

Reproducing Kernel in Krein Spaces

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Abstract: - This article describes a new form to introduce a reproducing kernel for a Krein space based on orthogonal projectors enabling to describe the kernel of a Krein space as the difference between the kernel of definite positive subspace and the kernel of definite negative subspace corresponding to kernel of the associated Hilbert space. As application, the authors obtain some basic properties of both kernels for Krein spaces and exhibit that each kernel is uniquely determined by the Krein space given. The methods and results employed generalize the notion of reproducing kernel given in Hilbert spaces to the context of spaces endowed with indefinite metric.

Key-Words: - Hilbert space, Krein space, reproducing kernel, indefinite metric.

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1 Introduction

A Hilbert space endowed with a function such that reproduces every element of space through the internal product is called a Hilbert space with reproducing kernel or a reproducing kernel Hilbert space. The theory of these spaces plays an important role in many branches of mathematics such that functional analysis, harmonic analysis, operator theory and quantum mechanics [1], [2] and [3]. The concept of reproducing kernel was used first at the early twentieth century by Zaremba [4]. The same author studied other functions that satisfy a property of reproducing in the theory of integral equations, called definite positive kernels [5]. Later, Bergman [6] introduced reproducing kernel in one and several variables for the class of harmonic functions and the class of analytic functions, called kernel functions. After them many authors have studied the same idea as Szego [7], even in recent years there has been considerable interest in studying the reproducing properties in functional analysis, harmonic analysis, operator theory, quantum mechanics, among other fields; which can be seen in [8], [9], [10] and [11]. In [12] a reproducing kernel structure was introduced for Hilbert spaces with functions where the differences of point evaluations are limited and in [13] they used the reproducing kernel to study model spaces generated by certain types of singular inner functions. Motivated by these researches, in this paper we extend the notion of reproducing kernel to more general spaces such that Krein spaces [14], [15], [16] and [17]. The principal research method used in this work is the hypothetical-

deductive method. In this work we characterize the existence of a reproducing kernel for a Krein space through two important Hilbert subspaces, called definite positive subspace and definite negative subspace. By using these subspaces, we describe the kernel of a Krein space as the difference between the kernel of definite positive subspace and the kernel of definite negative subspace, while the sum of these subspaces corresponds to kernel of the associated Hilbert space. Also, we prove some basic properties of both kernels for spaces with indefinite metric and we exhibit that each kernel is uniquely determined by the Krein space given. From this, we generalize the notion of reproducing kernel given in Hilbert spaces to the context of spaces endowed with indefinite metric. The results of our research can be potentially important for mathematicians who working in areas such that functional analysis, harmonic analysis, operator theory and quantum mechanics and others affine areas.

2 Problem Formulation

Starting from the reproducing kernel defined in Hilbert spaces, the generalization of this concept was pursued in spaces with undefined metrics, more specifically in Krein spaces. It was even sought to verify some basic properties for both kernel and in turn show that each kernels is uniquely determined by the given Krein space.

2.1 Kernels on Hilbert Spaces

Definition 2.1. Let \mathcal{H} be a Hilbert space of complex-valued functions on a set X . Given $f, g \in \mathcal{H}$, we denote by $\langle f, g \rangle$ the internal product of f and g . The complex-valued function $\mathcal{N}: X \times X \rightarrow \mathbb{C}$ denoted by $\mathcal{N}_y(x) := \mathcal{N}(x, y)$ is called a reproducing kernel for $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, if this function satisfies:

i) For each $y \in X$ the function \mathcal{N}_y defined by

$$\mathcal{N}_y(x) := \mathcal{N}(x, y) \text{ is an element of } \mathcal{H}.$$

ii) For each $x \in X$ and $f \in \mathcal{H}$, the reproducing property $f(x) = \langle f, \mathcal{N}_x \rangle$ is fulfilled.

Proposition 2.2. Let \mathcal{H} be a Hilbert space with reproducing kernel $\mathcal{N}: X \times X \rightarrow \mathbb{C}$, then $\|\mathcal{N}_x\|^2 = \mathcal{N}(x, x), \quad \forall x \in X$

A Hilbert space \mathcal{H} of complex-valued functions on a set X is called a Hilbert space with a reproducing kernel (often abbreviated EHNR) if there exists a reproducing kernel \mathcal{N} associated to \mathcal{H} .

Proposition 2.3. Let \mathcal{H} be a Hilbert space with reproducing kernel \mathcal{N} , then \mathcal{N} satisfies the following properties:

i) $\mathcal{N}(x, x) \geq 0$

ii) $\mathcal{N}(x, y) = \overline{\mathcal{N}(y, x)}$

iii) $|\mathcal{N}(x, y)|^2 \leq \mathcal{N}(x, x)\mathcal{N}(y, y)$

Also, iv) for all $x_0 \in X$ the following conditions are equivalent:

- a) $\mathcal{N}(x_0, x_0) = 0$
- b) $\mathcal{N}(x, x_0) = 0, \quad \forall x \in X$
- c) $f(x_0) = 0, \quad \forall f \in \mathcal{H}$

Theorem 2.4. Let \mathcal{H} be a Hilbert space with a reproducing kernel $\mathcal{N}(x, y)$, the \mathcal{N} is uniquely determined by \mathcal{H} .

The proofs of above results can be seen in [4].

2.2 Krein Spaces

Definition 2.5. A topological space with an internal product $(\mathfrak{K}, [\cdot, \cdot])$ and a fundamental decomposition $\mathfrak{K} = \mathfrak{K}^+ [+] \mathfrak{K}^-$ is called a Krein space, if the topology on the orthogonal sum coincides with the original topology on the space and the subspaces $(\mathfrak{K}^+, [\cdot, \cdot]), (\mathfrak{K}^-, -[\cdot, \cdot])$ are Hilbert spaces.

Note that if a Krein space \mathfrak{K} admits a decomposition $\mathfrak{K} = \mathfrak{K}^+ [+] \mathfrak{K}^-$, we can define an internal product $(k, h) := [k^+, h^+] - [k^-, h^-]$. In addition, there exist projections P^+ and P^- associated to the spaces \mathfrak{K}^+ and \mathfrak{K}^- respectively, such that $(P^\pm)^2 = P^\pm$ and $[P^\pm k, h] = [k, P^\pm h]$ for all $k, h \in \mathfrak{K}$. The operator $J: \mathfrak{K} \rightarrow \mathfrak{K}$ defined by $J := P^+ - P^-$ is called fundamental symmetry. Observe that $J^2 = I_{\mathfrak{K}}$, also

$$[k, h]_J := [Jk, h] = (k, h) \text{ and } \|k\|_J := \sqrt{[k, k]_J} \quad k, h \in \mathfrak{K} \quad (1)$$

The product $[\cdot, \cdot]_J$ is called J-internal product and $(\mathfrak{K}, [\cdot, \cdot]_J)$ is used to study the linear operators acting on the Krein space $(\mathfrak{K}, [\cdot, \cdot])$. For further information on Krein spaces, we refer to [15] and [16].

Definition 2.6. If $(\mathfrak{K}, [\cdot, \cdot]_{\mathfrak{K}})$ and $(\mathfrak{G}, [\cdot, \cdot]_{\mathfrak{G}})$ are Krein spaces with fundamental symmetries $J_{\mathfrak{K}}$ and $J_{\mathfrak{G}}$ respectively, then

$$\mathfrak{B}(\mathfrak{K}, \mathfrak{G}) := \left\{ T: \mathfrak{K} \rightarrow \mathfrak{G}, \text{ linear and } \|T\| = \sup_{k \in \mathfrak{K} \setminus \{0\}} \frac{\|Tk\|_{J_{\mathfrak{G}}}}{\|k\|_{J_{\mathfrak{K}}}} < \infty \right\}$$

Observe that the adjoint operator T^* of an operator $T: \mathfrak{K} \rightarrow \mathfrak{K}$ is given in natural form and the equality $[Tk, h] = [k, T^*h]$ is valid. More precisely if we consider the internal J-product on the Hilbert space $(\mathfrak{K}, [\cdot, \cdot]_J)$ we denoted the adjoint by $T^{[*]}$ where is the fundamental symmetry of the Krein space.

Definition 2.7. Let T be a linear operator with dense domain on $(\mathfrak{K}, [\cdot, \cdot]_{\mathfrak{K}})$ and codomain on $(\mathfrak{G}, [\cdot, \cdot]_{\mathfrak{G}})$. The adjoint operator T^* is defined by

$$[Tk, h]_{\mathfrak{G}} = [k, h^*]_{\mathfrak{K}}, \quad \forall k \in \text{dom},$$

where $T^*h := h^*$ is given by

$$[Tk, h]_{\mathfrak{G}} = [k, T^*h]_{\mathfrak{K}},$$

for all $k \in \text{dom}(T)$ and $h \in \text{dom}(T^*)$.

The adjoint operator T^* can be expressed through the adjoint $T^{[*]}$ which is called J-adjoint and satisfy

$$[Tk, g]_{\mathfrak{G}} = [k, T^{[*]}g]_{\mathfrak{K}}$$

for all $k \in \text{dom}(T)$ and $g \in \text{dom}(T^{[*]})$. Moreover [16], we have the equality

$$T^* = J_{\mathfrak{K}} T^{[*]} J_{\mathfrak{G}} \quad (2)$$

A linear operator T is called self-adjointed if $T = T^*$, and J-self adjoint if $T = T^{[*]}$. In addition, a linear operator T is invertible if its range and domain are the entire space. The following result can be seen in [14].

Lemma 2.8. Let $(\mathfrak{K}, [\cdot, \cdot]_J)$ be a Krein space with associated fundamental symmetry J and P an orthogonal projection commuting with J , then both $P\mathfrak{K}$ and $(I - P)\mathfrak{K}$ are Krein spaces with fundamental symmetries PJ and $(I - P)J$ respectively.

3 Problem Solution

The reproducing kernel for a Kerin spaces is defined from the kernels of the defined subspaces that form the associated Hilbert space.

3.1 Existence of Kernel for Krein Spaces

Proposition 3.1. Let $(\mathfrak{K}, [\cdot, \cdot]_J)$ be a Krein space of complex-valued functions on a set X . If \mathfrak{K} admits a decomposition $\mathfrak{K} = \mathfrak{K}^+ \dot{+} \mathfrak{K}^-$ with reproducing kernels $\mathcal{N}^+: X \times X \rightarrow \mathbb{C}$ and $\mathcal{N}^-: X \times X \rightarrow \mathbb{C}$ for the Hilbert spaces $(\mathfrak{K}^+, [\cdot, \cdot]_J)$ and $(\mathfrak{K}^-, -[\cdot, \cdot]_J)$ respectively, then $\mathcal{N}^+ + \mathcal{N}^-$ is a reproducing kernel for the Hilbert space $(\mathfrak{K}, [\cdot, \cdot]_J)$ associated to the fundamental symmetry J .

Proof. For each function k in \mathfrak{K} , there exist functions k^+ , k^- in the spaces \mathfrak{K}^+ and \mathfrak{K}^- respectively, such $k = k^+ + k^-$ and $\mathfrak{K}^+ \perp \mathfrak{K}^-$.

Suppose that \mathcal{N}^+ is a reproducing kernel for \mathfrak{K}^+ and that \mathcal{N}^- is a reproducing kernel for \mathfrak{K}^- , then for

each $y \in X$, \mathcal{N}_y^+ and \mathcal{N}_y^- are elements in \mathfrak{K}^+ and \mathfrak{K}^- respectively. From which we concluded that

$$(\mathcal{N}_y^+ + \mathcal{N}_y^-) \quad \text{given} \quad \text{by}$$

$$(\mathcal{N}_y^+ + \mathcal{N}_y^-)(x) = \mathcal{N}_y^+(x) + \mathcal{N}_y^-(x) \text{ belong to } \mathfrak{K}.$$

Also, the reproducing property is fulfilled for \mathcal{N}^+ and \mathcal{N}^- in $(\mathfrak{K}^+, [\cdot, \cdot]_J)$ and $(\mathfrak{K}^-, -[\cdot, \cdot]_J)$ respectively.

Therefore, for any $g \in \mathfrak{K}^+$, $h \in \mathfrak{K}^-$ and for each

$x \in X$ we have that $g(x) = [g, \mathcal{N}_x^+]$ and

$h(x) = -[h, \mathcal{N}_x^-]$. Hence, for each $k \in \mathfrak{K}$ with

$k = k^+ + k^-$ and any $x \in X$

$$\begin{aligned} k(x) &= k^+(x) + k^-(x) \\ &= [k^+ + k^-, \mathcal{N}_x^+ - \mathcal{N}_x^-] \end{aligned}$$

$$= [k, J^2(\mathcal{N}_x^+ - \mathcal{N}_x^-)]$$

$$= [k, (\mathcal{N}_x^+ + \mathcal{N}_x^-)]_J$$

Consequently $k(x) = [k, (\mathcal{N}_x^+ + \mathcal{N}_x^-)]_J$ and then

$\mathcal{N}_x^+ + \mathcal{N}_x^-$ is a reproducing kernel for $(\mathfrak{K}, [\cdot, \cdot]_J)$. Observe that, in the above reasoning, $k(x) = [k, \mathfrak{N}_x^+ - \mathfrak{N}_x^-]$ define a reproducing kernel in the following form.

Definition 3.2. Let $(\mathfrak{K}, [\cdot, \cdot])$ be a Krein space of complex-valued functions on a set X . The function $\mathfrak{N}: X \times X \rightarrow \mathbb{C}$ is said to be a reproducing kernel for $(\mathfrak{K}, [\cdot, \cdot])$ if the following conditions are true:

- i) For each $y \in X$ the function \mathfrak{N}_y defined by $\mathfrak{N}_y(x) = \mathfrak{N}(x, y)$ belong to \mathfrak{K} .
- ii) For any $x \in X$ and $k \in \mathfrak{K}$, the reproducing property is fulfilled for $k(x) = [k, \mathfrak{N}_x]$.

We can observe that the above definition generalizes the notion of kernel given in the second

section for a Hilbert space \mathfrak{K} , taking the fundamental symmetry as the identity operator.

Example 3.3. Consider $\ell_2(\mathbb{N})$ with the standard basis $\{e_n\}_{n \in \mathbb{N}}$, where $e_n = (\delta_{i,n})_{i \in \mathbb{N}}$ is the sequence whose n -th term is 1 and the other terms are zero. Now, let \mathfrak{K}_2 be the class of functions $k: \mathbb{N} \rightarrow \mathbb{C}$ such that $k = (k_i)_{i \in \mathbb{N}} \in \ell_2(\mathbb{N})$ and $k(i) := k_i, i \in \mathbb{N}$. If we take the internal product given by $(k, g \in \mathfrak{K}_2(\mathbb{N}))$

$$[k, g]_{\mathfrak{K}_2} = \sum_{i \in \mathbb{N}} (-1)^i k_i \bar{g}_i$$

Then $(\mathfrak{K}_2(\mathbb{N}), [\cdot, \cdot]_{\mathfrak{K}_2})$ is a Krein space with fundamental decomposition

$$\mathfrak{K}_2(\mathbb{N}) = \mathfrak{K}_2^+[\dot{+}] \mathfrak{K}_2^-(\mathbb{N}), \text{ where}$$

$$\mathfrak{K}_2^+(\mathbb{N}) = \overline{\text{span}\{e_{2n}\}_{n \in \mathbb{N}}} \text{ and}$$

$$\mathfrak{K}_2^-(\mathbb{N}) = \overline{\text{span}\{e_{2n-1}\}_{n \in \mathbb{N}}}$$

Thus, each $k \in \mathfrak{K}$ can be written in the form

$$k = \sum_{i \in \mathbb{N}} k_{2i} e_{2i} + \sum_{i \in \mathbb{N}} k_{2i-1} e_{2i-1}$$

Also, for the fundamental symmetry $J_{\mathfrak{K}}: \mathfrak{K}_2(\mathbb{N}) \rightarrow \mathfrak{K}_2(\mathbb{N})$ we have that

$$J_{\mathfrak{K}}(k) = ((-1)^i k_i)_{i \in \mathbb{N}}. \text{ That is,}$$

$$J_{\mathfrak{K}}k = \sum_{i \in \mathbb{N}} k_{2i} e_{2i} + \sum_{i \in \mathbb{N}} k_{2i-1} e_{2i-1}$$

Finally, if $\mathfrak{N}^{2+}: X \times X \rightarrow \mathbb{C}$ and $\mathfrak{N}^{2-}: X \times X \rightarrow \mathbb{C}$ is defined as:

$$\mathfrak{N}^{2+}(i, n) = \begin{cases} \delta_{i,n} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$\mathfrak{N}^{2-}(i, n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \delta_{i,n} & \text{if } n \text{ is odd} \end{cases}$$

Then, both \mathfrak{N}^{2+} and \mathfrak{N}^{2-} functions are reproducing kernels for the Hilbert spaces $(\mathfrak{K}^{2+}(\mathbb{N}), [\cdot, \cdot]_{\mathfrak{K}_2})$ and

$(\mathfrak{K}^{2-}(\mathbb{N}), -[\cdot, \cdot]_{\mathfrak{K}_2})$, respectively. For each $n \in \mathbb{N}$,

the function \mathfrak{N}_n^{\pm} defined by $\mathfrak{N}_n^{\pm}(i) := \mathfrak{N}^{\pm}(i, n)$

belong to $\mathfrak{K}^{\pm}(\mathbb{N})$ and satisfy (for n odd and even)

$$\sum_{i \in \mathbb{N}} |\mathfrak{N}_n^{2+}(i)|^2 = \sum_{i \in \mathbb{N}} |\delta_{i,n}|^2 = |\delta_{n,n}|^2 = 1 < \infty,$$

and zero in the other cases. In addition, the reproducing property is verified immediately for \mathbf{n} even and $\mathbf{k} \in \mathfrak{K}_2^+(\mathbb{N})$, because

$$\begin{aligned} [k, \mathfrak{N}_n^{2+}]_{\mathfrak{K}^2} &= \sum_{i \in \mathbb{N}} (-1)^i k_i \overline{\mathfrak{N}_n^2(i)} = (-1)^n k_n \delta_{n,n} \\ &= k_n = k(n) \end{aligned}$$

and $[k, \mathfrak{N}_n^{2+}]_{\mathfrak{K}^2} = 0 = k(n)$, in case odd. In a similar form we can prove this property for the kernel \mathfrak{N}_n^{2-} .

Finally, the function $\mathfrak{N}^2 := \mathfrak{N}^{2+} + \mathfrak{N}^{2-}$ is a reproducing kernel for the Hilbert space $(\mathfrak{K}_2(\mathbb{N}), [\cdot, \cdot]_{\mathfrak{K}^2})$ according the Proposition 3.1, where for each $\mathbf{n} \in \mathbb{N}$, we can deduce that

$\mathfrak{N}_n^2 = (\delta_i, \mathbf{n})_{i \in \mathbb{N}}$. Indeed,

$$\begin{aligned} [k, \mathfrak{N}_n^{2+}]_{\mathfrak{K}^2} &= \sum_{i \in \mathbb{N}} (-1)^i (-1)^i k_i \overline{\mathfrak{N}_n^2(i)} = k_n \delta_{n,n} \\ &= k_n = k(n) \end{aligned}$$

In consequence \mathfrak{N}^2 is a reproducing kernel for the Krein space $(\mathfrak{K}_2(\mathbb{N}), [\cdot, \cdot]_{\mathfrak{K}^2})$.

3.2 Kernel for Krein Spaces

In this section we prove some useful results to establish relations between reproducing kernel for the Krein spaces and the orthogonal projectors [14].

Lemma 3.4. Let $(\mathfrak{K}, [\cdot, \cdot])$ be a Krein space of complex-valued functions on a non-empty set X with fundamental symmetry J and P an orthogonal projection commuting with J . If $\mathfrak{N}: X \times X \rightarrow \mathbb{C}$ is a

reproducing kernel for $(\mathfrak{K}, [\cdot, \cdot])$, then $P\mathfrak{N}$ is a reproducing kernel for $P\mathfrak{K}$.

Proof. If \mathfrak{N} is a reproducing kernel for \mathfrak{K} then, for each $y \in X$ \mathfrak{N}_y belong to \mathfrak{K} and hence $P\mathfrak{N}_y$ belong to $P\mathfrak{K}$ for all $y \in X$. On the other hand, if $x \in X$ and $k \in \mathfrak{K}$ then $k(x) = [k, \mathfrak{N}_x]$. Also, if $g = Pk \in P\mathfrak{K}$

then g belong to \mathfrak{K} . Thus, P is a projection in \mathfrak{K} and $g(x) = [Pk, \mathfrak{N}_x] = [Pk, P\mathfrak{N}_x] = [g, P\mathfrak{N}_x]$. Hence, $P\mathfrak{N}$ satisfies the reproducing property and then is a reproducing kernel for $P\mathfrak{K}$.

Proposition 3.5. If $(\mathfrak{K}, [\cdot, \cdot])$ is a Krein space of complex-valued functions on X with a fundamental symmetry J , then the following are equivalent:

- i) \mathfrak{N} is a reproducing kernel for the Krein space $(\mathfrak{K}, [\cdot, \cdot])$
- ii) $J\mathfrak{N}$ is a reproducing kernel for the Hilbert space $(\mathfrak{K}, [\cdot, \cdot]_J)$

Proof. Let x be an element of X , then $\mathfrak{N}_x \in \mathfrak{K}$ if and only if $J\mathfrak{N}_x \in \mathfrak{K}$. Since J is an operator acting from \mathfrak{K} into itself and satisfy the equality $J^2 = \text{Id}$, this property implies that $[k, \mathfrak{N}_x] = [k, J\mathfrak{N}_x]_J$. Therefore, the equivalence of kernel is proved because by hypotheses, for each $k \in \mathfrak{K}$,

$$k(x) = [k, \mathfrak{N}_x] = [k, J\mathfrak{N}_x]_J$$

$$= \|\mathfrak{N}_x\|_J^2 \|\mathfrak{N}_y\|_J^2$$

The following result generalizes the Proposition 2.3 given for Hilbert spaces.

$$= [\mathfrak{N}_x, J\mathfrak{N}_x][\mathfrak{N}_y, J\mathfrak{N}_y]$$

$$= (J\mathfrak{N}_x)(x)(J\mathfrak{N}_y)(y)$$

$$= (J\mathcal{N})(x,x)(J\mathcal{N})(y,y)$$

Proposition 3.6. Let $(\mathfrak{K}, [\cdot, \cdot])$ be a Krein space of complex-valued functions on a set X and let $\mathfrak{N}: X \rightarrow \mathbb{C}$ be a reproducing kernel. Then, for each x and y in X , \mathfrak{N} satisfies the following properties:

i) $J\mathfrak{N}(x,x) \geq 0$

ii) $\mathfrak{N}(x,y) = \overline{\mathcal{N}(y,x)}$

iii) $|\mathfrak{N}(x,y)|^2 \leq (J\mathfrak{N})(x,x)(J\mathfrak{N})(y,y)$

Also,

iv) For each $x_0 \in X$, the following conditions are equivalent:

a) $\mathfrak{N}(x_0, x_0) = 0$

b) $\mathfrak{N}(x, x_0) = 0, \forall x \in X$

c) $k(x_0) = 0, \forall k \in \mathfrak{K}$.

Proof. Let \mathfrak{N} be a reproducing kernel for a Krein space $(\mathfrak{K}, [\cdot, \cdot])$ and $x, y \in X$. The properties ii) and iv) are clear. Also, according to Proposition 3.5,

$$J\mathfrak{N}(x,x) = [J\mathfrak{N}_x, J\mathfrak{N}_x]_J = \|\mathfrak{N}_x\|_J \geq 0$$

Finally, by Cauchy Schwarz's inequality, it follows iv),

$$|\mathcal{N}(x,y)|^2 = |[\mathfrak{N}_y, \mathfrak{N}_x]| \leq \|\mathfrak{N}_x\|_J^2 \|\mathfrak{N}_y\|_J^2$$

The Example 3.3 exhibits relationship among the reproducing kernels of \mathfrak{K} , \mathfrak{K}^+ and \mathfrak{K}^- . The relationship among these kernels is proved in the following theorem.

Theorem 3.7. Let $(\mathfrak{K}, [\cdot, \cdot])$ be a Krein space of complex-valued functions on a set X with a decomposition $\mathfrak{K} = \mathfrak{K}^+ \dot{+} \mathfrak{K}^-$. If $\mathfrak{N}: X \times X \rightarrow \mathbb{C}$ is a reproducing kernel for \mathfrak{K} , then there exist reproducing kernels \mathfrak{N}^+ , \mathfrak{N}^- for Hilbert spaces \mathfrak{K}^+ and \mathfrak{K}^- respectively, such $\mathfrak{N} = \mathfrak{N}^+ - \mathfrak{N}^-$.

Proof. Let \mathfrak{N} be a reproducing kernel for the Krein space $\mathfrak{K} = \mathfrak{K}^+ \dot{+} \mathfrak{K}^-$ and y an element of X , then the function \mathfrak{N}_y defined by $\mathfrak{N}_y(x) = \mathfrak{N}(x,y)$ belong to \mathfrak{K} and there exist $\mathfrak{N}_y^1, \mathfrak{N}_y^2$ in the Hilbert spaces \mathfrak{K}^+ and \mathfrak{K}^- respectively, such $\mathfrak{N}_y = \mathfrak{N}_y^1 + \mathfrak{N}_y^2$. Now it must be proved that the functions $\mathfrak{N}^+: X \times X \rightarrow \mathbb{C}$ and $\mathfrak{N}^-: X \times X \rightarrow \mathbb{C}$ given by

$$\mathfrak{N}^+(x,y) := \mathfrak{N}_y^1(x) \text{ and } \mathfrak{N}^-(x,y) := -\mathfrak{N}_y^2(x)$$

are reproducing kernels for the spaces $(\mathfrak{K}^+, [\cdot, \cdot])$ and $(\mathfrak{K}^-, -[\cdot, \cdot])$ respectively. Indeed, for each $y \in X$ the

functions \mathfrak{N}_y^+ and \mathfrak{N}_y^- defined by $\mathfrak{N}_y^+(x) := \mathfrak{N}^+(x, y)$ and $\mathfrak{N}_y^-(x) := \mathfrak{N}^-(x, y)$ meet the following

$$\mathfrak{N}_y^+(x) = \mathfrak{N}_y^1(x) \text{ and } \mathfrak{N}_y^-(x) = -\mathfrak{N}_y^2(x), \forall x \in X$$

whereby for each $y \in X$ the functions \mathfrak{N}_y^+ and \mathfrak{N}_y^- are elements of the Hilbert spaces \mathfrak{K}^+ and \mathfrak{K}^- respectively. Moreover, from this it is easily seen that $\mathfrak{N} = \mathfrak{N}^+ - \mathfrak{N}^-$. As well if $h \in \mathfrak{K}^+ \subset \mathfrak{K}$ then $h(x) = [h, \mathfrak{N}_x]$ for each $x \in X$ by using the reproducing property of kernel N. Therefore,

$$\begin{aligned} h(x) &= [h, \mathfrak{N}_x^+ - \mathfrak{N}_x^-] = [h, \mathfrak{N}_x^+] - [h, \mathfrak{N}_x^-] \\ &= [h, \mathfrak{N}_x^+] - 0 \\ &= [h, \mathfrak{N}_x^+] \end{aligned}$$

Similarly, if $g \in \mathfrak{K}^- \subset \mathfrak{K}$ then $g(x) = [g, \mathfrak{N}_x^-]$ and

$$\begin{aligned} g(x) &= [g, \mathfrak{N}_x^+ - \mathfrak{N}_x^-] = [g, \mathfrak{N}_x^+] - [g, \mathfrak{N}_x^-] \\ &= 0 - [g, \mathfrak{N}_x^-] \\ &= -[g, \mathfrak{N}_x^-] \end{aligned}$$

Thus, the reproducing properties of reproducing kernels \mathfrak{N}^+ and \mathfrak{N}^- are proved for the Hilbert spaces $(\mathfrak{K}^+, [\cdot, \cdot])$ and $(\mathfrak{K}^-, -[\cdot, \cdot])$, respectively.

In other words, Theorem 3.7 guarantees us a natural way to define the nucleus of reproduction for a Krein space. Even with theorem 3.8 we have the implication that said nucleus of reproduction is independent of the fundamental symmetry since any other nucleus is equivalent to that of the previous theorem.

Theorem 3.8. Let $(\mathfrak{K}, [\cdot, \cdot])$ be a Krein space of complex-valued functions on a set X. The reproducing kernel \mathfrak{N} is uniquely determined by the Krein space \mathfrak{K} .

Proof. Suppose that a Krein space \mathfrak{K} with fundamental symmetry J has a reproducing kernel \mathfrak{N}^1 . If \mathfrak{N}^2 is another reproducing kernel for \mathfrak{K} , by Proposition 3.6 it follows that for all $y \in X$,

$$\begin{aligned} \|\mathfrak{N}_y^1 - \mathfrak{N}_y^2\|_J^2 &= [\mathfrak{N}_y^1 - \mathfrak{N}_y^2, \mathfrak{N}_y^1 - \mathfrak{N}_y^2]_J \\ &= [J(\mathfrak{N}_y^1 - \mathfrak{N}_y^2), \mathfrak{N}_y^1] - [J(\mathfrak{N}_y^1 - \mathfrak{N}_y^2), \mathfrak{N}_y^2] \end{aligned}$$

Additionally, by the reproducing property of \mathfrak{N}^1 and \mathfrak{N}^2 we obtain the equalities

$$\begin{aligned} \|\mathfrak{N}_y^1 - \mathfrak{N}_y^2\|_J^2 &= J(\mathfrak{N}_y^1 - \mathfrak{N}_y^2)(y) - J(\mathfrak{N}_y^1 - \mathfrak{N}_y^2)(y) \\ &= 0 \end{aligned}$$

Then, $\mathfrak{N}_y^1 = \mathfrak{N}_y^2$ for each $y \in X$. Thus, for any $x, y \in X$,

$$\mathfrak{N}^1(x, y) = \mathfrak{N}^2(x, y)$$

4 Conclusion

We describe the kernel of a Krein space as the difference between the kernel of definite positive subspace and the kernel of definite negative subspace. Also, we prove some basic properties of both kernels for Krein spaces and we exhibit that each kernel is uniquely determined by the Krein space given. In consequence, we generalize the notion of reproducing kernel given in Hilbert spaces

to the context of spaces endowed with indefinite metric.

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Osmin Ferrer carried out the conceptualization, methodology and supervision.

Diego Carrillo was responsible for the writing of the original draft.

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