Bi-univalent Function Subfamilies Defined by q - Analogue of Bessel Functions Subordinate to (p, q) - Lucas Polynomials

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Abstract: - In the theory of bi-univalent functions,variety of special polynomials and special functions have been used. Using the q - analogue of Bessel functions, two families of regular and bi-univalent functions subordinate to (p, q) - Lucas polynomials are introduced in this paper. For elements in these defined families, we derive estimates for $|a_2|$, $|a_3|$ and for δ a real number we consider Fekete-Szegö problem $|a_3 - \delta a_2^2|$. We also provide relevent connection to existing result and discuss few interesting observations of the results investigated.

Key-Words: - Fekete-Szegö inequality, Bessel function, Bi-univalent functions, q - derivative operator, (p, q)-Lucas polynomials.

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1 Introduction

Let $\mathbb{C}$ be the set of complex numbers and $\Delta = \{\varsigma : \varsigma \in \mathbb{C}$ and $|\varsigma| < 1\}$ be the unit disk. Let $\mathbb{R}$ and $\mathbb{N} := \{1, 2, 3, ...\} = \mathbb{N}_0 \backslash \{0\}$ be the sets of real numbers and natural numbers, respectively. Let $\mathcal{A}$ denote the family of functions of the form

$$f(\varsigma) = \varsigma + \sum_{n=2}^{\infty} a_n \varsigma^n \quad (1)$$

which are holomorphic in Δ. Further, we denote the subfamily of $\mathcal{A}$ which are univalent in Δ by $\mathcal{S}$. According to the well-known theorem of Koebe, every function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus, every $f \in \mathcal{S}$ has an inverse f^{-1} satisfying

$$f^{-1}(f(\varsigma)) = \varsigma \ (\varsigma \in \Delta) \ \ and \ \ f(f^{-1}(w)) = w$$

where

$$|w| < r_0(f), r_0(f) \geqq \frac{1}{4}$$

and is in fact given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots := g(w). \quad (2)$$

If a function f and its inverse f^{-1} are both univalent in Δ, then a member f of $\mathcal{A}$ is called bi-univalent (or bi-schlicht) in Δ . The family of bi-univalent (or bi-schlicht) functions in Δ given by (1) is indicated by σ. The functions $-log(1-\varsigma)$, $\frac{1}{2} \, log \left(\frac{1+\varsigma}{1-\varsigma}\right)$, $\frac{\varsigma}{1-\varsigma}$ and so on are members of the class σ. However, the familiar Koebe function as well as $\varsigma - \frac{\varsigma^2}{2}$, $\frac{\varsigma}{1-\varsigma^2}$ (members of $\mathcal{S}$) are not members the class σ.

Lewin [18] examined the family σ and proved that $|a_2| < 1.51$ for elements in the family σ. Later, Brannan and Clunie[6] claimed that $|a_2| \leq \sqrt{2}$ for $f \in \sigma$. Subsequently, Tan [30] found the initial coefficient bounds of bi-univalent functions. Brannan and Taha [5] proposed bi-convex and bi-starlike functions which are similar to well-known subfamilies of $\mathcal{S}$. The momentum on investigation of the family σ was gained in recent years, which is due to the paper of Srivastava et al.][23] and that has led to a large number of papers in recent times. Some interesting results concerning initial bounds for certain special sets of σ have been examined in ([2],[7],[10],[11],[16],[17],[21] and [24]) on subfamilies of σ. They have obtained estimates on $|a_2|$, $|a_3|$ and $|a_3 - \delta a_2^2|$, $\delta \in \mathbb{R}$, a_2 and a_3 being the first two coefficients of Taylor-Maclaurin's expansion. However, the problem of finding the bounds on $|a_n|$ $(n = 3, 4, \cdots)$ for members of σ is still open.

Let Γ be the Gamma function. The first kind Bessel function of order ν is defined by (see [20])

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n (\varsigma/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)}, (\varsigma \in \mathbb{C},\ \nu \in \mathbb{R}). \quad (3)$$

Recently, Szasz and Kupan [25] found the univalence of the first kind Bessel function $\kappa_\nu : \Delta \to \mathbb{C}$ defined by

$$\begin{aligned} \kappa_\nu(z) &:= 2^\nu \Gamma(\nu+1)\varsigma^{1-\nu/2} J_\nu(\varsigma^{1/2}) \\ &= z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)} \varsigma^n, \quad (4) \end{aligned}$$

where $\varsigma \in \Delta,\ \nu \in \mathbb{R}$.
For κ_ν, the $q-$derivative operator ($0 < q < 1$) is defined by

$$\begin{aligned} \partial_q \kappa_\nu(\varsigma) &:= \frac{\kappa_\nu(q\varsigma) - \kappa_\nu(\varsigma)}{\varsigma(q-1)} \\ &= \partial_q \left[\varsigma + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)} \varsigma^n \right] \\ &= 1 + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)} [n,q]\varsigma^{n-1} \quad (5) \end{aligned}$$

where $\varsigma \in \Delta$,

$$[n,\ q] := \frac{1-q^n}{1-q} = 1 + \sum_{j=1}^{n-1} q^j, \qquad [0,\ q] := 0. \quad (6)$$

Using (6), we will define the following:

1. For any $n \in \mathbb{N}_0$,

$$[n,\ q]! := \begin{cases} 1, & \text{if } n = 0 \\ [1,q][2,q]\ldots[n,q] & \text{if } n \in \mathbb{N}. \end{cases} \quad (7)$$

   is the q - shifted factorial.

2. For any $n \in \mathbb{N}_0$,

$$[r,\ q]_n := \begin{cases} 1, & \text{if } n = 0 \\ [r,\ q][r+1,q]\ldots[r+n-1,q] & \text{if } n \in \mathbb{N}. \end{cases} \quad (8)$$

   is the q - generalized Pochhammer symbol.

For $0 < q < 1, \nu > 0$ and $\lambda > -1$, El-Deeb and Bulboacă [9] defined the function $\mathcal{J}^\lambda_{\nu,\,q} : \Delta \to \mathbb{C}$ by

$$\mathcal{J}^\lambda_{\nu,\,q}(\varsigma) := \varsigma + \quad (9)$$

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)} \frac{[n,q]!}{[\lambda+1,q]_{n-1}} \varsigma^n,\ \varsigma \in \Delta.$$

A computation shows that

$$\mathcal{J}^\lambda_{\nu,\,q}(\varsigma) * \mathcal{M}_{q,\,\lambda+1}(\varsigma) = z\partial_q \kappa_\nu(\varsigma), \quad \varsigma \in \Delta, \quad (10)$$

where $\mathcal{M}_{q,\,\lambda+1}(\varsigma)$ is given by

$$\mathcal{M}_{q,\,\lambda+1}(\varsigma) := \varsigma + \sum_{n=2}^{\infty} \frac{[\lambda+1,q]_{n-1}}{[n-1,q]!} \varsigma^n, \quad \varsigma \in \Delta. \quad (11)$$

Using the idea of convolutions and the definition of q - derivative, El-Deeb and Bulboacă [9] examined the operator $\mathcal{N}^\lambda_{\nu,\,q} : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned} \mathcal{N}^\lambda_{\nu,\,q} f(\varsigma) &:= \mathcal{J}^\lambda_{\nu,\,q}(\varsigma) * f(z) \\ &= \varsigma + \sum_{n=2}^{\infty} \psi_n a_n \varsigma^n, \quad (12) \end{aligned}$$

where $0 < q < 1, \nu > 0,\ \lambda > -1,\ \varsigma \in \Delta$ and

$$\psi_n := \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)} \frac{[n,q]!}{[\lambda+1,q]_{n-1}} \quad (13)$$

Remark 1.1. One can verify from (12) that the following identity hold for all $f \in \mathcal{A}$:

$$[\lambda+1,\ q]\mathcal{N}^\lambda_{\nu,q} f(\varsigma) = [\lambda,\ q]\mathcal{N}^{\lambda+1}_{\nu,q} f(\varsigma) + \quad (14)$$

$$q^\lambda \varsigma \partial_q \left([\lambda+1,\ q]\mathcal{N}^{\lambda+1}_{\nu,q} f(\varsigma) \right), \varsigma \in \Delta$$

and

$$\lim_{q \to 1^-} \mathcal{N}^\lambda_{\nu,\,q} f(\varsigma) = \mathcal{J}^\lambda_{\nu,\,1} f(\varsigma) =: \mathcal{J}^\lambda_\nu f(\varsigma) = \varsigma \quad (15)$$

$$+ \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)} \frac{n!}{(\lambda+1)_{n-1}} a^n_n,\ z \in \Delta.$$

The (p,q)-Lucas polynomials $\mathcal{L}_n(p(\varkappa), q(\varkappa), \varkappa)$, (or $\mathcal{L}_n(\varkappa)$) are given by the recurrence relation (see [13, 14]):

$$\mathcal{L}_n(\varkappa) = p(\varkappa)\mathcal{L}_{n-1}(\varkappa) + q(\varkappa)\mathcal{L}_{n-2}(\varkappa)(n \in \mathbb{N}\backslash\{1\}), \quad (16)$$

with

$$\mathcal{L}_0(\varkappa) = 2 \ \textit{and}\ \mathcal{L}_1(\varkappa) = p(\varkappa),$$

where $p(\varkappa)$ and $q(\varkappa)$ be polynomials with real coefficients. One can find from (16) that $\mathcal{L}_2(\varkappa) = p^2(\varkappa) + 2q(\varkappa),\ \mathcal{L}_3(\varkappa) = p^3(\varkappa) + 3p(\varkappa)q(\varkappa)$. Note that for particular choices of $p(\varkappa)$ and $q(\varkappa)$, the (p,q) - Lucas polynomials $\mathcal{L}_n(p(\varkappa), q(\varkappa), \varkappa)$, leads to the following polynomials:

1. $\mathcal{L}_n(2\varkappa, 1, \varkappa) = P_n(\varkappa)$ the Pell-Lucas polynomials.

2. $\mathcal{L}_n(\varkappa, 1, \varkappa) = L_n(\varkappa)$ the Lucas polynomials.
3. $\mathcal{L}_n(2\varkappa, -1, \varkappa) = T_n(\varkappa)$ the first kind Chebyshev polynomials.
4. $\mathcal{L}_n(3\varkappa, -2, \varkappa) = F_n(\varkappa)$ the Fermat-Lucas polynomials.
5. $\mathcal{L}_n(1, 2\varkappa, \varkappa) = Q_n(\varkappa)$ the Jacobsthal-Lucas polynomials.

It is known from [19] that

$$\mathcal{G}_{\mathcal{L}_n(\varkappa)}(\varsigma) := \sum_{n=0}^{\infty} \mathcal{L}_n(\varkappa) z^n = \frac{2 - p(\varkappa)\varsigma}{1 - p(\varkappa)\varsigma - q(\varkappa)\varsigma^2}\,. \tag{17}$$

is the generating function of the (p, q)-Lucas polynomials $\mathcal{L}_n(\varkappa)$.

It is well-known that these polynomials have potential applications in branches such as approximation theory, architecture, engineering sciences , statistics, mathematical and physical sciences. For more details about the above mentioned polynomials one can refer [13], [14],[15] and [19]. The recent research trends on functions $\in \sigma$ linked with (p, q) - Lucas polynomial can be seen in [1], [3], [4], [27], [28] and [29].

Motivated essentially by the fruitful usages of above mentioned polynomials and Bassel functions in Geometric function theory and the recent papers [8], [12] and [26], we present two subfamilies of bi-univalent functions defined by making use of q - analogue of Bessel functions subordinate to (p, q) - Lucas polynomials. Throughout this paper, the function $f^{-1}(w) = g(w)$ is as in (2) and the generating function $\mathcal{G}$ is as in (17).

The subordination principle for holomorphic functions f and g in Δ, is due to Miller and Mocanu (see [22]). f is said to be subordinate to g, if there exists a Schwarz function ω such that $f(\varsigma) = g(\omega(\varsigma))$ $(\varsigma \in \Delta), \omega(0) = 0$ and $|\omega(\varsigma)| < 1$. This subordination will be indicated by $f \prec g$ $(\varsigma \in \Delta)$ (or $f(\varsigma) \prec g(\varsigma))$ $(\varsigma \in \Delta)$. Further, if g is univalent in Δ, $f \prec g$ $(\varsigma \in \Delta) \Leftrightarrow f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Definition 1.1. For $\tau \geq 1, \mu \geq 0, 0 \leq \gamma \leq 1, 0 < q < 1, \nu > 0, \lambda > -1$ a function $f \in \boldsymbol{\sigma}$ of the form (1) is said to be in the class $\mathcal{S}^*_{\boldsymbol{\sigma}}(\tau,\ \gamma,\ \mu,\ \lambda,\ \nu,\ q,\ \varkappa)$, if

$$\frac{\varsigma\left(\left(\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)\right)'\right)^{\tau} + \mu\varsigma^2\left(\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)\right)''}{(1-\gamma)\varsigma + \gamma\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)} \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(\varsigma) - 1$$

and

$$\frac{w\left(\left(\mathcal{N}^{\lambda}_{\nu,\,q}g(w)\right)'\right)^{\tau} + \mu w^2\left(\mathcal{N}^{\lambda}_{\nu,\,q}g(w)\right)''}{(1-\gamma)w + \gamma\mathcal{N}^{\lambda}_{\nu,\,q}g(w)} \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(w) - 1,$$

where $\varsigma, w \in \Delta$.

Remark 1.2. Putting $q \to 1^-$, we obtain

$$\lim_{q \to 1^-} \mathcal{S}^*_{\boldsymbol{\sigma}}(\tau, \gamma, \mu, \lambda, \nu, q, \varkappa) =: \mathcal{S}^*_{\boldsymbol{\sigma}}(\tau, \gamma, \mu, \lambda, \nu, \varkappa),$$

the class of $f \in \sigma$ satisfying the following two conditions

$$\frac{\varsigma\left(\left(\mathcal{J}^{\lambda}_{\nu}f(\varsigma)\right)'\right)^{\tau} + \mu\varsigma^2\left(\mathcal{J}^{\lambda}_{\nu}f(\varsigma)\right)''}{(1-\gamma)\varsigma + \gamma\mathcal{J}^{\lambda}_{\nu}f(\varsigma)} \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(\varsigma) - 1,$$

and

$$\frac{w\left(\left(\mathcal{J}^{\lambda}_{\nu}g(w)\right)'\right)^{\tau} + \mu w^2\left(\mathcal{J}^{\lambda}_{\nu}g(w)\right)''}{(1-\gamma)w + \gamma\mathcal{J}^{\lambda}_{\nu,}g(w)} \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(w) - 1,$$

where

$$\tau \geq 1, \mu \geq 0, 0 \leq \gamma \leq 1, \nu > 0, \lambda > -1, \varsigma, w \in \Delta.$$

The family $\mathcal{S}^*_{\boldsymbol{\sigma}}(\tau,\ \gamma,\ \mu,\ \lambda,\ \nu,\ q,\ \varkappa)$, is of special interest for it contains many new subfamilies of $\boldsymbol{\sigma}$ for particular choices of γ and μ, as illustrated below:

1. $\mathcal{S}^*_{\boldsymbol{\sigma}}(\tau, 0, \mu, \lambda, \nu, q, \varkappa) \equiv \mathfrak{I}^*_{\sum}(\tau, \mu, \lambda, \nu, q, \varkappa)$ is the collection of functions $f \in \boldsymbol{\sigma}$ satisfying

$$\left(\left(\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)\right)'\right)^{\tau} + \mu\varsigma\left(\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)\right)'' \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(\varsigma) - 1$$

and

$$\left(\left(\mathcal{N}^{\lambda}_{\nu,\,q}g(w)\right)'\right)^{\tau} + \mu w\left(\mathcal{N}^{\lambda}_{\nu,\,q}g(w)\right)'' \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(w) - 1,$$

where $\varsigma, w \in \Delta$.

2. $\mathcal{S}^*_{\boldsymbol{\sigma}}(\tau, 1, \mu, \lambda, \nu, q, \varkappa) \equiv \mathfrak{K}^*_{\sum}(\tau, \mu, \lambda, \nu, q, \varkappa)$ is the set of functions $f \in \boldsymbol{\sigma}$ satisfying

$$\frac{\varsigma\left(\left(\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)\right)'\right)^{\tau}}{\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)} + \mu\left(\frac{\varsigma^2(\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma))''}{\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)}\right) \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(\varsigma) - 1$$

and

$$\frac{w\left(\left(\mathcal{N}^{\lambda}_{\nu,\,q}g(w)\right)'\right)^{\tau}}{\mathcal{N}^{\lambda}_{\nu,\,q}f(w)} + \mu\left(\frac{w^2(\mathcal{N}^{\lambda}_{\nu,\,q}g(w))''}{\mathcal{N}^{\lambda}_{\nu,\,q}g(w)}\right) \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(w) - 1,$$

where $\varsigma, w \in \Delta$.

3. $\mathcal{S}^*_{\boldsymbol{\sigma}}(\tau,\gamma,1,\lambda,\nu,q,\varkappa) \equiv \mathfrak{L}^*_{\sum}(\tau,\gamma,\lambda,\nu,q,\varkappa)$ is the collection of functions $\in \boldsymbol{\sigma}$ satisfying

$$\frac{\varsigma\left(\left(\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)\right)'\right)^{\tau}+\varsigma^2\left(\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)\right)''}{(1-\gamma)\varsigma+\gamma\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)} \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(\varsigma)-1$$

and

$$\frac{w\left(\left(\mathcal{N}^{\lambda}_{\nu,\,q}g(w)\right)'\right)^{\tau}+w^2\left(\mathcal{N}^{\lambda}_{\nu,\,q}g(w)\right)''}{(1-\gamma)w+\gamma\mathcal{N}^{\lambda}_{\nu,\,q}g(w)} \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(w)-1,$$

where $\varsigma, w \in \Delta$.

Definition 1.2. For $\xi \geq 1, \tau \geq 1, 0 < q < 1, \nu > 0,$ *and* $\lambda > -1$ a function $f \in \boldsymbol{\sigma}$ of the form (1) is said to be in the class $\mathcal{M}^*_{\boldsymbol{\sigma}}(\xi,\ \tau,\ \lambda,\ \nu,\ q,\ \varkappa)$, if

$$\frac{\xi\left[\left(\varsigma(\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma))'\right)'\right]^{\tau}+(1-\xi)}{\left(\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)\right)'} \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(\varsigma)-1$$

and

$$\frac{\xi\left[\left(w(\mathcal{N}^{\lambda}_{\nu,\,q}g(w))'\right)'\right]^{\tau}+(1-\xi)}{\left(\mathcal{N}^{\lambda}_{\nu,\,q}g(w)\right)'} \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(w)-1,$$

where $\varsigma, w \in \Delta$.

Remark 1.3. Putting $q \to 1^-$, we obtain

$$\lim_{q\to 1^-}\mathcal{M}^*_{\boldsymbol{\sigma}}(\xi,\ \tau,\ \lambda,\ \nu,\ q,\ \varkappa) =: \mathcal{M}^*_{\boldsymbol{\sigma}}(\xi,\ \tau,\ \lambda,\ \nu,\ \varkappa),$$

the class of $f \in \sigma$ satisfying the following two conditions

$$\frac{\xi\left[\left(\varsigma(\mathcal{J}^{\lambda}_{\nu}f(z))'\right)'\right]^{\tau}+(1-\xi)}{\left(\mathcal{J}^{\lambda}_{\nu}f(\varsigma)\right)'} \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(\varsigma)-1$$

and

$$\frac{\xi\left[\left(w(\mathcal{J}^{\lambda}_{\nu}g(w))'\right)'\right]^{\tau}+(1-\xi)}{\left(\mathcal{J}^{\lambda}_{\nu}g(w)\right)'} \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(w)-1,$$

where $\varsigma, w \in \Delta$.

We note that $\mathcal{M}^*_{\boldsymbol{\sigma}}(1,\tau,\lambda,\nu,q,\varkappa) \equiv \mathcal{U}^*_{\boldsymbol{\sigma}}(\tau,\lambda,\nu,q,\varkappa)$ is the family investigated in [26]. $\mathcal{M}_{\boldsymbol{\sigma}}(1,\ \tau,\ \lambda,\ \nu,\ q,\ \varkappa) \equiv \mathcal{T}^*_{\boldsymbol{\sigma}}(\tau,\ \lambda,\ \nu,\ q,\ \varkappa)$ is the collection of functions $f \in \boldsymbol{\sigma}$ satisfying

$$\frac{\left[\left(\varsigma(\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma))'\right)'\right]^{\tau}}{\left(\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)\right)'} \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(\varsigma)-1,\ \varsigma\in\Delta$$

and

$$\frac{\left[\left(w(\mathcal{N}^{\lambda}_{\nu,\,q}g(w))'\right)'\right]^{\tau}}{\left(\mathcal{N}^{\lambda}_{\nu,\,q}g(w)\right)'} \prec \mathcal{G}_{\mathcal{L}_n(\varkappa)}(w)-1,\ w\in\Delta.$$

2 The set of main results

In this section, we propose to find bounds on $|a_2|$, $|a_3|$ and $|a_3-\delta a_2^2|$ ($\delta \in \mathbb{R}$) for functions in the classes $\mathcal{S}^*_{\boldsymbol{\sigma}}(\tau,\gamma,\mu,\lambda,\nu,q,x)$ and $\mathcal{M}^*_{\boldsymbol{\sigma}}(\xi,\tau,\lambda,\nu,q,x)$, introduced in Definition1.1 and Definition 1.2 , respectively.

Theorem 2.1. *Let* $\tau \geq 1,\ \mu \geq 0,\ 0 \leq \gamma \leq 1,\ 0 < q < 1,\ \nu > 0,\ \lambda > -1$ *and* $f(\varsigma) = \varsigma + \sum\limits_{n=2}^{\infty} a_n\varsigma^n$ *be in the class* $\mathcal{S}^*_{\boldsymbol{\sigma}}(\tau,\ \gamma,\ \mu,\ \lambda,\ \nu,\ q,\ \varkappa)$. *Then*

$$|a_2| \leq \frac{|p(\varkappa)|\sqrt{|p(\varkappa)|}}{\sqrt{|\mathfrak{t}p^2(\varkappa)-2\mathfrak{s}\,q(\varkappa)|}},$$

$$|a_3| \leq \frac{|p(\varkappa)|}{(3(\eta+\mu)-\gamma)\psi_3}+\frac{p^2(\varkappa)}{\mathfrak{s}}$$

and for $\delta \in \mathbb{R}$

$$\left|a_3-\delta a_2^2\right| \leq \begin{cases} \dfrac{|p(\varkappa)|}{(3(\eta+\mu)-\gamma)\psi_3}, \text{if } |\delta-1| \leq J \\[2ex] \dfrac{|p(\varkappa)|^3\,|\delta-1|}{|\mathfrak{t}\,p^2(\varkappa)-2\mathfrak{s}\,q(\varkappa)|}, \text{if } |\delta-1| \geq J \end{cases}.$$

where

$$\eta = \tau+\mu, \tag{18}$$

$$\mathfrak{t} = [(3(\eta+\mu)-\gamma)\psi_3-2(\tau(\tau+1)+(2\mu-\gamma)\eta+2\mu\tau)\psi_2^2], \tag{19}$$

$$\mathfrak{s} = (2\eta-\gamma)^2\psi_2^2 \tag{20}$$

and

$$J = \frac{\left|tp^2(\varkappa)-2sq(\varkappa)\right|}{(3(\eta+\mu)-\gamma)\,\psi_3p^2(\varkappa)}. \tag{21}$$

Proof. Let $f \in \mathcal{S}^*_{\boldsymbol{\sigma}}(\tau,\ \gamma,\ \mu,\ \lambda,\ \nu,\ q,\ \varkappa)$, be given by (1). Then, for holomorphic functions u and v with

$$u(0)=0,\ v(0)=0, |u(\varsigma)| = \left|u_1\varsigma+u_2\varsigma^2+\ldots\right| < 1,$$

and

$$|v(w)| = \left|v_1w+v_2w^2+\ldots\right| < 1,\ \varsigma,\ w\in\Delta.$$

Therefore, on account of Definition 1.1, we can write

$$\frac{\varsigma\left(\left(\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)\right)'\right)^{\tau}+\mu\varsigma^2\left(\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)\right)''}{(1-\gamma)\varsigma+\gamma\mathcal{N}^{\lambda}_{\nu,\,q}f(\varsigma)} = \mathcal{G}_{\mathcal{L}_n(\varkappa)}(u(\varsigma))-1$$

and

$$\frac{w\left(\left(\mathcal{N}^{\lambda}_{\nu,\,q}g(w)\right)'\right)^{\tau}+\mu w^2\left(\mathcal{N}^{\lambda}_{\nu,\,q}g(w)\right)''}{(1-\gamma)w+\gamma\mathcal{N}^{\lambda}_{\nu,\,q}g(w)} =$$

$$\mathcal{G}_{\mathcal{L}_n(\varkappa)}(v(w))-1.$$

Or, equivalently,

$$\frac{\varsigma\left(\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(\varsigma)\right)'\right)^{\tau}+\mu\varsigma^2\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(\varsigma)\right)''}{(1-\gamma)\varsigma+\gamma\mathcal{N}_{\nu,\,q}^{\lambda}f(\varsigma)}=$$

$$-1+\mathcal{L}_0(\varkappa)+\mathcal{L}_1(\varkappa)u(\varsigma)+\mathcal{L}_2(\varkappa)[u(\varsigma)]^2+\ldots$$

and

$$\frac{w\left(\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)'\right)^{\tau}+\mu w^2\left(\mathcal{N}_{\nu,\,q}^{\lambda}g(w)\right)''}{(1-\gamma)w+\gamma\mathcal{N}_{\nu,\,q}^{\lambda}g(w)}=$$

$$-1+\mathcal{L}_0(\varkappa)+\mathcal{L}_1(\varkappa)v(w)+\mathcal{L}_2(\varkappa)[v(w)]^2+\ldots.$$

We obtain, from the above equalities

$$\frac{\varsigma\left(\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(\varsigma)\right)'\right)^{\tau}+\mu\varsigma^2\left(\mathcal{N}_{\nu,\,q}^{\lambda}f(\varsigma)\right)''}{(1-\gamma)\varsigma+\gamma\mathcal{N}_{\nu,\,q}^{\lambda}f(\varsigma)}=$$

$$1+\mathcal{L}_1(\varkappa)u_1\varsigma+[\mathcal{L}_1(\varkappa)u_2+\mathcal{L}_2(\varkappa)u_1^2]\varsigma^2+\ldots \quad (22)$$

and

$$\frac{w\left(\left(\mathcal{J}_{\nu,\,q}^{\lambda}g(w)\right)'\right)^{\tau}+\mu w^2\left(\mathcal{J}_{\nu,\,q}^{\lambda}g(w)\right)''}{(1-\gamma)w+\gamma\mathcal{J}_{\nu,\,q}^{\lambda}g(w)}=$$

$$1+\mathcal{L}_1(\varkappa)v_1w+[\mathcal{L}_1(\varkappa)v_2+\mathcal{L}_2(\varkappa)v_1^2]w^2+\ldots. \quad (23)$$

It is known that

$$|u_k|\leq 1,\qquad |v_k|\leq 1\qquad (k\in\mathbb{N}). \quad (24)$$

Comparing (22) and (23), we have

$$(2\eta-\gamma)\psi_2a_2=\mathcal{L}_1(\varkappa)u_1 \quad (25)$$

$$(3(\eta+\mu)-\gamma)\,\psi_3a_3+\left(\gamma^2+2\tau(\tau-1)-2\gamma\eta\right)\psi_2^2a_2^2=$$
$$\mathcal{L}_1(\varkappa)u_2+\mathcal{L}_2(\varkappa)u_1^2 \quad (26)$$

$$-(2\eta-\gamma)\psi_2\,a_2=\mathcal{L}_1(\varkappa)v_1 \quad (27)$$

and

$$(3(\eta+\mu)-\gamma)\,\psi_3\left(2a_2^2-a_3\right)+$$
$$\left(\gamma^2+2\tau(\tau-1)-2\gamma\eta\right)\psi_2^2a_2^2=\mathcal{L}_1(\varkappa)v_2+\mathcal{L}_2(\varkappa)v_1^2, \quad (28)$$

where η is as in (18).
From (25) and (27), we get

$$u_1=-v_1 \quad (29)$$

and

$$2\,\mathfrak{s}\,a_2^2=[\mathcal{L}_1(\varkappa)]^2(u_1^2+v_1^2) \quad (30)$$

where $\mathfrak{s}$ is given by (20).
If we add (26) to (28), we obtain

$$2\,b\,a_2^2=\mathcal{L}_1(\varkappa)(u_2+v_2)+\mathcal{L}_2(\varkappa)(u_1^2+v_1^2), \quad (31)$$

where

$$\mathfrak{b}=[(3(\eta+\mu)-\gamma)\psi_3+\left(\gamma^2+2\tau(\tau-1)-2\gamma\eta\right)\psi_2^2]. \quad (32)$$

From (30) and (31), we deduce that

$$2a_2^2=\frac{[\mathcal{L}_1(\varkappa)]^3\,(u_2+v_2)}{\mathfrak{b}\mathcal{L}_1^2(\varkappa)-\mathfrak{s}\,\mathcal{L}_2(\varkappa)}. \quad (33)$$

Putting the values of $\mathcal{L}_1(\varkappa)$, $\mathcal{L}_2(\varkappa)$ and applying (24) for $|u_2|$ and $|v_2|$, we get

$$|a_2|\leq\frac{|p(\varkappa)|\sqrt{|p(\varkappa)|}}{\sqrt{|\mathfrak{t}\,p^2(\varkappa)-2\,\mathfrak{s}\,q(\varkappa)|}}.$$

where $\mathfrak{t}$ is as mentioned in (19).

To find the estimate on $|a_3|$, first we subtract (28) from (26) and then in view of (29), we obtain

$$2(3(\eta+\mu)-\gamma)\psi_3\,a_3-2(3(\eta+\mu)-\gamma)\psi_3\,a_2^2=$$
$$\mathcal{L}_1(\varkappa)\,(u_2-v_2)+\mathcal{L}_2(\varkappa)\left(u_1^2-v_1^2\right)$$
$$a_3=\frac{\mathcal{L}_1(\varkappa)\,(u_2-v_2)}{2(3(\eta+\mu)-\gamma)\psi_3}+a_2^2. \quad (34)$$

Then in view of (30), (34) becomes

$$a_3=\frac{\mathcal{L}_1(\varkappa)\,(u_2-v_2)}{2(3(\eta+\mu)-\gamma)\psi_3}+\frac{[\mathcal{L}_1(\varkappa)]^2(u_1^2+v_1^2)}{2\,\mathfrak{s}}.$$

Applying (24), we deduce that

$$|a_3|\leq\frac{|p(\varkappa)|}{(3(\eta+\mu)-\gamma)\psi_3}+\frac{p^2(\varkappa)}{\mathfrak{s}}.$$

From (34), for $\delta\in\mathbb{R}$, we write

$$a_3-\delta a_2^2=\frac{\mathcal{L}_1(\varkappa)\,(u_2-v_2)}{2(3(\eta+\mu)-\gamma)\psi_3}+(1-\delta)\,a_2^2. \quad (35)$$

Substituting the value of a_2^2 from (33) in (35), we have

$$a_3-\delta a_2^2=$$
$$\mathcal{L}_1(x)\left\{\left(\Omega(\delta,\varkappa)+\frac{1}{\beta}\right)u_2+\left(\Omega(\delta,\varkappa)-\frac{1}{\beta}\right)v_2\right\}, \quad (36)$$

where

$$\beta=2(3(\eta+\mu)-\gamma)\psi_3$$
$$\Omega(\delta,\varkappa)=\frac{(1-\delta)\,[\mathcal{L}_1(\varkappa)]^2}{2(\mathfrak{b}[\mathcal{L}_1(\varkappa)]^2-\mathfrak{s}\,\mathcal{L}_2(\varkappa))}$$

and $\mathfrak{b}$ is given by (32). Then

$$\left|a_3-\delta a_2^2\right| \leq \begin{cases} \dfrac{|\mathcal{L}_1(\varkappa)|}{(3(\eta+\mu)-\gamma)\psi_3}; & 0 \leq |\Omega(\delta,\ \varkappa)| \leq \dfrac{1}{\beta} \\ 2\,|\mathcal{L}_1(\varkappa)|\,|\Omega(\delta,\ \varkappa)|\,; & |\Omega(\delta,\ \varkappa)| \geq \dfrac{1}{\beta}, \end{cases}$$

which evidently completes the proof of Theorem 2.1. $\square$

In the next theorem, we determine the bounds for $|a_2|$, $|a_3|$ and $|a_3-\delta a_2^2|$ for function $f \in \mathcal{M}^*_{\boldsymbol{\sigma}}(\xi,\ \tau,\ \lambda,\ \nu,\ q,\ x)$, the proof of which is omitted as it is similar to that of Theorem 2.1.

Theorem 2.2. *Let $\xi \geq 1, \tau \geq 1$, $0<q<1$, $\lambda > -1$, $\nu > 0$ and $f(\varsigma)=\varsigma+\sum\limits_{n=2}^{\infty} a_n\varsigma^n$ be in the class $\mathcal{M}^*_{\boldsymbol{\sigma}}(\xi,\ \tau,\ \lambda,\ \nu,\ q,\ \varkappa)$. Then*

$$|a_2| \leq \frac{|p(x)|\sqrt{|p(x)|}}{\sqrt{|yp^2(x)-2zq(x)|}},$$
$$|a_3| \leq \frac{|p(x)|}{3(3\xi\tau-1)\psi_3}+\frac{p^2(x)}{z}$$

and for $\delta \in \mathbb{R}$

$$\left|a_3-\delta a_2^2\right| \leq \begin{cases} \dfrac{|p(x)|}{3(3\xi\tau-1)\psi_3}, & \text{if } |\delta-1| \leq \mathfrak{H} \\ \dfrac{|p(x)|^3\,|\delta-1|}{|yp^2(x)-2zq(x)|}, & \text{if } |\delta-1| \geq \mathfrak{H} \end{cases}$$

where

$$y=[(3(3\xi\tau-1)\psi_3-8\xi\tau^2(2\xi-1)\psi_2^2],$$
$$z=4(2\xi\tau-1)^2\psi_2^2$$

and

$$\mathfrak{H}=\frac{\left|yp^2(x)-2zq(x)\right|}{3\,(3\xi\tau-1)\,\psi_3 p^2(x)}.$$

3 Outcome of main results

We arrive at the following outcome when $\gamma=0$ in Theorem 2.1.

Corollary 3.1. Let $\tau \geq 1, \mu \geq 0, 0<q<1, \lambda > -1$, $\nu > 0$ and $f(\varsigma)=\varsigma+\sum\limits_{n=2}^{\infty} a_n\varsigma^n$ be in the family $\mathfrak{J}^*_{\sum}(\tau,\ \mu,\ \lambda,\ \nu,\ q,\ \varkappa)$. Then

$$|a_2| \leq \frac{|p(\varkappa)|\sqrt{|p(\varkappa)|}}{\sqrt{|\mathfrak{t}_1 p^2(\varkappa)-2\mathfrak{s}_1 q(\varkappa)|}},$$
$$|a_3| \leq \frac{|p(\varkappa)|}{3(\eta+\mu)\psi_3}+\frac{p^2(\varkappa)}{\mathfrak{s}_1}$$

and for $\delta \in \mathbb{R}$

$$\left|a_3-\delta a_2^2\right| \leq \begin{cases} \dfrac{|p(\varkappa)|}{3(\eta+\mu)\psi_3}, & \text{if } |\delta-1| \leq J_1 \\ \dfrac{|p(x)|^3\,|\delta-1|}{|\mathfrak{t}_1 p^2(\varkappa)-2\mathfrak{s}_1 q(\varkappa)|}, & \text{if } |\delta-1| \geq J_3 \end{cases}.$$

where η is as in (18),

$$\mathfrak{t}_1=[3(\eta+\mu)\psi_3-2(\tau(\tau+1)+2\mu(\eta+\tau))\psi_2^2],$$
$$\mathfrak{s}_1=4\eta^2\psi_2^2$$

and

$$J_1=\frac{\left|t_1 p^2(\varkappa)-2s_1 q(\varkappa)\right|}{3(\eta+\mu)\psi_3 p^2(\varkappa)}$$

We arrive at the following outcome by taking $\gamma=1$ in Theorem 2.1.

Corollary 3.2. Let $\tau \geq 1$, $\mu \geq 0$, $0<q<1$, $\lambda > -1$, $\nu > 0$ and $f(\varsigma)=\varsigma+\sum\limits_{n=2}^{\infty} a_n\varsigma^n$ be in the set $\mathfrak{K}^*_{\sum}(\tau,\ \mu,\ \lambda,\ \nu,\ q,\ \varkappa)$. Then

$$|a_2| \leq \frac{|p(\varkappa)|\sqrt{|p(x)|}}{\sqrt{|\mathfrak{t}_2 p^2(\varkappa)-2\mathfrak{s}_2 q(\varkappa)|}},$$
$$|a_3| \leq \frac{|p(\varkappa)|}{3(\eta+\mu)-1)\psi_3}+\frac{p^2(\varkappa)}{\mathfrak{s}_2}$$

and for $\delta \in \mathbb{R}$

$$\left|a_3-\delta a_2^2\right| \leq \begin{cases} \dfrac{|p(x)|}{(3(\eta+\mu)-1)\psi_3}, & \text{if } |\delta-1| \leq J_2 \\ \dfrac{|p(\varkappa)|^3\,|\delta-1|}{|\mathfrak{t}_2 p^2(\varkappa)-2\mathfrak{s}_2 q(\varkappa)|}, & \text{if } |\delta-1| \geq J_2 \end{cases}.$$

where η is as in (18),

$$\mathfrak{t}_2=[(3(\eta+\mu)-1)\psi_3-2(\tau(\tau+1)+2\mu(\eta+\tau)-\eta)\psi_2^2],$$

$$\mathfrak{s}_2 = (2\eta - 1)^2 \psi_2^2$$

and

$$J_2 = \frac{\left|\mathfrak{t}_2 p^2(\varkappa) - 2\mathfrak{s}_2 q(\varkappa)\right|}{(3(\eta+\mu)-1)\,\psi_3 p^2(\varkappa)}.$$

We arrive at the following outcome by taking $\mu = 1$ in Theorem 2.1.

Corollary 3.3. Let $\tau \geq 1,\ 0 \leq \gamma \leq 1,\ 0 < q < 1,\ \lambda > -1,\ \nu > 0$ and $f(\varsigma) = \varsigma + \sum_{n=2}^{\infty} a_n \varsigma^n$ be in the family $\mathfrak{L}^*_{\sum}(\tau,\ \gamma,\ \lambda,\ \nu,\ q,\ \varkappa)$. Then

$$|a_2| \leq \frac{|p(\varkappa)|\sqrt{|p(\varkappa)|}}{\sqrt{|\mathfrak{t}_3 p^2(\varkappa) - 2\mathfrak{s}_3 q(\varkappa)|}},$$

$$|a_3| \leq \frac{|p(\varkappa)|}{(3\tau+6-\gamma)\psi_3} + \frac{p^2(\varkappa)}{\mathfrak{s}_3}$$

and for $\delta \in \mathbb{R}$

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \dfrac{|p(\varkappa)|}{(3\tau+6-\gamma)\psi_3}, & \text{if } |\delta - 1| \leq J_3 \\[2ex] \dfrac{|p(\varkappa)|^3\,|\delta-1|}{|\mathfrak{t}_3 p^2(\varkappa) - 2\mathfrak{s}_3 q(\varkappa)|}, & \\ & \text{if } |\delta - 1| \geq J_3 \end{cases}$$

where

$$t_3 = [(3\tau+6-\gamma)\psi_3 - 2(\tau^2+5\tau+2-\gamma(\tau+1))\psi_2^2],$$

$$s_3 = (2\tau+2-\gamma)^2\psi_2^2$$

and

$$J_3 = \frac{\left|t_3 p^2(\varkappa) - 2s_3 q(\varkappa)\right|}{(3\tau+6-\gamma)\,\psi_3 p^2(\varkappa)}.$$

Setting $\xi = 1$ in Theorem 2.2,we obtain

Corollary 3.4. Let $\tau \geq 1,\ 0 < q < 1,\ \lambda > -1,\ \nu > 0$ and $f(\varsigma) = \varsigma + \sum_{n=2}^{\infty} a_n \varsigma^n$ be in the set $\mathcal{T}^*_{\boldsymbol{\sigma}}(\tau,\ \lambda,\ \nu,\ q,\ \varkappa)$. Then

$$|a_2| \leq \frac{|p(\varkappa)|\sqrt{|p(\varkappa)|}}{\sqrt{|y_1 p^2(\varkappa) - 2z_1 q(\varkappa)|}},$$

$$|a_3| \leq \frac{|p(\varkappa)|}{3(3\tau-1)\psi_3} + \frac{p^2(\varkappa)}{z_1}$$

and for $\delta \in \mathbb{R}$

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \dfrac{|p(\varkappa)|}{3(3\tau-1)\psi_3}, & \text{if } |\delta - 1| \leq \mathfrak{H}_1 \\[2ex] \dfrac{|p(\varkappa)|^3\,|\delta-1|}{|y_1 p^2(\varkappa) - 2z_1 q(\varkappa)|}, & \\ & \text{if } |\delta - 1| \geq \mathfrak{H}_1 \end{cases}$$

where

$$y_1 = [(3(3\tau-1)\psi_3 - 8\tau^2\psi_2^2],$$

$$z_1 = 4(2\tau-1)^2\psi_2^2$$

and

$$\mathfrak{H}_1 = \frac{\left|y_1 p^2(\varkappa) - 2z_1 q(\varkappa)\right|}{3\,(3\tau-1)\,\psi_3 p^2(\varkappa)}.$$

4 Conclusions

Our investigation is motivated by the fruitful usage of certain special polynomials and Bassel functions, in the theory of bi-univalent functions. Making use of the q-analoguue of Bassel functions, we have introduced two subfamilies of bi-univalent (or bi-schlicht) functions subordinate to (p, q)-Lucas polynomials. For functions belonging to these subfamilies, we have found the upper bounds of $|a_2|$, $|a_3|$ and for δ a real number the Fekete- Szegö functional $|a_3 - \delta a_2^2|$ is considered. The special cases and implications of the main results have been identified. Finding estimate on the bound of $|a_n|,\ n \in \mathbb{R} - \{1, 2, 3\}$ is an open problem.

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Contribution of individual authors to the creation of a scientific article (Ghostwriting Policy)

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