Fixed Point Theorems in Partial 2-Metric Spaces

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Abstract: - In this paper we aim to introduce the concept of partial 2-metric spaces and study some fixed point theorems for self mappings defined on partial 2-metric spaces.

Key-Words: - Partial metric spaces, 2-metric spaces, partial 2-metric spaces, fixed point theorems, contractive conditions.

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1 Introduction

In recent years, several works on domain theory are created so as to equip the semantics domain with a notion of distance. Especially,the notion of fixed point theory and contraction mapping was extended and elaborated with the introduction of contraction principle by Banach [5]. Definition of 2-metric spaces was initiated by Gahler in a series of papers ([6]-[8]). The 2-metric space have a unique nonlinear structure, which is different from metric spaces. More several of the authors studied and generalize some theorems in 2-metric spaces. Fixed point theorem is an important tool in the theory of metric spaces, it guarantees the existence and uniqueness of fixed point of self maps of metric spaces. Iseki ([9]-[11]) obtained basic results on fixed point of operators in 2-metric spaces. After this work for Iseki, several authors studied and generalized fixed point theorems in 2-metric spaces. The notion of partial metric space was introduced by Matthews ([14],[15]). A partial metric space is obtained from metric space by replacing the equality $d(x, x) = 0$ in the definition of metric with the inequality $d(x, x) \leq d(x, y)$ for all x, y. This notion features a big range of applications not solely in several branches of mathematics, also within the field of computer domain and semantics ([1]-[4],[12],[13]). Recently, authors have targeted on partial metric spaces and its topological properties, and generalized fixed point theorems from the category of metric spaces to the class of partial metric spaces ([7]–[10]). In this paper we introduce the concept of partial 2-metric spaces and study the fixed point theorem under contraction self mapping on partial 2-metric spaces.

1.1 Preliminaries

Definition 1 [6]. *A 2-metric space is a set X with a non negative real valued function d on $X \times X \times X$ satisfying the following conditions: For every $x, y, z, u \in X$, we have:*

(M1) for two distinct point x, y in X there exist a point z in X such that $d(x, y, z) = 0$,

(M2) $d(x, y, z) = 0$ when at least two of x, y and z are equals,

(M3) $d(x, y, z) = d(x, z, y) = d(z, y, x)$,

(M4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$, and then the function d is called a 2-metric function on X.

Example 2 [13]. *Let a mapping $d : \mathbb{R}^3 \to [0, \infty)$ be defined by*

$$d(x, y, z) = min\{|x - y|, |y - z|, |z - x|\}.$$

Then d is a 2-metric on $\mathbb{R}$.

Definition 3 [6]. *A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be convergent to a point $x \in X$, that is $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} d(x_n, x, z) = 0$ for all $z \in X$, and the point x is called the limit of the sequence $\{x_n\}$ in X.*

Definition 4 [6]. *A sequence $\{x_n\}$ in a 2-metric space (X, d) is called a Cauchy sequence if $\lim_{m,n\to\infty} d(x_n, x_m, a) = 0$ for all $a \in X$.*

Definition 5 [6]. *A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.*

Remark 6 [6]. *Every convergent sequence in a 2-metric space is a Cauchy sequence .*

Definition 7 [14]. *Let X be a nonempty set. The mapping $p : X \times X \to [0, \infty)$ is said to be a partial metric on X if the following conditions are true. For any $x, y, z \in X$, we have:*

(PM-1) $x = y$ if and only if $p(x, x) = p(y, y) = p(x, y)$,

(PM-2) $p(x, x) \leq p(x, y)$,
(PM-3) $p(x, y) = p(y, x)$,
(PM-4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$, *and then the pair* (X, p) *is called a partial metric space, (for short* PMS*).*

2 Partial 2-metric spaces

In this section we have introduce the concept of partial 2-metric spaces and some properties.

Definition 8 *A mapping* $\rho : X^3 \to \mathbb{R}^+$ *where* X *is a non-empty set, is said to be a partial 2-metric on* X *if the following conditions are true. For every* $x, y, z, u \in X$*, we have:*
(P2M-1) $\rho(x, x, x) = \rho(y, y, y) = \rho(z, z, z) = \rho(x, y, z)$ *when at least two of* x, y *and* z *are equals,*
(P2M-2) $\rho(x, x, x) \leq \rho(x, y, z)$,
(P2M-3) $\rho(x, y, z) = \rho(x, z, y) = \rho(z, y, x)$,
(P2M-4) $\rho(x, y, z) \leq \rho(x, y, u) + \rho(x, u, z) + \rho(u, y, z) - \rho(u, u, u)$. *Then the pair* (X, ρ) *is called a partial 2-metric space; for short we write* $P2M$*-space.*

Example 9 *Let* $X = \{0, 1\}$*, and let* $\rho(x, y, z) = \begin{cases} 2 & \text{if } x = y = z = 0 \\ 1 & \text{otherwise} \end{cases}$*, then* (X, ρ) *is a* $P2M$*-space.*

Theorem 10 *Every 2-metric space is a* $P2M$*-space.*

Proof. Let (X, d) be a 2-metric space, then from the condition (M2) we obtain,

$$d(x, x, x) = d(y, y, y) = d(z, z, z) = 0,$$

when at least two point of any x, y, z are equals, that is (P2M-1) is satisfied. Since $d(x, y, z) \geq 0$, and

$$d(x, x, x) = 0 \leq d(x, y, z),$$

so,

$$d(x, x, x) \leq d(x, y, z),$$

which is the condition (P2M-2). Also from condition (M2), we have

$$d(x, y, z) = d(x, z, y) = d(z, y, x),$$

which is the condition (P2M-3). From the condition (M3), we have

$$d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z),$$

and $d(u, u, u) = 0$, then we can write

$$d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z) - d(u, u, u).$$

So (X, d) is a $P2M$-space. ■

From Example 9 shows that the inverse is not true, then

we have (X, ρ) is a $P2M$-space but it is not 2-metric space.

Definition 11 *A modified* $P2M$*-space indefined by replacing the condition (P2M-1) in Definition 8 by the following:*
*(P2M-1)** $x = y = z$, *if and only if* $\rho(x, x, x) = \rho(y, y, y) = \rho(z, z, z) = \rho(x, y, z)$, *then the pair* (X, ρ) *is called a modified partial 2-metric space; for short we write (*$P2M$*)*-space.*

Example 12 *Let* $\rho : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *defined by* $\rho(x, y, z) = \max\{x, y, z\}$ *for any* $x, y, z \in \mathbb{R}^+$*. Then the pair* $(\mathbb{R}^+, \rho)$ *is a (*$P2M$*)*-space.*

Example 13 *Let* $\mathbb{R}^+ = (0, \infty)$ *and* $\mathbb{R}^- = (-\infty, 0)$*. Consider the function* $\rho : \mathbb{R}^- \times \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}^+$ *defined by* $\rho(x, y, z) = -\min\{x, y, z\}$ *for any* $x, y, z \in \mathbb{R}$*. Then the pair* $(\mathbb{R}^-, \rho)$ *is a (*$P2M$*)*-space.*

Example 14 *Consider* $I = \{[a, b] : a \leq b; a, b \in \mathbb{R}\}$ *is the set of all closed intervals in* $\mathbb{R}$ *, let the function* $\rho : I^3 \to \mathbb{R}^+$*, which in defined by* $\rho([a, b], [c, d], f, g]) = \max\{b, d, g\} - \min\{a, c, f\}$*. Then the pair* (I, ρ) *is a (*$P2M$*)*-space.*

Example 15 *Let* $X = [0, a]$ *and* $\alpha \geq a \geq 3$*. Define the mapping* $\rho : X^3 \to \mathbb{R}^+$ *by* $\rho(x, y, z) =$

$$\begin{cases} x\,, & \text{if } \; x = y = z = 0\,, \\ \frac{4\alpha + x + y + z}{3}, & \text{if } \; x, y, z \in \{1, 2, 3\} \text{ and } x \neq y \neq z, \\ \frac{\alpha + x + y + z}{3}\,, & \text{otherwise.} \end{cases}$$

Then the pair (X, ρ) *is a (*$P2M$*)*-space .*

Remark 16 *Every* $P2M$*-space is (*$P2M$*)*-space and the inverse is not true.*

Theorem 17 *Let* (X, ρ) *be a* $P2M$*-space, and the function* $d_\rho : X^3 \to [0, \infty)$*, defined by* $d_\rho(x, y, z) = 3\rho(x, y, z) - \rho(x, x, x) - \rho(y, y, y) - \rho(z, z, z)$*, then* (X, d_ρ) *is a 2-metric space.*

Proof. It's clear that for all distinct elements $x, y, z \in X$, we have

$$d_\rho(x, y, z) \neq 0,$$

and from the condtion (P2M-1), we get

$$d_\rho(x, y, z) = 3\rho(x, y, z) - \rho(x, x, x) - \rho(y, y, y) - \rho(z, z, z) = 0,$$

when at least two points x, y, z in X are equals. From the condtion (P2M-3), we obtain

$$d_\rho(x, y, z) = d_\rho(x, z, y) = d_\rho(z, y, x).$$

Also from the condtion (P2M-4), we have

$$\begin{aligned} & d_\rho(x, y, z) \\ & = 3\rho(x, y, z) - \rho(x, x, x) - \rho(y, y, y) \\ & - \rho(z, z, z). \\ & \leq 3 \times [\rho(x, y, u) + \rho(x, u, z) \\ & + \rho(u, y, z) - \rho(u, u, u)] \\ & - \rho(x, x, x) - \rho(y, y, y) - \rho(z, z, z). \\ & = 3\rho(x, y, u) - \rho(u, u, u) - \rho(x, x, x) \\ & - \rho(y, y, y) + 3\rho(x, u, z) - \rho(u, u, u) \\ & - \rho(x, x, x) - \rho(z, z, z) + 3\rho(u, y, z) \\ & - \rho(u, u, u) - \rho(y, y, y) - \rho(z, z, z) \\ & + \rho(x, x, x) + \rho(y, y, y) + \rho(z, z, z). \end{aligned}$$

This implies,

$$d_\rho(x, y, z) \leq d_\rho(x, y, u) + d_\rho(x, u, z) + d_\rho(u, y, z).$$

Then d_ρ is a 2-metric on X, which the proof . ■

Definition 18 *A sequence* $\{x_n\}$ *in a* $P2M$*-space* (X, ρ) *converges to a point* x *in* X *if* $\lim_{n\to\infty} \rho(x_n, x, z) = \rho(x, x, x)$ *for all* z *in* X.

Definition 19 *A sequence* $\{x_n\}$ *in a* $P2M$*-space* (X, ρ) *is said to be a Cauchy sequence if the* $\lim_{n,m\to\infty} \rho(x_m, x_n, z)$ *exists and finite, for all* z *in* X.

Definition 20 *A* $P2M$*-space* (X, ρ) *is said to be complete if every Cauchy sequence in* X *converges to an element* x *in* X *such that* $\lim_{n,m\to\infty} \rho(x_m, x_n, z) = \rho(x, x, x)$*, for all* z *in* X.

Definition 21 *Let* (X, ρ) *be a* $P2M$*-space, a function* $f : X \to X$ *is said to be a contraction if there exist a constant* $c \in [0, 1)$*, such that* $\rho(fx, fy, z) \leq c \times \rho(x, y, z)$ *for all* $x, y, z \in X$.

Theorem 22 *Let* (X, ρ) *be a complete* $P2M$*-space and* f *be a self-mapping on* X *satisfying the condition* $\rho(fx, fy, z) \leq c \times \rho(x, y, z)$ *for all* $x, y, z \in X$ *and* $c \in [0, 1)$*. Then* f *has a unique fixed point.*

Proof. Let $x \in X$ and for a fixed $z \in X$, it is clear that from the condtion (P2M-4) for each $n, k \in \mathbb{N}$, we have

$$\begin{aligned} & \rho(f^{n+k+1}(x), f^n(x), z) \\ & \leq \rho(f^{n+k+1}(x), f^n(x), f^{n+k}(x)) \\ & + \rho(f^{n+k+1}(x), f^{n+k}(x), z) \\ & + \rho(f^{n+k}(x), f^n(x), z) \\ & - \rho(f^{n+k}(x), f^{n+k}(x), f^{n+k}(x)) \, . \end{aligned}$$

This implies,

$$\begin{aligned} & \rho(f^{n+k+1}(x), f^n(x), z) \\ & \leq \rho(f^{n+k+1}(x), f^n(x), f^{n+k}(x)) \\ & + \rho(f^{n+k+1}(x), f^{n+k}(x), z) \\ & + \rho(f^{n+k}(x), f^n(x), z). \end{aligned}$$

Using the contraction condition it follows that

$$\begin{aligned} & \rho(f^{n+k+1}(x), f^n(x), f^{n+k}(x)) \\ & = \rho(f(f^{n+k}(x)), f^{n+k}(x), f^n(x)) \\ & \leq c^{n+k} \times \rho(f(x), x, f^n(x)); \end{aligned}$$

and

$$\begin{aligned} & \rho(f^{n+k+1}(x), f^{n+k}(x), z) \\ & = \rho(f(f^{n+k})(x), f^{n+k}(x), z) \\ & \leq c^{n+k} \times \rho(f(x), x, z). \end{aligned}$$

From (2) and (3) into (1), we get

$$\begin{aligned} & \rho(f^{n+k+1}(x), f^n(x), z) \\ & \leq c^{n+k} \times \rho(f(x), x, f^n(x)) + c^{n+k} \times \rho(f(x), x, z) \\ & + \rho(f^{n+k}(x), f^n(x), z). \end{aligned}$$

Also we have,

$$\begin{aligned} & \rho(f^{n+k}(x), f^n(x), z) \\ & \leq \rho(f^{n+k}(x), f^{n+k-1}(x), z) \\ & + \rho(f^{n+k}(x), f^n(x), f^{n+k-1}(x)) \\ & + \rho(f^{n+k-1}(x), f^n(x), z) \\ & - \rho(f^{n+k-1}(x), f^{n+k-1}(x), f^{n+k-1}(x)). \end{aligned}$$

So,

$$\begin{aligned} & \rho(f^{n+k}(x), f^n(x), z) \\ & \leq \rho(f^{n+k}(x), f^{n+k-1}(x), z) \\ & + \rho(f^{n+k}(x), f^n(x), f^{n+k-1}(x)) \\ & + \rho(f^{n+k-1}(x), f^n(x), z). \\ & \leq c^{n+k-1} \times \rho(f(x), x, f^n(x)) \\ & + c^{n+k-1} \times \rho(f(x), x, z) \\ & + \rho(f^{n+k-1}(x), f^n(x), z). \end{aligned}$$

Thus,

$$\begin{aligned}&\rho(f^{n+k+1}(x), f^{n+k}(x), z)\\ &\le c^{n+k} \times \rho(f(x), x, z) + c^{n+k-1} \times \rho(f(x), x, z) + ...\\ &+ c^{n} \times \rho(f(x), x, z) + c^{n+k} \times \rho(f(x), x, f^{n}(x))\\ &+ c^{n+k-1} \times \rho(f(x), x, f^{n}(x)) + ...\\ &+ c^{n} \times \rho(f(x), x, f^{n}(x)) + \rho(f^{n}(x), f^{n}(x), z).\end{aligned}$$

Then,

$$\begin{aligned}&\rho(f^{n+k+1}(x), f^{n+k}(x), z)\\ &\le [c^{n+k} + c^{n+k-1} + ... + c^{n}]\\ &\times [\rho(f(x), x, z) + \rho(f(x), x, f^{n}(x))]\\ &+ c^{n} \times \rho(f(x), x, z).\\ &\le [c^{n+k} + c^{n+k-1} + ... + c^{n}]\\ &\times [\rho(f(x), x, z) + \rho(f(x), x, f^{n}(x))]\\ &+ c^{n} \times \rho(f(x), x, z).\end{aligned}$$

Which implies,

$$\begin{aligned}&\rho(f^{n+k+1}(x), f^{n+k}(x), z)\\ &\le c^{n} \times \frac{1-c^{k+1}}{1-c}\\ &\times [\rho(f(x), x, z) + \rho(f(x), x, f^{n}(x))]\\ &+ c^{n} \times \rho(x, x, z).\\ &\le c^{n} \times \frac{c^{n}}{1-c}\\ &\times [\rho(f(x), x, z) + \rho(f(x), x, f^{n}(x))]\\ &+ c^{n} \times \rho(x, x, z).\end{aligned}$$

Consequently $\{f^{n}(x)\}$ is a Cauchy sequence in the $P2M$-space (X, ρ), and

$$\lim_{n,m\to\infty} \rho(f^{m}(x), f^{n}(x), z) = 0 \ \ \forall\, z \in X\ .$$

Since X is Complete $P2M$-space. Then we can choose $a \in X$ such that $f^{n}(x)$ converges to a, so,

$$\begin{aligned}\lim_{n,m\to\infty} \rho(f^{m}(x), f^{n}(x), z) &= \lim_{n,m\to\infty} \rho(f^{n}(x), a, z)\\ &= \rho(a, a, a) = 0 \ \ \forall\, z \in X\ .\end{aligned}$$

In the following we will show that $a \in X$ is a fixed point for f. We have,

$$\begin{aligned}&\rho(f(a), a, z)\\ &\le \rho(f(a), f^{n+1}(x), z) + \rho(f(a), a, f^{n+1}(x))\\ &+ \rho(f^{n+1}(x), a, z).\\ &= \rho(f^{n+1}(x), f(a), z) + \rho(f^{n+1}(x), f(a), a)\\ &+ \rho(f^{n+1}(x), a, z).\\ &\le c \times [\rho(f^{n}(x), a, z) + \rho(f^{n}(x), a, a)]\\ &+ \rho(f^{n+1}(x), a, z).\end{aligned}$$

As $n \to \infty$, we get

$$\rho(f(a), a, z) = 0, \forall z \in X,$$

so, $f(a) = a$. For the uniqueness proof, we assume that there exist another fixed point $b \in X$, so, $f(b) = b$. Now,

$$\rho(b, a, z) = \rho(f(b), f(a), z) \le c \times \rho(b, a, z).$$

This means that $c \ge 1$, which contradicts that $c \in [0, 1)$. So, we must have $\rho(b, a, z) = 0,\ \forall\ z \in X$. Then $a = b$, which the proof. ■

Lemma 23 *Assume that $x_n \to x$ as $n \to \infty$ in a $P2M$-space such that $\rho(x, x, x) = 0$ then $\lim_{n\to\infty} \rho(x_n, y, z) = \rho(x, y, z)$ for every x, y and z in X.*

Proof. First note that $\lim_{n\to\infty} \rho(x_n, x, z) = \rho(x, x, x) = 0\ \forall\ z \in X$.

From the condition (P2M-4) we find that

$$\begin{aligned}&\rho(x_n, y, z)\\ &\le \rho(x_n, y, x) + \rho(x_n, x, z) + \rho(x, y, z)\\ &- \rho(x, x, x),\\ &\le \rho(x_n, y, x) + \rho(x_n, x, z) + \rho(x, y, z).\end{aligned}$$

$$\rho(x_n, y, z) - \rho(x, y, z) \le \rho(x_n, y, x) + \rho(x_n, x, z).$$

Also,

$$\begin{aligned}&\rho(x, y, z)\\ &\le \rho(x, y, x_n) + \rho(x, x_n, z)\\ &+ \rho(x_n, y, z) - \rho(x_n, x_n, x_n),\\ &\le \rho(x, y, x_n) + \rho(x, x_n, z) + \rho(x_n, y, z).\end{aligned}$$

$$\rho(x, y, z) - \rho(x_n, y, z) \le \rho(x, y, x_n) + \rho(x, x_n, z).$$

Hence,

$$\begin{aligned}&\rho(x_n, y, x) + \rho(x_n, x, z)\\ &\le |\rho(x, y, z) - \rho(x_n, y, z)|\\ &\le \rho(x, y, x_n) + \rho(x, x_n, z).\end{aligned}$$

Let $n \to \infty$, and from the condition of (P2M-1) we conclude the claim. ■

Lemma 24 *(1) A sequence $\{x_n\}$ is a Cauchy sequence in the a $P2M$-space (X, ρ) if and only if $\{x_n\}$ is also a Cauchy sequence in the 2-metric space (X, d_ρ).*

(2) (X, ρ) is complete if and only if (X, d_ρ) is also complete. Moreover

$$\lim_{n\to\infty} \rho(x, x_n, z) = \lim_{n,m\to\infty} \rho(x_n, x_m, z) = \rho(x, x, x)$$

$$\Leftrightarrow \lim_{n\to\infty} d_\rho(x, x_n, z) = 0.$$

Proof. (1) Let $\{x_n\}$ is a Cauchy sequence in the $P2M$-space (X, ρ), then there exist $\alpha \in \mathbb{R}$ such that $\varepsilon > 0$, there is $n(\varepsilon) \in \mathbb{N}$, we have

$$|\rho(x_n, x_m, z) - \alpha| \leq \frac{\varepsilon}{3} \text{ for all } n, m \geq n(\varepsilon) \text{ and } z \in X.$$

Since,

$$\begin{aligned} & d_\rho(x_n, x_m, z) \\ & = 3\rho(x_n, x_m, z) - \rho(x_n, x_n, x_n) \\ & - \rho(x_m, x_m, x_m) - \rho(z, z, z) \\ & = [3\rho(x_n, x_m, z) - 3\alpha] - [\rho(x_n, x_n, x_n) - \alpha] \\ & - [\rho(x_m, x_m, x_m) - \alpha] - [\rho(z, z, z) - \alpha]. \end{aligned}$$

From P2M-2, we get $\rho(x_n, x_n, x_n) \leq \rho(x_n, x_m, z)$, then,

$$|\rho(x_n, x_n, x_n) - \alpha| \leq \frac{\varepsilon}{3}.$$

Then,

$$\begin{aligned} & d_\rho(x_n, x_m, z) \\ & \leq 3\,|\rho(x_n, x_m, z) - \alpha| + |\rho(x_n, x_n, x_n) - \alpha| \\ & + |\rho(x_m, x_m, x_m) - \alpha| + |\rho(z, z, z) - \alpha| \end{aligned}$$

$$d_\rho(x_n, x_m, z) \leq \varepsilon \quad \forall\, n, m \geq n(\varepsilon).$$

Then $\{x_n\}$ is a Cauchy sequence in (X, d_ρ). Now, we prove that every Cauchy sequence$\{x_n\}$ in (X, d_ρ) is a Cauchy sequence in (X, ρ). Since $\{x_n\}$ is a Cauchy sequence in (X, d_ρ), then

$$d_\rho(x_n, x_m, z) \leq \varepsilon \;\; \forall \;\; n, m \geq n_0(\varepsilon).$$

Take $\varepsilon = \frac{1}{2}$ then there exist $n_0(\varepsilon) \in \mathbb{N}$ such that

$$d_\rho(x_n, x_m, z) \leq \frac{1}{2} \;\; \forall\, n, m \geq n_0(\varepsilon).$$

Since,

$$\begin{aligned} & d_\rho(x_n, x_{n_0}, z) + \rho(x_n, x_n, z) \\ & = d_\rho(x_{n_0}, x_n, z) + \rho(x_{n_0}, x_{n_0}, z), \end{aligned}$$

then we have,

$$\begin{aligned} & |\rho(x_n, x_n, z)| \\ & = |d_\rho(x_{n_0}, x_n, z) - d_\rho(x_n, x_{n_0}, z) + \rho(x_{n_0}, x_{n_0}, z)| \\ & \leq 2\, d_\rho(x_{n_0}, x_n, z) + |\rho(x_{n_0}, x_{n_0}, z)| \\ & < 1 + |\rho(x_{n_0}, x_{n_0}, z)|\,. \end{aligned}$$

Consequentely, the sequence $\{\rho(x_n, x_n, z)\}_n$ is bounded in $\mathbb{R}$, and so there exist $a \in \mathbb{R}$, such that a subsequence $\{\rho(x_{n_k}, x_{n_k}, z)\}_k$ is convergent to a, i.e $\lim_{k\to\infty} \rho(x_{n_k}, x_{n_k}, z) = a$. It remins to prove that $\{\rho(x_n, x_n, z)\}_n$ is a Cauchy sequence in $\mathbb{R}$. Since $\{x_n\}$ is Cauchy sequence (X, d_ρ) given $\varepsilon > 0$ $\exists$ $n(\varepsilon) \in \mathbb{N}$ such that $d_\rho(x_n, x_m, z) < \frac{3\,\varepsilon}{2}$ $\forall\, n, m \geq n(\varepsilon)$. Thus,

$$\begin{aligned} & |\rho(x_n, x_n, z) - \rho(x_m, x_m, z)| \\ & = \frac{1}{3}\,|d_\rho(x_m, x_n, z) - d_\rho(x_n, x_m, z)| \\ & \leq \frac{2}{3}\, d_\rho(x_m, x_n, z) = \varepsilon. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \rho(x_n, x_n, z) = a, \text{ and we get}$$

$$\begin{aligned} & |\rho(x_n, x_m, z) - \alpha| \\ & = |\rho(x_n, x_m, z) - \rho(x_n, x_n, z) + \rho(x_n, x_n, z) - \alpha| \\ & \leq |\rho(x_n, x_m, z) - \rho(x_n, x_n, z)| \\ & + |\rho(x_n, x_n, z) - \alpha| \\ & < d_\rho(x_n, x_m, z) + |\rho(x_n, x_n, z) - \alpha| \\ & < \varepsilon,\ \forall n, m \geq n(\varepsilon). \end{aligned}$$

Then $\{x_n\}$ is a Cauchy sequence in (X, ρ).

(2) Now, we prove completeness of (X, d_ρ) implies completeness of (X, ρ). If $\{x_n\}$ is a Cauchy sequence in (X, ρ) then it is also a Cauchy sequence in (X, d_ρ). Since (X, d_ρ) is a complete 2-metric space, then there exist $y \in X$ such that

$$\begin{aligned} \lim_{n\to\infty} d_\rho(y, x_n, z) = \lim_{n\to\infty} & 3\rho(y, x_n, z) - \rho(y, y, y) \\ & - \rho(x_n, x_n, x_n) - \rho(z, z, z). \end{aligned}$$

Since $\{x_n\}$ is a Cauchy sequence in (X, ρ), let $\varepsilon > 0$ then $\exists$ $n_0 \in \mathbb{N}$, such that $d_\rho(x_m, x_n, z) \leq \frac{3\varepsilon}{2}$. Then,

$$\begin{aligned} & |\rho(x_n, x_n, z) - \rho(x_m, x_m, z)| \\ & = \frac{1}{3}\,|d_\rho(x_m, x_n, z) - d_\rho(x_n, x_m, z)| \\ & \leq \frac{2}{3}\, d_\rho(x_m, x_n, z) = \varepsilon \;\; \forall\, n, m > n_0. \end{aligned}$$

This shows that (X, ρ) is complete.

Conversely, we prove (X, d_ρ) is complete. Let $\{x_n\}$ is a Cauchy sequence in (X, d_ρ). Then $\{x_n\}$ is a Cauchy sequence in (X, ρ) and it is convergent to $y \in X$ with

$$\lim_{n,m\to\infty} \rho(x_n, x_m, z) = \lim_{n\to\infty} \rho(y, x_n, z) = \rho(y, y, y).$$

Then,

$$\rho(y, x_n, z) - \rho(y, y, y) < \varepsilon; n \geq n(\varepsilon).$$

Since,

$$\begin{aligned} & d_\rho(y,x_n,z) \\ &= 3\rho(y,x_n,z)-\rho(y,y,y) \\ &- \rho(x_n,x_n,x_n)-\rho(z,z,z) \\ &< |\rho(y,x_n,z)-\rho(y,y,y)| \\ &+ |\rho(y,x_n,z)-\rho(x_n,x_n,x_n)| \\ &+ |\rho(y,x_n,z)-\rho(z,z,z)| \\ &= 3\varepsilon < \varepsilon. \end{aligned}$$

Then $d_\rho(y,x_n,z) < \varepsilon \ \forall\, n \geq n(\varepsilon)$. Then (X,d_ρ) is complete. Finally, it's a simple to check that $\lim_{n\to\infty} d_\rho(x,x_n,z)=0$ if and only if

$$\lim_{n\to\infty}\rho(x,x_n,z)=\lim_{n,m\to\infty}\rho(x_n,x_m,z)=\rho(x,x,x).$$

■

Theorem 25 *Let (X,ρ) be a $P2M$-space, let $T : X\to X$ be a map for which the inequality*

$$\begin{aligned} & a\rho(Tx,Ty,z)+b[\rho(x,Tx,z)+\rho(y,Ty,z) \\ &+c[\rho(x,Ty,z)+\rho(y,Ty,z) \qquad (1) \\ &\leq s\rho(x,y,z)+r\rho(x,T^2x,z), \qquad (4) \end{aligned}$$

holds for all x,y in X where the constants a,b,c,r and s satsisfy

$$\begin{aligned} & 0 \leq \frac{s-b}{a+b} < 1, \\ & a+b\neq 0,\ a+b+c>0,\ c-r>0,\ c>0. \end{aligned}$$

Then T has at least one fixed point.

Proof. Take an arbitrary point $x_0\in X$, define the

sequence $x_{n+1}=Tx_n$, $n=0,1,2,3\ldots$. Sustituting $x=x_n,\ y=\ x_{n+1}$ into equation (4), we have

$$\begin{aligned} & a\rho(Tx_n,Tx_{n+1},z)+b[\rho(x_n,Tx_n,z) \\ &+\rho(x_{n+1},Tx_{n+1},z)]+c[\rho(x_n,Tx_{n+1},z) \\ &+\rho(x_{n+1},Tx_{n+1},z)] \\ &\leq s\rho(x_n,x_{n+1},z)+r\rho(x_n,T^2x_n,z), \end{aligned}$$

which implies

$$\begin{aligned} & a\rho(x_{n+1},x_{n+2},z)+b[\rho(x_n,x_{n+1},z) \\ &+\rho(x_{n+1},x_{n+2},z)]+c[\rho(x_n,x_{n+2},z) \\ &+\rho(x_{n+1},x_{n+1},z)] \\ &\leq s\rho(x_n,x_{n+1},z)+r\rho(x_n,x_{n+2},z). \end{aligned}$$

Rewriting this inequality as

$$\begin{aligned} & (a+b)\rho(x_{n+1},x_{n+2},z)+(c-r)\rho(x_n,x_{n+2},z) \\ &+c\rho(x_{n+1},x_{n+1},z) \\ &\leq (s-b)\rho(x_n,x_{n+1},z), \end{aligned}$$

and using the fact

$$(c-r)\rho(x_n,x_{n+2},z)+c\rho(x_{n+1},x_{n+1},z)\geq 0$$

where $c-r>0$, $c>0$. Then we obtain

$$\rho(x_{n+1},x_{n+2},z)\leq \alpha\rho(x_n,x_{n+1},z)$$

$\alpha=\frac{(s-b)}{(a+b)}$, $(a+b)\neq 0$, $0\leq\alpha\leq 1$. Thus,

$$\begin{aligned} \rho(x_{n+1},x_{n+2},z) &\leq \alpha\rho(x_n,x_{n+1},z) \\ &\leq \alpha^2\rho(x_{n-1},x_n,z) \\ &\leq \alpha^3\rho(x_{n-2},x_{n-1},z) \\ &\leq \ldots \leq \alpha^{n+1}\rho(x_0,x_1,z). \end{aligned}$$

We will show $\{x_n\}$ is a cauchy sequence. Since

$$\rho(x_n,x_{n+1},z)\leq \alpha^{n+1}\rho(x_0,x_1,z).$$

Taking $n\to\infty,\ 0\leq\alpha\leq 1,$
then $\lim_{n\to\infty}\rho(x_n,x_m,z)\to 0$ (exist and finite).

Then $\{x_n\}$ is a Cauchy sequence in (X,ρ). By Lemma 24, $\{x_n\}$ is also a Cauchy sequence in (X,d_ρ). Since (X,ρ) is complete then (X,d_ρ) is also complete. Thus there exists $x\in X$ such that $x_n\to x$ in (X,d_ρ); moreover, by Lemma 24,

$$\begin{aligned} & \lim_{n\to\infty}\rho(x,x_n,z) \\ &= \lim_{n,m\to\infty}\rho(x_n,x_m,z) \\ &= \rho(x,x,x)=0 \Leftrightarrow \lim_{n\to\infty} d_\rho(x,x_n,z)=0. \end{aligned}$$

Now, we will show that x be a fixed point of T. Substituting $x=x_n$ and $y=x$ into (4), we obtain

$$\begin{aligned} & a\rho(Tx_n,Tx,z)+b[\rho(x_n,Tx_n,z) \\ &+\rho(x,Tx,z)] \\ &+c[\rho(x_n,Tx,z)+\rho(x,Tx_n,z)] \\ &\leq s\rho(x_n,x,z)+r\rho(x,T^2x_n,z), \end{aligned}$$

which implies

$$\begin{aligned} & a\rho(x_{n+1},Tx,z)+b[\rho(x_n,x_{n+1},z) \\ &+\rho(x,Tx,z)]+c[\rho(x_n,Tx,z) \\ &+\rho(x_n,Tx_n,z)] \\ &\leq s\rho(x_n,x,z)+r\rho(x,x_{n+2},z). \end{aligned}$$

Taking the limit as $n\to\infty$ and using Lemma 23, we get

$$(a+b+c)\rho(x,Tx,z)\leq 0.$$

Since $(a+b+c)>0$,
then we have $0 \le (a+b+c)\rho(x,Tx,z) \le 0$,

this means that $\rho(x,Tx,z)=0$. From theorem 17, we get

$$0 \le d_\rho(x,Tx,z) = 3\,\rho(x,Tx,z) - \rho(x,x,x) - \rho(Tx,Tx,Tx) - \rho(z,z,z)$$

$$= -\rho(Tx,Tx,Tx) - \rho(z,z,z) \le 0,$$

hence $d_\rho(x,Tx,z)=0$, that is $x=Tx$, which the proof. ■

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