Graph Realizations Constrained by Connected Local Dimensions and Connected Local Bases

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Abstract: For an ordered set $W = \{w_1, w_2, ..., w_k\}$ of k distinct vertices in a connected graph G, the representation of a vertex v of G with respect to W is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$, where $d(v, w_i)$ is the distance from v to w_i for $1 \leq i \leq k$. The set W is called a connected local resolving set of G if the representations of every two adjacent vertices of G with respect to W are distinct and the subgraph $\langle W \rangle$ induced by W is connected. A connected local resolving set of G of minimum cardinality is a connected local basis of G. The connected local dimension $\text{cld}(G)$ of G is the cardinality of a connected local basis of G. In this paper, the connected local dimensions of some well-known graphs are determined. We study the relationship between connected local bases and local bases in a connected graph, and also present some realization results.

Key-Words: representation, connected local resolving set, connected local basis, connected local dimension.

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1 Introduction

For an ordered set $W = \{w_1, w_2, ..., w_k\}$ of k distinct vertices of a connected graph G, the *representation* of a vertex v of G with respect to W is the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k)),$$

where $d(v, w_i)$ is the distance between v and w_i for each integer i with $1 \leq i \leq k$. If representations of any pairs of vertices u and v with respect to W are distinct, then W is called a *resolving set* of G. A resolving set of minimum cardinality is a *minimum resolving set* or a *basis* of G. The cardinality of basis of G is the *dimension* of G, which is denoted by $\dim(G)$. To illustrate this concept, consider the graph G of Fig. 1. We consider the representations of vertices of G with

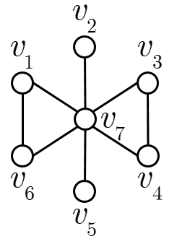

Figure 1: A connected graph G

respect to the ordered set $W_1 = \{v_1, v_3\}$. Therefore, their representations with respect to W_1 are

$$\begin{aligned} &r(v_1|W_1) = (0, 2), \quad r(v_2|W_1) = (2, 2), \\ &r(v_3|W_1) = (2, 0), \quad r(v_4|W_1) = (2, 1), \\ &r(v_5|W_1) = (2, 2), \quad r(v_6|W_1) = (1, 2), \\ &r(v_7|W_1) = (1, 1). \end{aligned}$$

Since $r(v_2|W_1) = (2, 2) = r(v_5|W_1)$, it follows that W_1 is not a resolving set of G. By considering the ordered set $W_2 = \{v_1, v_2, v_3\}$, the representations of vertices of G with respect to W_2 are

$$\begin{aligned} &r(v_1|W_2) = (0, 2, 2), \quad r(v_2|W_2) = (2, 0, 2), \\ &r(v_3|W_2) = (2, 2, 0), \quad r(v_4|W_2) = (2, 2, 1), \\ &r(v_5|W_2) = (2, 2, 2), \quad r(v_6|W_2) = (1, 2, 2), \\ &r(v_7|W_2) = (1, 1, 1). \end{aligned}$$

Since these representations are distinct, it follows that W_2 is a resolving set of G. In fact, W_2 is a basis of G and so $\dim(G) = 3$.

The concept of resolving sets was introduced by Slater in [13] and [14]. He used a locating set for what we have called a resolving set and referred to the cardinality of a basis of a connected graph as its location number. He described the usefulness of this idea when working with U.S. sonar and coast guard LORAN (long range aids to navigation) stations. Following Slater and others [4], [5] and [6], we can think of a resolving set as the set W of vertices in a connected graph G so that each vertex in G is uniquely determined by its distances to the vertices of W. Independently, Harary and Melter [3] discovered this concept as well but used the term metric dimension rather than location number. This concept was rediscovered by Johnson [8] of the Pharmacia Company while attempting to develop a capability of large datasets of chemical graphs. In [1], Chartrand and others used the term resolving set for locating set and used metric dimension for location number. Wang, Miao and Liu [15] characterized connected graphs of order n with dimension $n-3$ by using metric matrix. An application of resolving set was presented in [11]. Resolving sets in graphs have been studied further in [7], [9] and [10].

Let W be an ordered set of vertices of a connected graph G. For every pair of adjacent vertices u and v in G, if the representations of u and v with respect to W are distinct, then W is called a *local resolving set* of G. A local resolving set of G having minimum cardinality is a *minimum local resolving set* or a *local basis* of G and this cardinality is the *local dimension* of G, which is denoted by $\mathrm{ld}(G)$. A subgraph H of a graph G is called an *induced subgraph* of G if whenever u and v are vertices of H and uv is an edge of G, then uv is an edge of H as well. If S is a nonempty set of vertices of G, then the *subgraph of G induced by S* is the induced subgraph with vertex set S. This induced subgraph is denoted by $\langle S\rangle$. A resolving set W of a connected graph G is called a *connected resolving set* of G if the induced subgraph $\langle W\rangle$ is connected. The minimum cardinality of a connected resolving set of G is the *connected dimension* of G, which is denoted by $\mathrm{cd}(G)$, and a resolving set of G having this cardinality is called a *minimum connected resolving set* of a *connected basis* of G. To illustrate these concepts, consider the graph G of Fig. 1. Recall that $W_1 = \{v_1, v_3\}$ is not a resolving set of G. However, since the representations of any two adjacent vertices of G with respect to W_1 are distinct, it follows that W_1 is a local resolving set of G. Clearly, there is no a local resolving set of G consisting of one vertex. Therefore, W_1 is a local basis of G and so $\mathrm{ld}(G) = 2$. For an ordered set $W_2 = \{v_1, v_2, v_3\}$, we know that W_2 is a resolving set of G. Since $\langle W_2\rangle$ is not connected, it follows that W_2 is not a connected resolving set of G. It is routine to verify that $W_3 = \{v_1, v_2, v_3, v_7\}$ is a connected resolving set of G. Indeed, W_3 is a connected basis of G, that is, $\mathrm{cd}(G) = 4$.

The concept of local resolving sets was introduced by Okamoto and others in [2]. They characterized all nontrivial connected graphs of order n having the local dimension 1, $n-2$ or $n-1$. The idea of connected resolving sets has appeared in [12] and used the connected resolving number $\mathrm{cr}(G)$ of a connected graph G for what we have called the connected dimension of G. The local dimension and the connected dimension of some well-known graphs have been determined in [2] and [12], respectively. We state these results in the next three theorems.

Theorem 1.1 ([2]). *Let G be a nontrivial connected graph of order n. Then*

(i) $\mathrm{ld}(G) = n-1$ *if and only if* $G = K_n$ *and*

(ii) $\mathrm{ld}(G) = 1$ *if and only if* G *is bipartite.*

Theorem 1.2 ([12]). *Let G be a nontrivial connected graph of order n. Then*

(i) $\mathrm{cd}(G) = 1$ *if and only if* $G = P_n$ *and*

(ii) *if* $G = C_n$, *then* $\mathrm{cd}(G) = 2$.

Theorem 1.3 ([12]). *Let G be a connected graph of order $n \geq 3$. Then* $\mathrm{cd}(G) = n-1$ *if and only if* $G = K_n$ *or* $G = K_{1,n-1}$.

In this paper, we study a local resolving set W of a connected graph G such that the induced subgraph $\langle W\rangle$ is connected in G. In order to do this, let us introduce some definitions and notation. Let W be an ordered set of vertices of a connected graph G. Then W is called a *connected local resolving set* of G if W is a local resolving set of G such that the induced subgraph $\langle W\rangle$ of G is connected. A connected local resolving set of G having minimum cardinality is a *minimum connected local resolving set* or a *connected local basis* of G and this cardinality is the *connected local dimension* of G, which is denoted by $\mathrm{cld}(G)$. To illustrate this concept, consider the graph G of Fig. 1. We know that $W_1 = \{v_1, v_3\}$ is a local resolving set of G. However, since the induced subgraph $\langle W_1\rangle$ of G is not connected, it follows that W_1 is not a connected local resolving set of G. Then consider the ordered set $W' = \{v_1, v_3, v_7\}$. The representations of vertices of G with respect to W' are

$$\begin{array}{ll} r(v_1|W') = (0,2,1), & r(v_2|W') = (2,2,1), \\ r(v_3|W') = (2,0,1), & r(v_4|W') = (2,1,1), \\ r(v_5|W') = (2,2,1), & r(v_6|W') = (1,2,1), \\ r(v_7|W') = (1,1,0). & \end{array}$$

Since the representations of any two adjacent vertices of G with respect to W' are distinct, if follows that W' is a local resolving set of G. Moreover, the induced subgraph $\langle W'\rangle$ is connected and so W' is a connected local resolving set of G. By a case-by-case analysis, it can be shown that every connected local resolving set of G must contain at least two vertices, that is, one of $\{v_1, v_6\}$ and one of $\{v_3, v_4\}$. Thus, there is no connected resolving set of G having cardinality 2 and so W' is a connected local basis of G. Hence, $\mathrm{cld}(G) = 3$.

Observe that every connected local resolving set of a connected graph G is also a local resolving set of G but every local resolving set of G may or may not be a connected local resolving set of G. This implies that

$$1 \leq \mathrm{ld}(G) \leq \mathrm{cld}(G) \leq n-1. \qquad (1.1)$$

If W is a connected local resolving set of G, then $\langle W \rangle$ is connected. However, since the representations of any two vertices of G with respect to W need not be distinct, it follows that W is not necessarily a connected resolving set of G. In fact, every connected resolving set of G is a connected local resolving set of G, that is,

$$1 \leq \text{cld}(G) \leq \text{cd}(G) \leq n-1. \tag{1.2}$$

From Eq.(1.1) and Eq.(1.2), we obtain that

$$1 \leq \text{ld}(G) \leq \text{cld}(G) \leq \text{cd}(G) \leq n-1. \tag{1.3}$$

For every ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices of a connected graph G, recall that the only vertex of G whose representation with respect to W contains 0 in its i^{th} coordinate is w_i, that it, the vertices of W necessarily have distinct representations with respect to W. On the other hand, the representations of vertices of G that do not belong to W have elements, all of which are positive. Indeed, to determine whether an ordered set W is a connected local resolving set of G, we only need to show that any two adjacent vertices in $V(G) - W$ have distinct representations with respect to W and $\langle W \rangle$ is connected.

2 The connected local dimensions of some well-known graphs

In this section, we determined the connected local dimensions of some well-known graphs.

Theorem 2.1. *Let G be a connected graph of order $n \geq 2$. Then*

(i) $\text{cld}(G) = 1$ *if and only if G is a bipartite graph,*

(ii) $\text{cld}(G) = n - 1$ *if and only if $G = K_n$, a complete graph of order n.*

Proof. (i) Assume that $\text{cld}(G) = 1$. Then $\text{ld}(G) = 1$ by Eq.(1.3). Therefore, G is bipartite by Theorem 1.1 (ii). For converse, suppose that G is bipartite. By Theorem 1.1 (ii), $\text{ld}(G) = 1$ and so there is a 1-element local basis W of G. Indeed, W is also connected local basis of G, that is, $\text{cld}(G) = 1$.

(ii) Suppose that $\text{cld}(G) = n-1$. Eq.(1.2) implies that $\text{cd}(G) = n - 1$. Thus, by Theorem 1.3, G is complete or star. If G is a star that is not complete, then G is bipartite of order at least 3. By (i), $\text{cld}(G) = 1$, a contradiction. Hence, G is complete. On the other hand, if $G = K_n$, then by Theorem 1.1, $\text{ld}(G) = n - 1$, and so $\text{cld}(G) = n - 1$ by Eq.(1.3). □

Since every bipartite graph contains no odd cycle, it follows that the connected local dimension of an even cycle is 1. In fact, the connected local dimension of an odd cycle is 2, as we present next.

Theorem 2.2. *For an integer $n \geq 3$, the connected local dimension of a cycle C_n is*

$$\text{cld}(C_n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If n is even, then C_n is bipartite. By Theorem 2.1 (i), $\text{cld}(G) = 1$. We may assume that n is odd. Let $C_n = (v_1, v_2, ..., v_n, v_1)$ and let $W = \{v_1, v_2\}$. Consider the representations of vertices in $V(C_n) - W$. If $3 \leq i \leq \frac{n+1}{2}$, $r(v_i|W) = (i-1, i-2)$. If $i = \frac{n+3}{2}$, $r(v_i|W) = (\frac{n-1}{2}, \frac{n-1}{2})$. If $\frac{n+5}{2} \leq i \leq n$, $r(v_i|W) = (n-i+1, n-i+2)$. Thus, W is a local resolving set of C_n. Since $\langle W \rangle$ is connected, it follows that W is a connected local resolving set of C_n and so $\text{cld}(C_n) \leq 2$. Since C_n is not bipartite, it follows by Theorem 2.1 (i) that $\text{cld}(C_n) \geq 2$. Hence, $\text{cld}(C_n) = 2$. □

Observe that if G' is a graph obtained by adding a pendant edge to a connected graph G, then it is easy to verify that $\text{cld}(G') = \text{cld}(G)$. However, if a vertex v is added to a connected graph G such that more than one edge is incident with v, then the connected local dimension of the resulting graph can stay the same, decrease, or increase significantly. For example, for $n \geq 3$, $1 \leq \text{cld}(C_n) \leq 2$. Consider the connected local dimension of a *wheel* $W_n = C_n + K_1$, where $n \geq 3$. Clearly, $\text{cld}(W_3) = 3$, $\text{cld}(W_4) = \text{cld}(W_5) = 2$ and $\text{cld}(W_6) = 3$. However, for $n \geq 7$, the connected local dimension of a wheel W_n increases with n as we show next.

In $W_n = C_n + K_1$, let $C_n = (v_1, v_2, ..., v_n, v_1)$, where $n \geq 7$, and let v be the central vertex of W_n. Let S be a set of two or more vertices of C_n, let v_i and v_j be two distinct vertices of S, and let P and P' denote the two distinct $v_i - v_j$ paths determined by C_n. If either P or P', say P, contains only two vertices of S (namely, v_i and v_j), then we refer to v_i and v_j as *neighboring vertices* of S and the set of vertices of P that belong to $C_n - \{v_i, v_j\}$ as the *gap* of S (determined by v_i and v_j). The two gaps of S determined by a vertex of S and its two neighboring vertices will be referred to as *neighboring gaps*. Consequently, if $|S| = r$, then S has r gaps, some of which may be empty.

The next theorem presents a necessary and sufficient condition for a set W to be a local resolving set of W_n.

Theorem 2.3. *Let W be a set of vertices of a wheel $W_n = C_n + K_1$, where $n \geq 7$. Then W is a local resolving set of W_n if and only if every gap of W contains at most three vertices of C_n.*

Proof. Assume, to the contrary, that there is a gap of W containing at least four vertices of C_n. Then there are two adjacent vertices u and u' in this gap such that $d(u,w) = d(u',w) = 2$ for all $w \in W - \{v\}$. Therefore, $r(u|W) = r(u'|W)$, which is impossible. To show the converse, suppose that every gap of W contains at most three vertices of C_n. Since $n \geq 7$, it follows that W contains at least two vertices of C_n. For any two adjacent vertices x and y contained in a gap of W, there exists a vertex in W adjacent to only one of $\{x, y\}$. Hence the representation of x and y with respect to W are distinct. If the central vertex $v \in W$, we are done. Suppose that $v \notin W$.

Case 1. $|W| \geq 3$.
Since v is adjacent to every vertex of C_n, it follows that the representations of v and any vertices of C_n with respect to W are distinct.

Case 2. $|W| = 2$.
Suppose that there is a vertex $w' \in V(C_n)$ such that $r(w'|W) = (1,1) = r(v|W)$. Since $n \geq 7$, W contains a gap of at least four vertices, which is impossible. Hence the representations of v and any vertices of C_n with respect to W are distinct. □

An immediate consequence from Theorem 2.3 is that if W is a local resolving of W_n, where $n \geq 7$, $W - \{v\}$ is also a local resolving set. It follows that, for $n \geq 7$, every local basis of W_n contains no central vertex. However, every connected local basis of W_n must contain the central vertex. It is shown in the next result.

Lemma 2.4. *Every connected local basis of a wheel W_n, where $n \geq 7$ must contain the central vertex.*

Proof. Assume, to the contrary, that there is a connected local basis W of W_n not containing the central vertex v. Then W consists of consecutive vertices in C_n. Without loss of generality, let $W = \{v_1, v_2, ..., v_k\}$. By Theorem 2.3, it implies that $k \geq n - 3$. By the argument similar to the one used for the proof of Theorem 2.3, the set $W' = \{v, v_1, v_4, v_5, ..., v_k\}$ is a local resolving set of W_n. Moreover, $\langle W' \rangle$ is connected, that is, W' is also a connected local resolving set of W_n having cardinality $k - 1$, contradicting the assumption that W is a connected local basis of W_n. □

We are now prepared to present the connected local dimension of a wheel W_n, where $n \geq 7$.

Theorem 2.5. *Let W_n be a wheel, where $n \geq 7$, Then* $\text{cld}(W_n) = \lceil \frac{n}{4} \rceil + 1$.

Proof. By Theorem 2.3 and Lemma 2.4, we obtain that $\text{cld}(W_n) \geq \lceil \frac{n}{4} \rceil + 1$. It remains to verify that $\text{cld}(W_n) \leq \lceil \frac{n}{4} \rceil + 1$. Let $W = \{v_i \in V(C_n) \mid i \equiv 1(\text{mod}4)\} \cup \{v\}$ with $|W| = \lceil \frac{n}{4} \rceil + 1$. Since every gap of W contains at most three vertices from C_n, it follows by Theorem 2.3 that W is a local resolving set of W_n. Moreover, since W contains the central vertex v, it follows that $\langle W \rangle$ is connected and so W is a connected local resolving set of W_n. Therefore, $\text{cld}(W_n) \leq \lceil \frac{n}{4} \rceil + 1$. Hence, $\text{cld}(W_n) = \lceil \frac{n}{4} \rceil + 1$. □

3 Graphs with prescribed connected local dimensions and other parameters

The *open neighborhood* or the *neighborhood* of a vertex u of a connected graph G is the set of all vertices that are adjacent to u, which is denoted by $N(u) = \{v \in V(G) \mid uv \in E(G)\}$. The *closed neighborhood* $N[u]$ of u is defined as $N(u) \cup \{u\}$. Two vertices u and v of G are *twins* if either (i) $uv \notin E(G)$ and $N(u) = N(v)$ or (ii) $uv \in E(G)$ and $N[u] = N[v]$. In particular, if the condition (ii) holds, then u and v are called *true twins*. Consequently, the relations twin and true twin are equivalence relations on $V(G)$ and, as such, these relations partition $V(G)$ into equivalence classes which are called *twin equivalence classes* and *true twin equivalence classes*, respectively or, more simply, *twin classes* and *true twin classes*, respectively. Observe that if G contains l distinct true twin classes $U_1, U_2, ..., U_l$ of G, then every local resolving set of G must contain at least $|U_i| - 1$ vertices from each U_i, where $1 \leq i \leq l$. This observation has been described in [2] as we state next.

Proposition 3.1 ([2]). *Let G be a connected graph having l true twin classes $U_1, U_2, ..., U_l$. Then every local resolving set of G must contain $|U_i| - 1$ vertices from each U_i, where $1 \leq i \leq l$. Moreover,* $\text{ld}(G) \geq \sum_{i=1}^{l} |U_i| - l$.

We have seen that if G is a connected graph of order n with $\text{ld}(G) = a$ and $\text{cld}(G) = b$, then $1 \leq a \leq b \leq n - 1$ by Eq.(1.1). In fact, any integers a, b and n with $1 \leq a \leq b \leq n - 1$ are realizable as the local dimension, connected local dimension and order of some graphs as we show next.

Theorem 3.2. *Let a, b, n be integers with $n \geq 4$. Then there exists a connected graph G of order n with* $\text{ld}(G) = a$ *and* $\text{cld}(G) = b$ *if and only if a, b, n satisfy one of the following:*

(i) $a = b = n - 1$,

(ii) $a = b = 1$, *and*

(iii) $2 \leq a \leq b \leq n - 2$.

Proof. Assume that there exists a connected graph G of order n with $\text{ld}(G) = a$ and $\text{cld}(G) = b$. By Eq.(1.1), we obtain that $1 \leq a \leq b \leq n - 1$. If $b = n - 1$, then G is a complete graph K_n. Thus,

$a = b = n-1$. If $a = 1$, then G is a bipartite graph. Therefore, $a = b = 1$. For otherwise, $2 \le a \le b \le c \le n-2$. Hence, if G is a connected graph of order n with $\mathrm{ld}(G) = a$ and $\mathrm{cld}(G) = b$, then a, b and n must satisfy one of (i), (ii) and (iii). It remains to verify the converse. If $a = b = n-1$, then let G be a complete graph K_n and the result is true. If $a = b = 1$, then let G be a path P_n. Thus, the graph G has the desired properties. We may assume that $2 \le a \le b \le n-2$. We consider two cases.

Case 1. $a = b$.

Let G' be a graph obtained from a complete graph K_a with vertex set $\{u_1, u_2, ..., u_a\}$ and a path $P_{n-a} = (v_1, v_2, ..., v_{n-a})$ by joining v_1 to every vertex of K_a. Since $V(K_a)$ is a true twin class of G', every local resolving set of G' must contain at least $a-1$ vertices from $V(K_a)$. However, if a set W contains only $a-1$ vertices from $V(K_a)$, then W does not contain u_i for some integer i with $1 \le i \le a$ and so $r(u_i|W) = r(v_1|W) = (1, 1, ..., 1)$. Therefore, G contains no local resolving set of cardinality $a-1$, that is, $\mathrm{ld}(G') \ge a$. Since $r(v_j|V(K_a)) = (j, j, ..., j)$, where $1 \le j \le n-a$, it follows that $V(K_a)$ is a local resolving set of G' having cardinality a, that is, $V(K_a)$ is a local basis of G'. Moreover, $V(K_a)$ is also a connected local basis of G'. Hence, $\mathrm{ld}(G') = \mathrm{cld}(G') = a$.

Case 2. $a < b$.

Let G be a graph obtained from a complete graph K_a with vertex set $\{u_1, u_2, ..., u_a\}$ and two paths $P_{b-a+1} = (v_1, v_2, ..., v_{b-a+1})$ and $P_{n-b-1} = (w_1, w_2, ..., w_{n-b-1})$ by joining v_1 to every vertex of K_a, and w_1 to both v_{b-a} and v_{b-a+1}. Since $V(K_a)$ is a true twin class of G, it follows by Proposition 3.1 that every local resolving set of G must contain at least $a-1$ vertices from $V(K_a)$. However, every set consisting of $a-1$ vertices from $V(K_a)$ is not a local resolving set of G since the representations of v_{b-a+1} and w_1 with respect to this set are the same. Thus, every local resolving set of G contains at least a vertices. It is routine to verify that every local resolving set of G must contain at least one vertex from $\{v_{b-a+1}\} \cup V(P_{n-b-1})$. Then the set $(V(K_a) - \{u_1\}) \cup \{v_{b-a+1}\}$ is a minimum local resolving set of G. Hence, $\mathrm{ld}(G) = a$. Since every connected local resolving set of G is also a local resolving set of G, it follows that every connected local resolving set of G must contain at least $a-1$ vertices from $V(K_a)$ and at least one vertex from $\{v_{b-a+1}\} \cup V(P_{n-b-1})$. Therefore, every connected local resolving set of G contains $v_1, v_2, ..., v_{b-a}$. In fact, the set $(V(K_a) - \{u_1\}) \cup V(P_{b-a+1})$ is a connected local basis of G, that is, $\mathrm{cld}(G) = b$. □

We know by Eq.(1.2) that if G is a connected graph of order n with $\mathrm{cld}(G) = b$ and $\mathrm{cd}(G) = c$, then $1 \le b \le c \le n-1$. Next, we show that for any integers b, c and n with $1 \le b \le c \le n-1$ are realizable as the connected local dimension, connected dimension and order of some graphs.

Theorem 3.3. *Let b, c, n be integers with $n \ge 4$. Then there exists a connected graph G of order n with $\mathrm{cld}(G) = b$ and $\mathrm{cd}(G) = c$ if and only if b, c, n satisfy one of the following:*

(i) $b = c = n-1$,

(ii) $b = 1$ *and* $1 \le c \le n-1$, *and*

(iii) $2 \le b \le c \le n-2$.

Proof. Assume that there exists a connected graph of order n with $\mathrm{cld}(G) = b$ and $\mathrm{cd}(G) = c$. By Eq.(1.2), we obtain that $1 \le b \le c \le n-1$. If $b = n-1$, then $c = n-1$ by Eq.(1.2). If $b = 1$, then $1 \le c \le n-1$ by Eq.(1.2). If $2 \le b \le n-2$, then G is neither a star nor a complete graph, and so $2 \le b \le c \le n-2$. Hence, if G is a connected graph of order n with $\mathrm{cld}(G) = b$ and $\mathrm{cd}(G) = c$, then b, c and n must satisfy one of (i), (ii) and (iii). It remains to verify the converse. If $b = c = n-1$, then let G be a complete graph K_n and the result is true. Next, assume that $b = 1$ and $1 \le c \le n-1$. For $c = 1$, let G be a path P_n; while for $c = n-1$ let G be a star $K_{1,n-1}$. Since $\mathrm{cld}(P_n) = \mathrm{cd}(P_n) = 1$, and $\mathrm{cld}(K_{1,n-1}) = 1$ and $\mathrm{cd}(K_{1,n-1}) = n-1$, it follows that the result holds for $b = 1$ and $c = 1, n-1$. For $2 \le c \le n-2$, let G be a graph obtained from a complete bipartite graph $K_{2,c-1}$ with partite sets $U = \{u_1, u_2\}$ and $U' = \{w_1, w_2, ..., w_{c-1}\}$, and a path $P_{n-c-1} = (v_1, v_2, ..., v_{n-c-1})$ by joining v_1 to both u_1 and u_2. Since G is bipartite, it follows that $\mathrm{cld}(G) = 1$. It is routine to show that the set $V(K_{2,c-1}) - \{u_2\}$ is a connected basis of G. Therefore, $\mathrm{cd}(G) = c$. Hence, the result holds for $b = 1$ and $2 \le c \le n-2$. Now assume that $2 \le b \le c \le n-2$. We consider two cases.

Case 1. $b = c$.

The graph G' of the proof for Theorem 3.2 has $\mathrm{cld}(G') = b$ with a connected local basis $V(K_b)$. In fact, $V(K_b)$ is also a connected basis of G', that is, $\mathrm{cd}(G') = b$.

Case 2. $b < c$.

Let G be a graph obtained from a complete graph K_b with vertex set $\{u_1, u_2, ..., u_b\}$, a star $K_{1,c-b}$ with vertex set $\{v, v_1, v_2, ..., v_{c-b}\}$ and a path $P_{n-c-1} = (w_1, w_2, ..., w_{n-c-1})$ by joining the central vertex v of $K_{1,c-b}$ to w_1 and every vertex of K_b. It is immediate that the set $V(K_b)$ is a connected local basis of G. Therefore, $\mathrm{cld}(G) = b$. Moreover, the set $(V(K_b) - \{u_1\}) \cup V(K_{1,c-b})$ is a connected basis of G, that is, $\mathrm{cd}(G) = c$. □

4 Connected local bases and local bases in graphs

In this section, we study the relationship between connected local bases and local bases in a connected graph G. Certainly, if W is a local resolving set of G, then a set W' containing W is also a local resolving set of G. Therefore, if W is a local basis of G such that $\langle W\rangle$ is disconnected, then surely there is a smallest superset W' of W for which $\langle W'\rangle$ is connected. This suggests the following question: Does there exist a graph with a connected local basis not containing any local bases? The answer to this question is given in the next result.

Theorem 4.1. *There is an infinite class of connected graphs G such that some connected local bases of G contain a local basis of G and others contain no local basis of G.*

Proof. Let G be a graph obtained from a complete graph K_a of order $a \geq 2$ with vertex set $\{u_1, u_2, ..., u_a\}$, a cycle $C_4 = (v_1, v_2, v_3, v_4, v_1)$ and a path $P_3 = (w_1, w_2, w_3)$ by joining v_1 to every vertex of K_a and joining w_1 and w_3 to v_1, v_4 and v_2, v_3, respectively. A graph G is shown in Fig. 2. We first

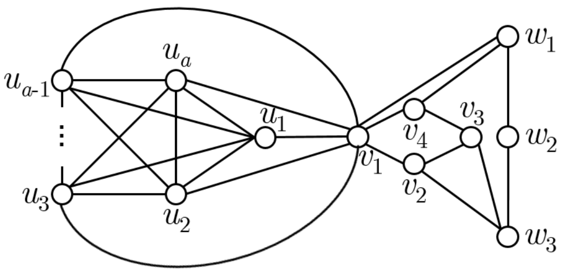

Figure 2: A graph G

verify that the set $B = \{u_1, u_2, ..., u_{a-1}\} \cup \{w_2\}$ is a local basis of G. We can show, by a case-by-case analysis, that B is a local resolving set of G. Next, we claim that B is a local resolving set of minimum cardinality. Assume, to the contrary, that there is a local resolving set W of G having cardinality at most $a-1$. Since $V(K_a)$ is a true twin class of G, it follows that every local resolving set of G must contain at least $a-1$ vertices of K_a. Therefore, W consists of $a-1$ vertices of K_a. However, v_4 and w_1 are adjacent and $d(v_4, u_i) = d(w_1, u_i)$ for each integer i with $1 \leq i \leq a$. This is a contradiction. Hence, B is a local basis of G and so $\text{ld}(G) = a$. Second, we determine that $\text{cld}(G) = a + 2$. In order to do this, we claim that $\text{cld}(G) \geq a + 2$. Suppose, contrary to our claim, that there is a connected local resolving set W' of G having cardinality $a + 1$. Recall that every connected local basis of G must contain at least $a-1$ vertices of K_a. We consider two cases.

Case 1. $V(K_a) \subseteq W'$.
Since $\langle W'\rangle$ is connected and $|W'| = a + 1$, it follows that $W' = V(K_a) \cup \{v_1\}$. However, since v_4 is adjacent to w_1 and $r(v_4|W') = r(w_1|W')$, it follows that W' is not a connected resolving set of G, which is a contradiction.

Case 2. $V(K_a) \nsubseteq W'$.
Since $\langle W'\rangle$ is connected and $|W'| = a + 1$, it follows that W' contains v_1 and one vertex from $\{v_2, v_4, w_1\}$. If W' contains v_2 or w_1, then $r(v_3|W') = r(w_3|W')$. If W' contains v_4, then $r(w_2|W') = r(w_3|W')$. Therefore, W' is not a connected local resolving set of G. This is also a contradiction.

Therefore, $\text{cld}(G) \geq a + 2$. On the other hand, the sets $S_1 = \{u_1, u_2, ..., u_{a-1}\} \cup \{v_1, w_1, w_2\}$ and $S_2 = \{u_1, u_2, ..., u_{a-1}\} \cup \{v_1, v_4, w_1\}$ are connected local resolving sets of G. Therefore, $\text{cld}(G) \leq a + 2$. Hence, $\text{cld}(G) = a + 2$.

Last, it can be verified that every local basis of G contains exactly $a-1$ vertices of K_a and exactly one vertex from $\{v_3, w_2\}$. Observe that the connected local basis S_1 contains the local basis B of G, while the connected local basis S_2 contains no local basis of G. □

From the previous theorem, there is a connected graph having many connected local bases. This leads us to determine a connected graph G having a unique connected local basis. It has been shown in [2] that there is a connected graph with a unique local basis. In fact, there is a connected graph with a unique connected local basis as we show next.

Theorem 4.2. *For $k \geq 3$, there exists a graph with a unique connected local basis of cardinality $k + 1$.*

Proof. Let G_1 be a complete graph K_{2^k} with vertex set $U = \{u_0, u_1, ..., u_{2^k-1}\}$, and let G_2 be a empty graph $\overline{K}_k$ with vertex set $W = \{w_{k-1}, w_{k-2}, ..., w_0\}$. Then the graph G is obtained from G_1 and G_2 by adding edges between U and W as follows. Let each integer j for $1 \leq j \leq 2^k - 1$ be expressed in its base 2 (binary) representation. Thus, each such j can be expressed as a sequence of k coordinates, that is, a k-vector, where the rightmost coordinate represents the value (either 0 or 1) in the 2^0 position, the coordinate to its immediate left is the value in the 2^1 position, etc. For integers i and j with $0 \leq i \leq k-1$ and $0 \leq j \leq 2^k - 1$, we join w_i and u_j if and only if the value in the 2^i position in the binary representation of j is 1. For example, Fig. 3 shows the edges joining between U and W in the graph G for $k = 3$. It was shown in [2] that W is a unique local basis of G. Therefore, there is no connected local basis of G having cardinality k, that is, $\text{cld}(G) \geq k + 1$. Since W is a local basis of G, it follows that $W' = W \cup \{u_{2^k-1}\}$

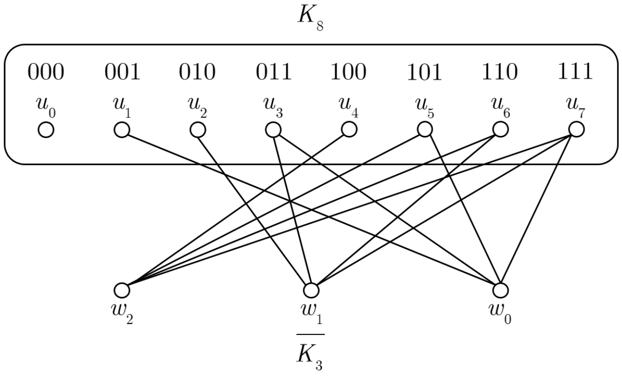

Figure 3: A graph G for $k = 3$

is a connected local resolving set of G. In fact, W' is a connected local basis of G.

It remains only to show that G has no other connected local basis. If $U' \subseteq U$ and $|U'| = k + 1$, then $|U - U'| = 2^k - k - 1 \geq 2$. Since the distance of every two vertices of U is 1, it follows that there are at least two adjacent vertices of $U - U'$ having the same representation with respect to U', and so U' is not a connected local resolving set of G. Thus, every connected local resolving set of G must contain at least one vertex of W. Suppose that $B \neq W'$ is a connected local basis of G. Therefore, $B = U'' \cup W''$, where $U'' \subseteq U$ and $W'' \subseteq W$. If $|W''| = k$, then B does not contain u_{2^k-1}. Therefore, $\langle B \rangle$ is not connected, which is impossible. If $|W''| \leq k - 1$, then U'' contains at least two vertices. We may therefore assume that $|U''| = i \geq 2$. Then $|W''| = k - i + 1$. Since every vertex of $U - U''$ has distance 1 from every vertex of U'', it follows that there are at most 2^{k-i+1} distinct representations of vertices of $U - U''$ with respect to B. However, since $2^k - i > 2^{k-i+1}$, there are two vertices of $U - U''$ such that their representation with respect to B are the same, contradicting the fact that B is a connected local basis of G. Hence, W' is a unique connected local basis of G. $\square$

5 Discussion and conclusions

By Eq.(1.3), it suggests the following question: For which quadruples a, b, c, n of integers with $1 \leq a \leq b \leq c \leq n-1$, does there exist a connected graph G of order n with $\mathrm{ld}(G) = a$, $\mathrm{cld}(G) = b$ and $\mathrm{cd}(G) = c$?

In this research, we have investigated the connected local dimensions of bipartite graphs, complete graphs, cycles and wheels. We show the realization results that any integers a, b and n with $1 \leq a \leq b \leq n-1$ are realizable as the local dimension, connected local dimension and order of some graphs. Moreover, for any integers b, c and n with $1 \leq b \leq c \leq n-1$ are realizable as the connected local dimension, connected dimension and order of some graphs. We present the relationship between connected local bases and local bases in a connected graph, that is, there is an infinite class of connected graphs G such that some connected local bases of G contain a local basis of G and others contain no local basis of G. We determine the stronger result that there is a connected graph with a unique connected local basis.

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