | 2.01941 | 2.02007 |
| 0.8 | 2.22763 | 2.27759 | 2.27329 | 2.22808 | 2.25366 | 2.25080 | 2.22718 | 2.23720 | 2.23530 |
| 0.9 | 2.46069 | 2.52650 | 2.52763 | 2.46093 | 2.49493 | 2.49614 | 2.46046 | 2.47265 | 2.47375 |

Table 1: Numerical results for Example 4.5 with comparison to ADM and CAS

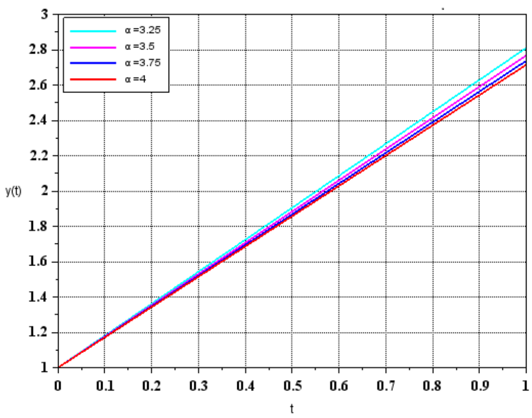


Figure 6: Approximate solutions for Example 4.5 for distinct values of $\alpha$.

## 5 Conclusion

In this article, the modification of Sumudu decomposition method based on combination of the Sumudu transform method and the iterative DGJ method is introduced to solve nonlinear fractional Volterra integro-differential equations. Several examples concerning nonlinear Volterra integro-differential equations of fractional order are tested to confirm the applicability and the advantages of the proposed method. The comparison made between exact solution and obtained solutions by MSDM. The achieved results are being well in agreement with exact solutions. The results show that the proposed method is powerful and efficient one for solving nonlinear fractional integro-differential equation in terms of its simplicity, implementation and high accuracy.

We expect that the proposed method can be applied to solve different type of fractional integro-differential problems and other fractional problems arising in science.

## Acknowledgment

This research project was financially supported by Mahasarakham University. The author is very grateful to the reviewers for their valuable comments and suggestions towards the improvement of this paper.

*References:*

[1] I. Podlubny, *Fractional Differential Equation, Mathematics in Science and Engineering*, Academic Press, New York, 1999.

[2] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin, Germany, 2010.

[3] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, San Diego, CA, 2006.

[4] S. Momani, M.A. Noor, Numerical methods for fourth-order fractional integro-differential equations, *Appl. Math. Comput*. Vol.182, 2006, 754-760.

[5] R. Mittal, R. Nigam, Solution of fractional integro-differential equations by Adomian decomposition method, *Int. J. Appl. Math. Mech*. Vol.4, No.2, 2008, 87-94.

[6] C. Yang, J. Hou, Numerical solution of integro-differential equations of fractional order by Laplace decomposition method, *WSEAS Trans. Math.*, Vol.12, No.12, 2013, 1173-1183.

[7] S. Tate, H.T. Dinde, A new modification of Adomian decomposition method for nonlinear fractional-order Volterra integro-differential equations, *World Journal of Modelling and Simulation*, Vol.15, No.1, 2019, pp. 33-41.

[8] A.A. Hamoud, K.P. Ghadle, Approximate solutions of Volterra integro-differential equations of fractional order by using analytical techniques, *Acta Univ. Apul.*, Vol.57, 2019, 63-74.

[9] K. Al-Khaled, M.H. Yousef, Sumudu decomposition method for solving higher-order nonlinear Volterra-Fredholm fractional integro-differential equations, *Nonlinear Dynamics and Systems Theory*, Vol.19, No.12, 2019, 348-361.

[10] X. Zhang, B. Tang and Y. He, Homotopy analysis method for higher-order fractional integro-differential equations, *Comput. lMath. Appl.*, Vol.62, 2011, 3194-3203.

[11] H. Saeedi, M.M. Moghadam, Numerical solution of nonlinear Volterra integro-differential equations of arbitrary order by CAS wavelets, *Commun Nonlinear Sci Numer Simulat*, Vol.16, 2011, 1216-1226.

[12] V. Daftardar-Gejji, H. Jafari, An iterative method for solving nonlinear functional equations, *J. Math. Analy. Appl*. 316 (2),2006, 753–763.

[13] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional integrals and derivatives: Theory and Applications*, Gordon and Breach, Amsterdam, 1993.

[14] G.K.Watugala, Sumudu transform: A new integral transform to solve differential equations and control engineering problems, *International Journal of Mathematical Education in Science and Technology*, Vol.24, No.1, 1993, pp. 35-43.

[15] F.B.M. Belgacem, A.A. Karaballi, Sumudu transform fundamental properties investigations and applications, *Journal of Applied Mathematics and Stochastic Analysis*, Vol.2006, Article ID 91083, 2006, pp. 1–23.

## Sources of funding for research presented in a scientific article or scientific article itself

**This research project was financially supported by Mahasarakham University.**

## Creative Commons Attribution License 4.0 (Attribution 4.0 International , CC BY 4.0)