Subdivisions of horned or spindle Dupin cyclides using Bézier curves with mass points

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Abstract: This paper shows the same algorithm is used for subdivisions of Dupin cyclides with singular points and quadratic Bézier curves passing through infinity. The mass points are usefull for any quadratic Bézier representation of a parabola or an hyperbola arc. The mass points are mixing weighted points and pure vectors. Any Dupin cyclide is considered in the Minkowski-Lorentz space. In that space, the Dupin cyclide is defined by the union of two conics laying on the unit pseudo-hypersphere, called the space of spheres. The subdivision of any Dupin cyclide, is equivalent to subdivide two Bézier curves of degree 2 with mass points, independently. The use of these two curves eases the subdivision of a Dupin cyclide patch or triangle.

Key-Words: Mass points, no bounded Rational Quadratic Bézier curves, Conics, Subdivisions, Space of spheres, Minkowski-Lorentz space, Dupin cyclides with singular point(s)

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1 Introduction

This article set up the second part of Dupin cyclides subdivisions work. In the first part [1], the authors give some algorithms to subdivide Bézier curves representing ellipse arcs or semi-ellipses with mass points. For the quadratic Lorentz form, these conics are circles on the space of spheres in the Minkowski-Lorentz space and represent ring Dupin cyclides. So, the same algorithms permit to subdivide Dupin cyclide patches. Moreover, on a ring torus or a ring quartic Dupin cyclide, one can find three kinds of circles: the meridians and the parallels which are curvature lines and Yvon-Villarceau circles. On the aforementioned surfaces, using a meridian arc, a parallel arc and an Yvon-Villarceau circle arc, one can generate rectangular 3D triangles [2], using a meridian arc or a parallel arc and two Yvon-Villarceau circle arcs one can generate isosceles 3D triangles [3]. With the other Dupin cyclides, one can construct 3D triangles with circular edges using one meridian and two parallels or one parallel and two meridians. In the paper, the subdivision of 3D triangles which are degenerated patches on horned or spindle Dupin cyclides is treated.

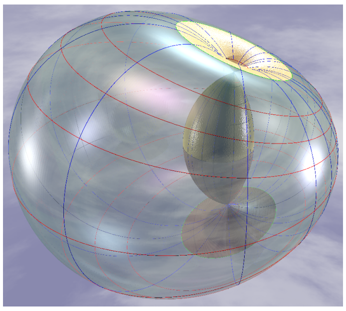
The paper provides methods to subdivide Dupin cyclides having at least one singular point. In the Minkowski-Lorentz space, a Dupin cyclide with one singular point (resp. two singular points) is represented by a connected circle which looks like an ellipse and an affine parabola isometric to a line (resp. a circle with two components which looks like a hyperbola). The rational quadratic Bézier curves are modelling these conics. Thus, to subdivide these Dupin cyclides is equivalent to subdivide rational quadratic Bézier curves in the forementioned representation based on mass points.

The paper is organized as follows: the section 2 presents Dupin cyclides with one or two singular points in the 3-dimensional Euclidean affine space $\mathcal{E}_3$ and their representation in the Minkowski-Lorentz space. The section 3 provides some theorems which permit the subdivision of Bézier curves representing parabola or hyperbola arc and the subdivision of Dupin cyclides. Before the conclusion and the perspective, in section 4, some examples of subdivision of Horned Dupin cyclides patches are presented. The Appendix A gives the representation of Dupin cyclides in the Minkowski-Lorentz space. The Appendix B recalls some Theorems defined in [1].

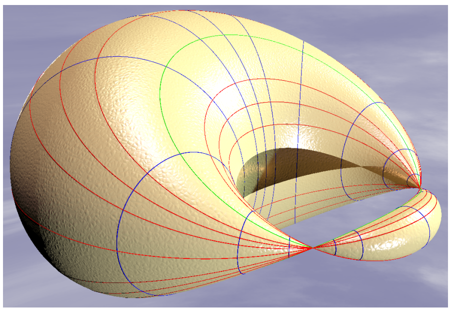
2 Dupin cyclides in the Minkowski-Lorentz space

The Dupin cyclides were invented by P. Dupin in 1822 [4] and introduced in CAD by R. Martin in 1982 [5]. A Dupin cyclide can be defined, in two different ways, as the envelope of an one-parameter family of oriented spheres. Each family of spheres can be seen as a conic in the space of spheres Λ^4 which is the 4-dimensional pseudo-unit hypersphere of the Minkowski-Lorentz space, see ??.

We can distinguish five kinds of Dupin cyclides [5, 6, 7] depending on the number of singular points: ring, [1] spindle, Figure 1(a); horned, Figure 1(b); one-singularity spindle, Figure 2(a) and singly horned, Figure 2(b).



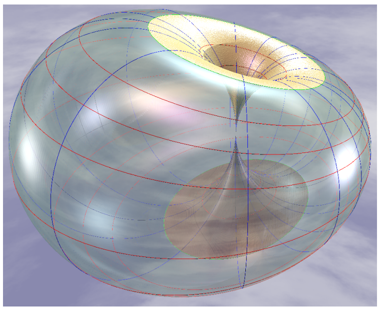
(a)



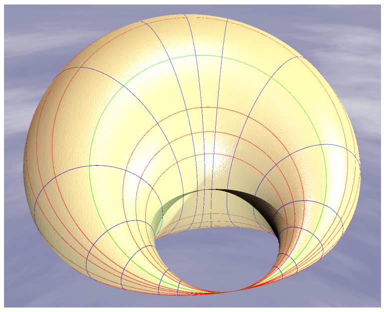
(b)

Figure 1: Dupin cyclides with two singular points in $\mathcal{E}_3$. (a): spindle. (b): horned.

The representation of a Dupin cyclide with singular point(s) on the space of spheres Λ^4 is the union of two circles, Figure 3 or the union of a circle and an affine parabola isometric to a line,



(a)



(b)

Figure 2: Dupin cyclides with an unique singular point in $\mathcal{E}_3$. (a): one-singularity spindle. (b): singly horned.

Figure 4 (see [8]).

Using an Euclidean point of view, we can see these conics as:

- an ellipse and a hyperbola, Figure 3, for a spindle or horned Dupin cyclide;
- an ellipse and a parabola isometric to a line, Figure 4, for a singly horned Dupin cyclide or one-singularity spindle Dupin cyclide.

3 Regular iterative subdivision of Dupin cyclides

From the Minkowski-Lorentz spaces, the set $\widetilde{\mathrm{L}_{4,1}}$ of mass points (A, a) are defined where

- $a = 0$ implies that A is a vector of $\overrightarrow{\mathrm{L}_{4,1}}$;
- $a \neq 0$ implies that A is a point of $\mathrm{L}_{4,1}$.

3.1 Common case with three weighted points : $(\omega_0; \omega_1; \omega_2) \in \{1\} \times]-1, +\infty[- \{0\} \times \{1\}$

In this paragraph, the three control points are weighted points, the arc is bounded. Any proper Euclidean conic can be obtained and theorems

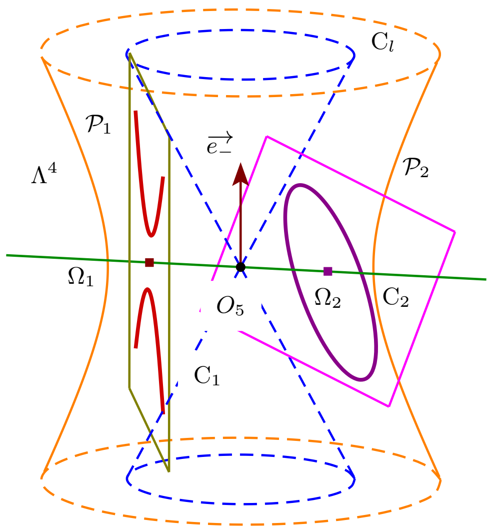


Figure 3: Representation of horned and spindle Dupin cyclides on Λ^4.by the union of two circles, one is connected, the other has two connected components, see Table 1.

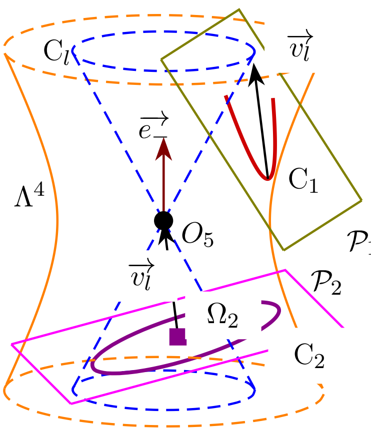


Figure 4: Representation of a Dupin cyclide with one only singular point is the union of a circle and a affine parabola isometric to a line, see Table 1.

given in [1] are used. In the other section, subdivision of unbounded arcs are now on consideration i.e. $\omega_1 = -1$ (resp. $\omega_1 < -1$) leads to a parabola arc (resp. hyperbola arc).

We have

$$\left(x_o \overrightarrow{e_o} + \sum_{i=1}^{3} x_i \overrightarrow{e_i} + x_\infty \overrightarrow{e_\infty} \right) \boldsymbol{.} \overrightarrow{e_\infty} = -x_o \quad (1)$$

and

$$\left(x_o \overrightarrow{e_o} + \sum_{i=1}^{3} x_i \overrightarrow{e_i} + x_\infty \overrightarrow{e_\infty} \right) \boldsymbol{.} \overrightarrow{e_o} = -x_\infty \quad (2)$$

and we introduce the following notation.

Notation 1 In the following, the notation $\overrightarrow{e_{o\vee\infty}}$ always means either $\overrightarrow{e_o}$ or $\overrightarrow{e_\infty}$ to compute the opposite of the first or of the last component of a light-like vector.

Let $\overrightarrow{m}$ be a light-like vector. Then, two cases arise:

1. $\overrightarrow{m} = \lambda \overrightarrow{e_\infty}$ with $\lambda \in \mathbb{R}^*$, and we have

$$\overrightarrow{m} \boldsymbol{.} \overrightarrow{e_{o\vee\infty}} = \overrightarrow{m} \boldsymbol{.} \overrightarrow{e_o} = -\lambda \neq 0 \quad (3)$$

and

$$\frac{1}{\lambda} \overrightarrow{m} = \overrightarrow{e_\infty} \quad (4)$$

2. $\overrightarrow{m} = x_o \overrightarrow{e_o} + \sum_{i=1}^{3} x_i \overrightarrow{e_i} + x_\infty \overrightarrow{e_\infty}$ with $(x_o, x_1, x_2, x_3) \neq (0, 0, 0, 0)$, and we have

$$\overrightarrow{m} \boldsymbol{.} \overrightarrow{e_{o\vee\infty}} = \overrightarrow{m} \boldsymbol{.} \overrightarrow{e_\infty} = -x_o \neq 0 \quad (5)$$

and

$$\frac{1}{x_o} \overrightarrow{m} = \overrightarrow{e_o} + \sum_{i=1}^{3} \frac{x_i}{x_o} \overrightarrow{e_i} + \frac{x_\infty}{x_o} \overrightarrow{e_\infty} \quad (6)$$

which represents the point $M\left(\frac{x_1}{x_o}, \frac{x_2}{x_o}, \frac{x_3}{x_o}\right)$ of $\mathcal{E}_3$.

In the canonical reference frame $(O_5; \overrightarrow{e_-}, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_+})$ of the Minkowski-Lorentz space, the coordinates of any point are noted $(x_-; x_1; x_2 : x_3; x_+)$ whereas, in the reference frame $(O_5; \overrightarrow{e_o}, \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_\infty})$, the point coordinates are noted $(x_o; x_1; x_2 : x_3; x_\infty)$.

The vector $\overrightarrow{e_{o\vee\infty}}$ denotes either the vector $\overrightarrow{e_o}$ or the vector $\overrightarrow{e_\infty}$ in order to get back the component of $\left(\overrightarrow{P_1\sigma_0} + \overrightarrow{P_1\sigma_2}\right)$ along vector $\overrightarrow{e_\infty}$ or vector $\overrightarrow{e_o}$.

In order to fill out the Bézier by a light-like vector representing a point of $\mathcal{E}_3$, we must take two cases : if locally the spheres radii are positives, we are in the half space whose equation is $x_+ = 2\,x_\infty - x_o > 0$ and the light-like vector must have a component equal to 1 along $\overrightarrow{e_{o\vee\infty}}$ while if the spheres radii are negatives we are in the half space whose equation is $x_+ = 2\,x_\infty - x_o < 0$ and the light-like vector must have a component equal to -1 along $\overrightarrow{e_{o\vee\infty}}$.

3.2 Parabola case

3.2.1 Case with three weighted points: $\omega_0 = \omega_2 = 1$ and $\omega_1 = -1$

In this paragraph, the Bézier curve is the complement of a bounded arc of parabola. It yields,

$$B_0(t) - B_1(t) + B_2 = (1-2\,t)^2$$

where B_0, B_1 and B_2 are the quadratic Bernstein polynomials. For any control points of the Bézier curve, the vector is obtained for $t = \frac{1}{2}$.

Theorem 1 : Let γ be a Bézier curve with mass control points $(\sigma_0;1)$, $(P_1;-1)$ and $(\sigma_2;1)$ laying on the conic C.

Denote

$$b_0 = \sqrt{\left|\frac{-1}{\left(\overrightarrow{P_1\sigma_0}+\overrightarrow{P_1\sigma_2}\right)\cdot\overrightarrow{e_{o\vee\infty}}}\right|} \tag{7}$$

Let h_0 defined by

$$h_0 : u \longmapsto \frac{b_0\,u}{(1-u)+2\,b_0\,u} \tag{8}$$

then $\gamma \circ h_0$ is a Bézier curve with mass control points $(\sigma_{00};1)$, $\left(\overrightarrow{P_{10}};0\right)$ and $(\overrightarrow{m_{20}};0)$ laying on the conic C with

$$\left\{\begin{array}{lcl}(\sigma_{00};1) & = & (\sigma_0;1)\\ \left(\overrightarrow{P_{10}};0\right) & = & \left(b_0\,\overrightarrow{P_1\sigma_0};0\right)\\ (\overrightarrow{m_{20}};0) & = & \left(b_0^2\left(\overrightarrow{P_1\sigma_0}+\overrightarrow{P_1\sigma_2}\right);0\right)\end{array}\right. \tag{9}$$

Proof : Use the relationship from formula (51) with b_0 and $c_0 = 1$.
■

For symmetry reasons, we can formulate

Theorem 2 : Let γ be a Bézier curve with mass control points $(\sigma_0;1)$, $(P_1;\omega_1)$ and $(\sigma_2;1)$ laying on the C.

Consider

$$a_1 = \sqrt{\left|\frac{-1}{\left(\overrightarrow{P_1\sigma_0}+\overrightarrow{P_1\sigma_2}\right)\cdot\overrightarrow{e_{o\vee\infty}}}\right|}$$

Let h_1 be defined by

$$h_1 : u \longmapsto \frac{a_1\,(1-u)+\,u}{2\,a_1\,(1-u)+\,u} \tag{10}$$

then $\gamma \circ h_1$ is the Bézier curve with mass control points and $(\sigma_{21};1)$ laying on the conic C with

$$\left\{\begin{array}{lcl}(\overrightarrow{m_{01}};0) & = & \left(a_1^2\left(\overrightarrow{P_1\sigma_0}+\overrightarrow{P_1\sigma_2}\right);0\right)\\ \left(\overrightarrow{P_{11}};0\right) & = & \left(a_1\,\overrightarrow{P_1\sigma_2};0\right)\\ (\sigma_{21};1) & = & (\sigma_2;1)\end{array}\right. \tag{11}$$

Notice that if the Dupin cyclide is represented by a parabola (it owns only one singular point or contains only the point at the infinite direction as a revolution cylinder) The spheres radii get a constant sign : positive, negative or null. The figure 5 shows an iteration of subdivision based on the Theorems 1 and 2 and we have the vector $\overrightarrow{m_{01}} = \overrightarrow{m_{20}}$ directing the parabola axis.

3.2.2 Case of a parabola where $(\omega_0;\omega_1;\omega_2) = (1;0;0)$

The first control mass point is a weighted point whereas the other control mass points are vectors, the Bézier curve is a parabola arc.

Theorem 3 Let a Bézier curve γ with control mass points $(\sigma_0;1)$, $\left(\overrightarrow{P_1};0\right)$ and $(\overrightarrow{m_2};0)$ laying on the conic C. Let h_0 defined by

$$h_0 : u \longmapsto \frac{u}{(1-u)+2\,u} \tag{12}$$

then $\gamma \circ h_0$ is a Bézier curve with control mass points $(\sigma_{00};1)$, $(P_{10};1)$ and $(\sigma_{20};1)$ laying on the conic C with

$$\left\{\begin{array}{lcl}(\sigma_{00};1) & = & (\sigma_0;1)\\ (P_{10};1) & = & \left(\mathcal{T}_{\overrightarrow{P_1}}(\sigma_0);1\right)\\ (\sigma_{20};1) & = & \left(\mathcal{T}_{\overrightarrow{m_2}+2\,\overrightarrow{P_1}}(\sigma_0);1\right)\end{array}\right. \tag{13}$$

Proof: by the use of Formula (51) with $b_0 = 1$ and $c_0 = 1$:
■

The new curve is polynomial, so, one can use the usual De Casteljau algorithm with $t = \frac{1}{2}$.

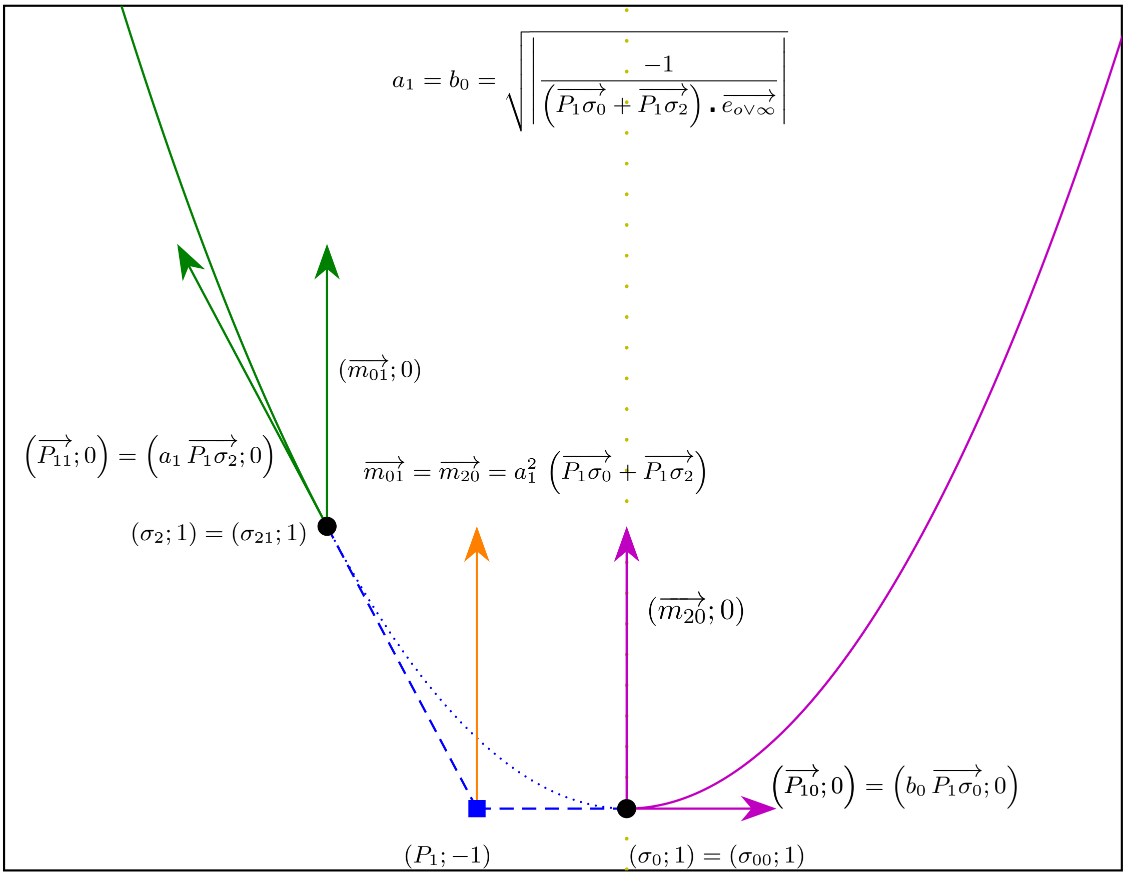


Figure 5: An iteration of an iterative construction of a parabola arc γ on Λ^4, with endpoints $(\sigma_0;1)$ and $(\sigma_2;1)$ with tangent lines $(\sigma_0 P_1)$ and $(\sigma_2 P_1)$ with $(P_1;-1)$. If the $\overrightarrow{m_{20}}$ vector represents a point of $\mathcal{E}_3$, its component along $\overrightarrow{e_o}$ with the same sign as the spheres radii of the canal surface. If the vector $\overrightarrow{m_{20}}$ performs the point in the infinite direction of $\mathcal{E}_3$, if the spheres radii are positive we have $\overrightarrow{m_{20}} = \overrightarrow{e_\infty}$, else we have $\overrightarrow{m_{20}} = -\overrightarrow{e_\infty}$.

Theorem 4 Let a Bézier curve γ with control mass points $(\sigma_0;1)$, $\left(\overrightarrow{P_1};0\right)$ and $(\overrightarrow{m_2};0)$ laying on the conic C. Let h_1 defined by

$$h_1 : u \longmapsto \frac{1}{2\,(1-u)+u} \tag{14}$$

then $\gamma \circ h_1$ is a Bézier curve with control mass points $(\sigma_{01};1)$, $\left(\overrightarrow{P_{11}};0\right)$ and $(\overrightarrow{m_{21}};0)$ laying on the conic C with

$$\left\{\begin{array}{rcl} (\sigma_{01};1) & = & \left(\mathcal{T}_{\overrightarrow{m_2}+2\,\overrightarrow{P_1}}(\sigma_0)\,;1\right) \\ \left(\overrightarrow{P_{11}};0\right) & = & \left(\overrightarrow{P_1}+\overrightarrow{m_2};0\right) \\ (\overrightarrow{m_{21}};0) & = & (\overrightarrow{m_2};0) \end{array}\right. \tag{15}$$

Proof: by the use of Formula (57) with $a_1 = 1$ and $b_1 = 1$. ■

The Figure 6 shows an iteration of the subdivision of a non-bounded parabola based on the Theorems 3 and 4.

3.3 Case of circles with two connected components or Hyperbola case

Theorems 15 and 16 provide a way to divide a circle arc with two connected components. In this section, we are focusing on other "standard" cases of arcs of circles with two connected components as: the case of a unbounded arc in 3.3.1, the case of one branch in 3.3.2, the case of a connected arc unbounded in 3.3.3.

3.3.1 Case with three weighted points: $(\omega_0;\omega_1;\omega_2) \in \{1\} \times \left]-\infty,-1\right[\times \{1\}$

In that case, the curve is the complement of a bounded arc of a circle sketched by a hyperbola. The denominator of the Bézier curve is a polyno-

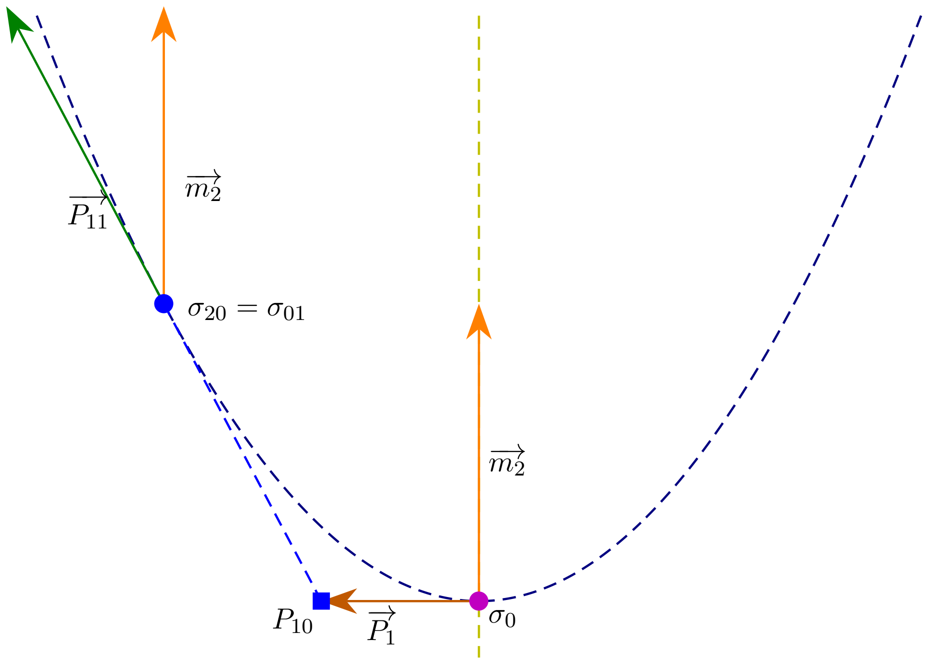


Figure 6: One iteration of a iterative subdivision of a non-bounded parabola arc.

mial of degree 2 with two roots satisfying

$$\begin{cases} t_1 = \dfrac{1-\omega_1-\sqrt{\omega_1^2-1}}{2\,(1-\omega_1)} \\ t_2 = \dfrac{1-\omega_1+\sqrt{\omega_1^2-1}}{2\,(1-\omega_1)} \end{cases} \tag{16}$$

given that $\omega_1 < -1 < 0$ implies the roots t_1 and t_2 belong to $[0;1]$ and $t_1 < t_2$.

We define three homographies h_2, h_3 and h_4 such that

$$\begin{cases} h_2\left([0;1]\right) = [0;t_1] \\ h_3\left([0;1]\right) = [t_1;t_2] \\ h_4\left([0;1]\right) = [t_2;1] \end{cases}$$

The use a the denominator roots gives directly the light vectors representing the singular points of the Dupin cyclide. These homographies have been chosen in order to limit the number of cases to study and to the classical cases described in paragraphs 3.3.2 and 3.3.3, the homography h_3 provides a circle branch.

Theorem 5 Let γ be a Bézier curve with control mass points $(\sigma_0;1)$, $(P_1;\omega_1)$ and $(\sigma_2;1)$ laying on the conic C. Let us introduce

$$\overrightarrow{v_1} = \left(B_0\left(t_1\right)\overrightarrow{P_1\sigma_0} + B_2\left(t_1\right)\overrightarrow{P_1\sigma_2}\right)$$

Let h_2 defined by

$$\begin{array}{rccc} h_2: & [0;1] & \longrightarrow & [0;t_1] \\ & u & \longmapsto & \dfrac{d_0\,t_1\,u}{(1-u)+d_0\,u} \end{array} \tag{17}$$

then $\gamma \circ h_2$ equals a Bézier curve with control mass points $(\sigma_{00};1)$, $(P_{10};\varpi_{10})$ and $(\overrightarrow{m_{20}};0)$ laying on the conic C with

$$\begin{cases} (\sigma_{00};1) = (\sigma_0;1) \\ P_{10} = \mathrm{B}\left\{(\sigma_0;1-t_1)\,;(P_1;t_1\omega_1)\right\} \\ \varpi_{10} = d_0\left(1+(\omega_1-1)\,t_1\right) \\ (\overrightarrow{m_{20}};0) = \left(d_0^2\,\overrightarrow{v_1};0\right) \end{cases} \tag{18}$$

where

$$d_0 = \sqrt{\left|\frac{-1}{\overrightarrow{v_1}\cdot\overrightarrow{e_{o\vee\infty}}}\right|} \tag{19}$$

Proof: By the use of relations of Formula (42) with $d = d_0$ given by Formula (19), $a = 0$, $b = d_0 t_1$ and $c = 1$.

■

The following theorem allows to obtain a branch of the circle.

Theorem 6 : Homographic parameter change

Let γ be a Bézier curve with control mass points $(\sigma_0; 1)$, $(P_1; \omega_1)$ and $(\sigma_2; 1)$ laying on the conic C.

Let us define

$$\left\{ \begin{array}{rcl} \overrightarrow{v_2} & = & B_0\left(t_1\right) \overrightarrow{P_1\sigma_0} + B_2\left(t_1\right) \overrightarrow{P_1\sigma_2} \\ \overrightarrow{v_3} & = & B_0\left(t_2\right) \overrightarrow{P_1\sigma_0} + B_2\left(t_1\right) \overrightarrow{P_2\sigma_2} \end{array} \right.$$

Let h_3 defined by

$$\begin{array}{rccc} h_3 : & [0;1] & \longrightarrow & [t_1;t_2] \\ & u & \longmapsto & \dfrac{t_1\,(1-u) + d_0\,t_2\,u}{(1-u) + d_0\,u} \end{array} \quad (20)$$

then $\gamma \circ h_3$ is the Bézier curve with control mass points $(\overrightarrow{m_{01}}; 0)$, $(Q_{11}; \varpi_{11})$ and $(\overrightarrow{m_{21}}; 0)$ laying on the conic C with

$$\left\{ \begin{array}{rcl} (\overrightarrow{m_{01}}; 0) & = & \left(c_1^2\, \overrightarrow{v_2}; 0\right) \\ (\Omega_{11}; \varpi_{11}) & = & \left(G_{11}; c_1\, d_1\, \left(1+\omega_1\right)\right) \\ (\overrightarrow{m_{21}}; 0) & = & \left(d_1^2\, \overrightarrow{v_3}; 0\right) \end{array} \right. \quad (21)$$

where G_{11} is the mass center of $\left(\sigma_0; \frac{1}{2(1-\omega_1)}\right)$, $\left(P_1; \frac{-\omega_1^2}{1-\omega_1}\right)$ and $\left(\sigma_2; \frac{1}{2(1-\omega_1)}\right)$ and

$$\left\{ \begin{array}{rcl} c_1 & = & \sqrt{\left|\dfrac{-1}{\overrightarrow{v_2} \cdot \overrightarrow{e_{o\vee\infty}}}\right|} \\ d_1 & = & \sqrt{\left|\dfrac{-1}{\overrightarrow{v_3} \cdot \overrightarrow{e_{o\vee\infty}}}\right|} \end{array} \right. \quad (22)$$

In addition Ω_{11} is the center of the circle defined by the Bézier curve.

Proof: By the use of relations of Formula (42) with $c = c_1$ and $d = d_1$ given by Formula (22), $a = t_1$ and $b = d_1 t_2$.

■

In Theorems 5 and 6, the light vectors $\overrightarrow{m_{20}}$ and $\overrightarrow{m_{01}}$ are identical but in Theorem 6, the center weight Ω_{11} is negative. Thus, both Bézier are not located on the same branch of circle. It is possible to restore a positive weight while taking $(-\overrightarrow{m_{01}}; 0)$, $(\Omega_{11}; -\varpi_1)$ and $(-\overrightarrow{m_{21}}; 0)$ as control mass points. We prove now that the point P_{10} belongs to one of both asymptotic lines of the circle :

Proposition 1 :

Let Ω_{11} be the center of the circle built thanks to Theorem 6.

Let P_{10} and $\overrightarrow{m_{20}}$ be two mass points built thanks to Theorem 5.

The vectors $\overrightarrow{m_{20}}$ and $\overrightarrow{P_{10}\Omega_{11}}$ are not vanishing and parallel vectors and verify

$$\overrightarrow{m_{20}} \cdot \overrightarrow{P_{10}\Omega_{11}} = 0 \quad (23)$$

implies the point P_{10} belongs to one of the asymptotic lines of the circle.

Proof: Left to the reader.

■

In Formula (23), the light-like parallel vectors $\overrightarrow{m_{20}}$ and $\overrightarrow{P_{10}\Omega_{11}}$ are perpendicular too. The last case follows.

Theorem 7 Let γ be a Bézier curve with control mass points $(\sigma_0; 1)$, $(P_1; \omega_1)$ and $(\sigma_2; 1)$ laying on the conic C.

Let us define

$$\overrightarrow{v_4} = B_0\left(t_2\right) \overrightarrow{P_1\sigma_0} + B_2\left(t_2\right) \overrightarrow{P_1\sigma_2}$$

Let h_4 be defined by

$$\begin{array}{rccc} h_4 : & [0;1] & \longrightarrow & [t_2;1] \\ & u & \longmapsto & \dfrac{c_2\,t_2\,(1-u) + u}{c_2\,(1-u) + u} \end{array} \quad (24)$$

then $\gamma \circ h_4$ is the Bézier curve with control mass points $(\overrightarrow{m_{20}}; 0)$, $(P_{12}; \varpi_{12})$ and $(\sigma_{22}; 1)$ laying on the conic C with

$$\left\{ \begin{array}{c} (\overrightarrow{m_{02}}; 0) = \left(c_2^2\, \overrightarrow{v_4}; 0\right) \\ P_{12} = \mathrm{B}\left\{\left(P_1; \left(1-t_2\right)\omega_1\right); \left(\sigma_2; t_2\right)\right\} \\ \varpi_{12} = c_2\left(\omega_1 + \left(1-\omega_1\right)t_2\right) \\ (\sigma_{22}; 1) = (\sigma_2; 1) \end{array} \right. \quad (25)$$

where

$$c_2 = \sqrt{\left|\dfrac{-1}{\overrightarrow{v_4} \cdot \overrightarrow{e_{o\vee\infty}}}\right|} \quad (26)$$

Proof : By the use of relations of Formula (42) with $c = c_2$ given by Formula (26), $a = t_2 c_2$ and $b = d = 1$.

■

Adapting the proof of Theorem 1, we could show that the point P_{12} belongs to one of the asymptotic lines of the circle.

The Figure 7 shows an iteration of a subdivision by the use of Theorems 5, 6 and 7. Thus we have on one hand $\overrightarrow{m_{20}} = \overrightarrow{m_{01}}$ and on the other hand $\overrightarrow{m_{21}} = \overrightarrow{m_{02}}$ and $\varpi_{11} < 0$. The Bézier curve defined by the control mass points $(\sigma_0; 1)$, $(P_1; \omega_1)$ and $(\sigma_2; 1)$ is subdivided in three Bézier curves, the control mass points are

- $(\sigma_0; 1)$, $(P_{10}; \varpi_{10})$ and $(\overrightarrow{m_{20}}; 0)$ for the first curve;
- $(\overrightarrow{m_{01}}; 0)$, $(\Omega_{11}; \varpi_{11})$ and $(\overrightarrow{m_{21}}; 0)$ or $\left(\overrightarrow{-m_{01}}; 0\right)$, $(\Omega_{11}; -\varpi_{11})$ and $(-\overrightarrow{m_{21}}; 0)$ for the second curve;
- $(\overrightarrow{m_{02}}; 0)$, $(P_{12}; \varpi_{12})$ and $(\sigma_2; 1)$ for the third curve.

3.3.2 Case where $(\omega_0; \omega_1; \omega_2) \in \{0\} \times \mathbb{R}^{+*} \times \{0\}$

The extrem mass control points are vectors and the intermediate control mass point is a weighted point with weight $\omega_1 > 0$. For $\omega_1 < 0$, multiplying all control mass points by -1 gives the same curve. That is equivalent to take the opposite of extrem vectors and the opposite of the intermediate weight. Thus the curve is a branch of an Euclidean hyperbola with Ω_1 as center. the curve is represented by the Bézier curve γ with control mass points $(\overrightarrow{m_0}; 0)$, $(\Omega_1; \omega_1)$ and $(\overrightarrow{m_2}; 0)$.

Theorem 8 Let γ be a Bézier curve with control mass points $(\overrightarrow{m_0}; 0)$, $(\Omega_1; \omega_1)$ and $(\overrightarrow{m_2}; 0)$ laying on the conic C. Let h_0 defined by

$$h_0 : u \longmapsto \frac{\dfrac{1}{\sqrt{2\omega_1}}\, u}{(1-u) + 2\, \dfrac{1}{\sqrt{2\omega_1}}\, u} \tag{27}$$

then $\gamma \circ h_0$ is a Bézier curve with control mass points $(\overrightarrow{m_{00}}; 0)$, $(P_{10}; \varpi_{10})$ and $(\sigma_{20}; 1)$ laying on the conic C with

$$\left\{ \begin{array}{c} (\overrightarrow{m_{00}}; 0) = (\overrightarrow{m_0}; 0) \\ (P_{10}; \varpi_{10}) = \left(\mathcal{T}_{\frac{1}{\omega_1}\overrightarrow{m_0}}(\Omega_1) ; \sqrt{\dfrac{\omega_1}{2}} \right) \\ (\sigma_{20}; 1) = \left(\mathcal{T}_{\frac{1}{2\,\omega_1}\left(\overrightarrow{m_0}+\overrightarrow{m_2}\right)}(\Omega_1) ; 1 \right) \end{array} \right. \tag{28}$$

Proof: by the use of Formula (51) with $c_0 = 1$ and $b_0 = \frac{1}{\sqrt{2\omega_1}}$.

■

In order to obtain the second part of the subdivision, for symmetric reasons, the use of Formula (57) with $a_1 = \frac{1}{\sqrt{2\omega_1}}$ and $b_1 = 1$ provides :

Theorem 9 : homographic parameter change
Let γ be a Bézier curve with control mass points $(\overrightarrow{m_0}; 0)$, $(\Omega_1; \omega_1)$ and $(\overrightarrow{m_2}; 0)$ laying on the conic C. Let h_1 defined by

$$h_1 : u \longmapsto \frac{\dfrac{1}{\sqrt{2\omega_1}}\,(1-u) + u}{2\, \dfrac{1}{\sqrt{2\omega_1}}\,(1-u) + u} \tag{29}$$

then $\gamma \circ h_1$ is the Bézier curve with control mass points $(\sigma_{01}; 1)$, $(P_{11}; \varpi_{01})$ and $(\overrightarrow{m_{21}}; 0)$ laying on the conic C with

$$\left\{ \begin{array}{c} (\sigma_{01}; 1) = \left(\mathcal{T}_{\frac{1}{2\,\omega_1}\left(\overrightarrow{m_0}+\overrightarrow{m_2}\right)}(\Omega_1) ; 1 \right) \\ (P_{11}; \varpi_{11}) = \left(\mathcal{T}_{\frac{1}{\omega_1}\overrightarrow{m_2}}(\Omega_1) ; \sqrt{\dfrac{\omega_1}{2}} \right) \\ (\overrightarrow{m_{21}}; 0) = (\overrightarrow{m_2}; 0) \end{array} \right. \tag{30}$$

The Figure 8 shows an iteration of the subdivision based on the Theorems 8 and 9, thus we have $\sigma_{20} = \sigma_{01}$.

3.3.3 Case where $(\omega_0; \omega_1; \omega_2) \in \{0\} \times \mathbb{R}^{+*} \times \{1\}$

If $\omega_1 > 0$, from Corollary 1 with $c = -1$ and $b = 1$ we obtain the arc complement. Thus, in the following, ω_1 is taken strictly positive, the first control point is a vector and the two others are weighted points. The curve is a hyperbola arc.

Theorem 10 : homographic parameter change
Let γ be a Bézier curve with control mass points $(\overrightarrow{m_0}; 0)$, $(P_1; \omega_1)$ and $(\sigma_2; 1)$ laying on the conic C. Let h_0 defined by

$$h_0 : u \longmapsto \frac{\dfrac{1}{\sqrt{1+2\omega_1}}\, u}{(1-u) + 2\, \dfrac{1}{\sqrt{1+2\omega_1}}\, u} \tag{31}$$

then $\gamma \circ h_0$ is a Bézier curve control mass points $(\overrightarrow{m_{00}}; 0)$, $(P_{10}; \varpi_{10})$ and $(\sigma_{20}; 1)$ laying on the

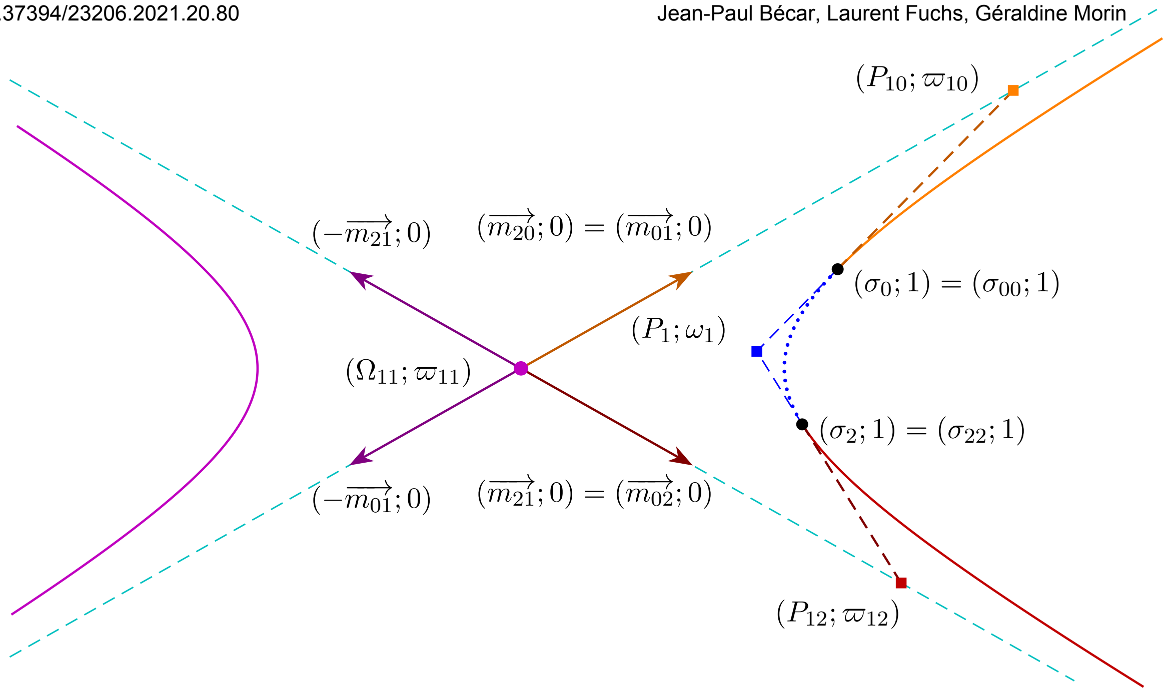


Figure 7: An iteration of an iterative building of a conic arc γ on Λ^4, with σ_0 et σ_2 as vertices and $(\sigma_0 P_1)$ and $(\sigma_2 P_1)$ as tangent lines with $(P_1;\omega_1)$ and $\omega_1 < -1$. Moreover $\varpi_{11} < 0$.

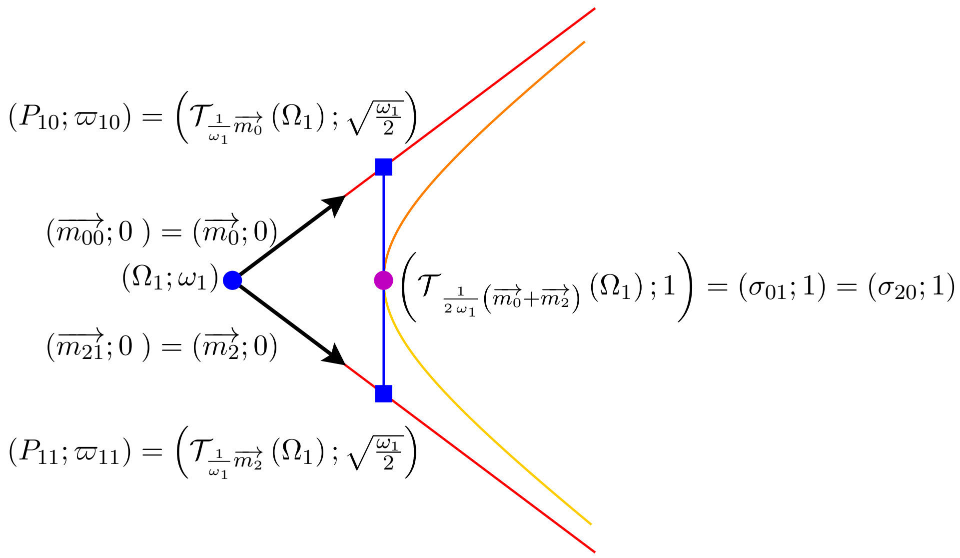


Figure 8: One iteration of an iterative subdivision on the spheres space of a circle branch which looks like a hyperbola.

conic C with :

$$\left\{\begin{array}{rcl} (\overrightarrow{m_{00}};0) & = & (\overrightarrow{m_0};0) \\ P_{10} & = & \mathcal{T}_{\frac{1}{\omega_1}\overrightarrow{m_0}}(P_1) \\ \varpi_{10} & = & \dfrac{\omega_1}{\sqrt{1+2\omega_1}} \\ (\sigma_{20};1) & = & \left(\mathcal{T}_{\frac{1}{2\,\omega_1}\overrightarrow{m_0}}(G_3)\,;1\right) \end{array}\right. \qquad (32)$$

where $G_3 = \mathrm{B}\left\{(P_1;2\,\omega_1)\,;(\sigma_2;1)\right\}$.

Proof: by the use of Formula (51) with $c_0 = 1$ and $b_0 = \frac{1}{\sqrt{1+2\omega_1}}$.

■

Notice that we recover the recurrence formula

$$\varpi_1 = \frac{\omega_1}{\sqrt{1+2\omega_1}}$$

given in [9].

Theorem 11 : homographic parameter change

Let γ be a Bézier curve with control mass points $(\overrightarrow{m_0};0)$, $(P_1;\omega_1)$ and $(\sigma_2;1)$ laying on the conic C. Let h_1 defined by

$$h_1 : u \longmapsto \frac{\dfrac{1}{\sqrt{1+2\omega_1}}\,(1-u)\,+u}{2\,\dfrac{1}{\sqrt{1+2\omega_1}}\,(1-u)+u} \tag{33}$$

then $\gamma \circ h_1$ is a Bézier curve with control mass points $(\sigma_{01};1)$, $(P_{11};\varpi_{11})$ and $(\sigma_{21};1)$ laying on the conic C with

$$\left\{\begin{array}{c} (\sigma_{01};1) = \left(\mathcal{T}_{\frac{1}{1+2\,\omega_1}\overrightarrow{m_0}}(G_3)\,;1\right) \\ (P_{11};\varpi_{11}) = \left(G_4;\dfrac{1+\omega_1}{\sqrt{1+2\omega_1}}\right) \\ (\sigma_{21};1) = (\sigma_2;1) \end{array}\right. \tag{34}$$

where $G_3 = \mathrm{B}\left\{(P_1;2\,\omega_1)\,(\sigma_2;1)\right\}$ and $G_4 = \mathrm{B}\left\{(P_1;\omega_1)\,(\sigma_2;1)\right\}$.

Proof: By the use of relations of Formula (57) with $a_1 = \frac{1}{\sqrt{1+2\omega_1}}$ and $b_1 = 1$.

■

The Figure 9 shows an iteration of the subdivision based on the theorems 10 and 11 and we have $\sigma_{20} = \sigma_{01}$.

3.3.4 Case where $(\omega_0;\omega_1;\omega_2) = (1;0;-1)$

The curve is a semi-circle which looks like an Euclidean semi-hyperbole. The endpoints of the Bézier curve are the weighted points $(\sigma_0;1)$ and $(\sigma_2;-1)$, the intermediate control mass point is the vector $\left(\overrightarrow{P_1};0\right)$ and we have

$$\left\{\begin{array}{rcl} 4\,\overrightarrow{P_1}^2 & = & -\overrightarrow{\sigma_2\sigma_0}^2 \\ \overrightarrow{P_1}\,\boldsymbol{.}\,\overrightarrow{\sigma_2\sigma_0} & = & 0 \end{array}\right.$$

Theorem 12 Let γ be a Bézier curve with control mass points $(\sigma_0;1)$, $\left(\overrightarrow{P_1};0\right)$ and $(\sigma_2;-1)$ laying on the conic C. Let b_0 defined by:

$$b_0 = \sqrt{\left|\frac{-1}{\left(\overrightarrow{\sigma_2\sigma_0}+2\,\overrightarrow{P_1}\right)\boldsymbol{.}\,\overrightarrow{e_{o\vee\infty}}}\right|} \tag{35}$$

Let h_0 defined by

$$h_0 : u \longmapsto \frac{b_0\,u}{(1-u)+2\,b_0\,u} \tag{36}$$

then $\gamma \circ h_0$ is a Bézier curve with control mass points $(\sigma_{00};1)$, $(P_{10};b_0)$ and $(\overrightarrow{m_{20}};0)$ laying on the conic C with

$$\left\{\begin{array}{rcl} (\sigma_{00};1) & = & (\sigma_0;1) \\ (P_{10};\varpi_1) & = & \left(\mathcal{T}_{\overrightarrow{P_1}}(\sigma_0)\,;b_0\right) \\ (\overrightarrow{m_{20}};0) & = & \left(b_0^2\left(\overrightarrow{\sigma_2\sigma_0}+2\,\overrightarrow{P_1}\right);0\right) \end{array}\right. \tag{37}$$

Proof: by the use of Formula (51) with b_0 and $c_0 = 1$:

■

In order to obtain the second part of the subdivision, we use the following theorem:

Theorem 13 Let γ be a Bézier curve with control mass points $(\sigma_0;1)$, $\left(\overrightarrow{P_1};0\right)$ and $(\sigma_2;-1)$ laying on the conic C. Let a_1 be defined by

$$a_1 = \sqrt{\left|\frac{-1}{\left(\overrightarrow{\sigma_2\sigma_0}+2\,\overrightarrow{P_1}\right)\boldsymbol{.}\,\overrightarrow{e_{o\vee\infty}}}\right|}$$

Let h_1 defined by

$$h_1 : u \longmapsto \frac{a_1\,(1-u)+u}{2\,a_1\,(1-u)+u} \tag{38}$$

then $\gamma \circ h_1$ is a Bézier curve with control mass points $(\overrightarrow{m_{01}};0)$, $(P_{11};-a_1)$ et $(\sigma_{21};-1)$, laying on the conic C with

$$\left\{\begin{array}{c} (\overrightarrow{m_{01}};0) = \left(a_0^2\left(\overrightarrow{\sigma_2\sigma_0}+2\,\overrightarrow{P_1}\right);0\right) \\ (P_{11};\varpi_{11}) = \left(\mathcal{T}_{-\overrightarrow{P_1}}(\sigma_2)\,;-a_1\right) \\ (\sigma_{21};-1) = (\sigma_2;-1) \end{array}\right. \tag{39}$$

and the vector $\overrightarrow{m_{01}}$ represents a point of the Alexandrov compactness of $\mathcal{E}_3$, its first component is pseudo-unitary or $\overrightarrow{m_{01}}$ is $\overrightarrow{e_\infty}$ or $-\overrightarrow{e_\infty}$.

Proof: by the use of Formula (57) with a_1 and $b_1 = 1$.

■

The Figure 10 shows an iteration of the subdivision based on the Theorems 12 and 13. Then, to obtain a standard form of the same conic arc, the control mass points $(\overrightarrow{m_{01}};0)$, $(P_{11};-a_1)$ and $(\sigma_{21};-1)$ are replaced by $(-\overrightarrow{m_{01}};0)$, $(P_{11};a_1)$ and $(\sigma_{21};1)$.

In the next section, we give a synthesis and the links between the previous theorems.

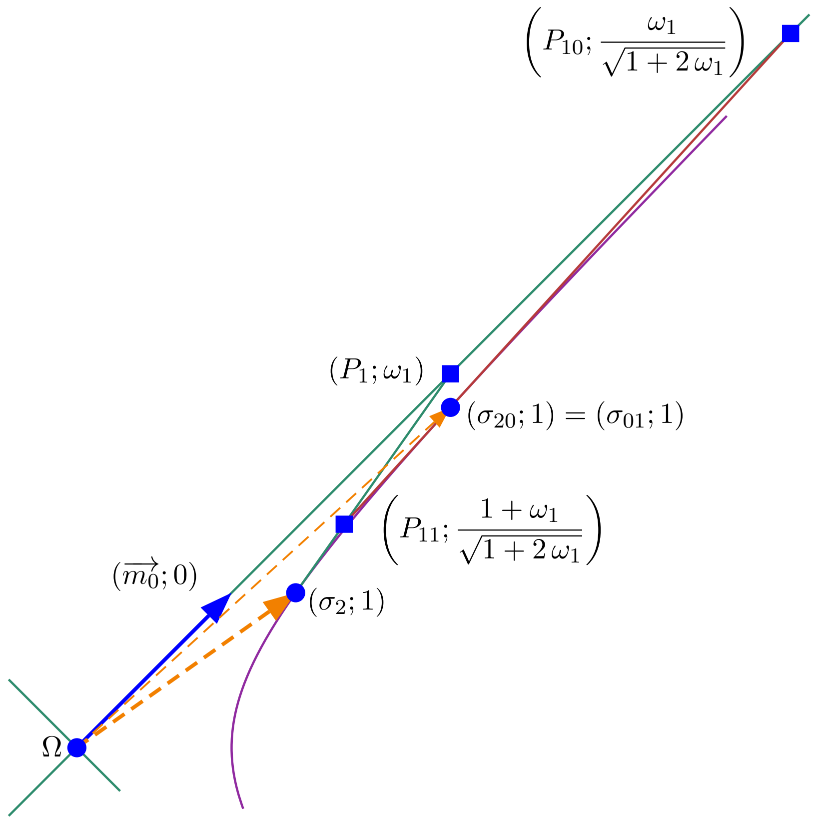


Figure 9: One iteration of an iterative construction of a circle arc from three mass points $(\overrightarrow{m_0}; 0)$, $(P_1; \omega_1)$ and $(\sigma_2; 1)$.

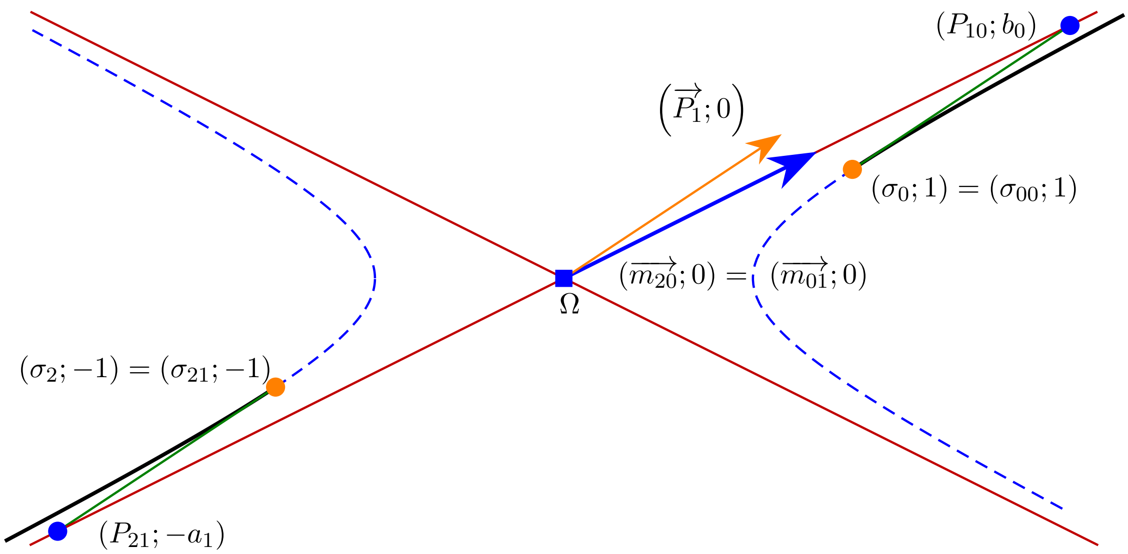


Figure 10: One iteration of a non-connected semi-circle defined by a Bézier curve of control mass points $(\sigma_0; 1)$, $\left(\overrightarrow{P_1}; 0\right)$ and $(\sigma_2; -1)$.

3.4 Synthesis

First, the subdivision of a parabola, isometric to a line, is on consideration. The Figure 11 shows a graph which synthesizes the links between the theorems.

Second, the subdivision of a non-connected circle (a hyperbola with an Euclidean point of view) is on consideration. The Figure 12 shows a graph which synthesizes the links between the theorems. The arrows in small discontinuous point illustrate multiplication of mass points by -1 whereas the arrows in broken lines highlight the swap of endpoints.

3.5 First example with a Horned Dupin cyclide

In this example, we are modeling a horned Dupin cyclide defined by :

- both extremal spheres S_0 and S_2 with centers $O_0\,(0;4;0)$ and $O_2\,(-5;0;0)$ and radii $\rho_0=1$ and $\rho_2=4$ respectively ;
- the intermediate sphere S_1 with center $O_1\left(-\frac{5}{2};2\sqrt{3};0\right)$ and radius $\rho_1=\frac{5}{2}$.

On Figure 13, the spheres S_0 and S_2 get a "bozo" texture while the sphere S_1 gets a red pinky texture with black strips.

The spheres S_0, S_1 and S_2 are represented on Λ^4 by the three points

$$\left\{\begin{array}{c}\sigma_0\left(1,0,4,0,\frac{15}{2}\right),\\ \sigma_1\left(\frac{2}{5},-1,\frac{4\sqrt{3}}{5},0,\frac{12}{5}\right)\\ \sigma_2\left(\frac{1}{4},-\frac{5}{4},0,0,\frac{9}{8}\right)\end{array}\right.$$

The circle arc modeling the Dupin cyclide bit is defined by the Bézier curve with control mass points. $(\sigma_0;1)$, $(P_1;\omega_1)$ and $(\sigma_2;1)$ with :

$$\left\{\begin{array}{c}P_1\left(\frac{1}{4};-\frac{5}{4};1;0;\frac{9}{8}\right)\\ \omega_1=\sqrt{2}\end{array}\right.$$

The derivative spheres from σ_0 and σ_2 are given respectively by

$$\left\{\begin{array}{c}\overset{\bullet}{\sigma_0}\left(-\frac{3}{4},-\frac{5}{4},-3,0,-\frac{51}{8}\right)\\ \overset{\bullet}{\sigma_2}\,(0;0;-1;0;0)\end{array}\right.$$

thus we obtain the extrem characteristic circles of the Dupin cyclide bit.

The roots of $B_0\,(t)+\omega_1\,B_1\,(t)+B_2\,(t)$ are

$$\left\{\begin{array}{l}t_1=1-\frac{\sqrt{2}}{2}\simeq 0,293\\ t_2=\frac{\sqrt{2}}{2}\simeq 0,707\end{array}\right.$$

The Theorems 5, 6 and 7 allow to start the subdivision of the complement of the previous arc that is the Bézier curve with control mass points $(\sigma_0;1)$, $(P_1;-\omega_1)$ and $(\sigma_2;1)$. It yields,

- Theorem 5

  We obtain $\sigma_{00}=\sigma_0$, $d_0=\frac{2\sqrt{6}}{3}\simeq 1,633$ where

$$\left\{\begin{array}{l}P_{10}\begin{pmatrix}\frac{3\sqrt{2}+4}{4}\\ \frac{5\sqrt{2}}{4}\\ 4+3\sqrt{2}\\ 0\\ \frac{51\sqrt{2}+60}{8}\end{pmatrix}\\ \omega_{10}=\frac{-2\sqrt{3}+2\sqrt{6}}{3}\end{array}\right.$$

  and

$$(\overrightarrow{m_{20}};0)=\left(\left(1;\frac{5}{3};8\frac{\sqrt{2}}{3};0;\frac{17}{2}\right);0\right)$$

  and the Dupin cyclide bit between the characteristic circle in magenta color and the singular point in red color is modeled by the Bézier curve with control mass points $(\overrightarrow{m_{20}};0)$, $(P_{10};\omega_{10})$ and $(\sigma_{00};1)$. The following subdivisions will carried out by the use of Theorems 10 and 11.

- Theorem 6

  We have $c_1=d_0$, $(\overrightarrow{m_{20}};0)=(\overrightarrow{m_{01}};0)$ and

$$d_1=\frac{2\sqrt{6}+4\sqrt{3}}{3}\simeq 3,942$$

  and

$$(\Omega_{11};\omega_{11})=\left(\left(-\frac{1}{8};-\frac{15}{8};0;0;-\frac{33}{16}\right);-\frac{8}{3}\right)$$

  and

$$(\overrightarrow{m_{21}};0)=\left(\left(1;\frac{5}{3};-8\frac{\sqrt{2}}{3};0;\frac{17}{2}\right);0\right)$$

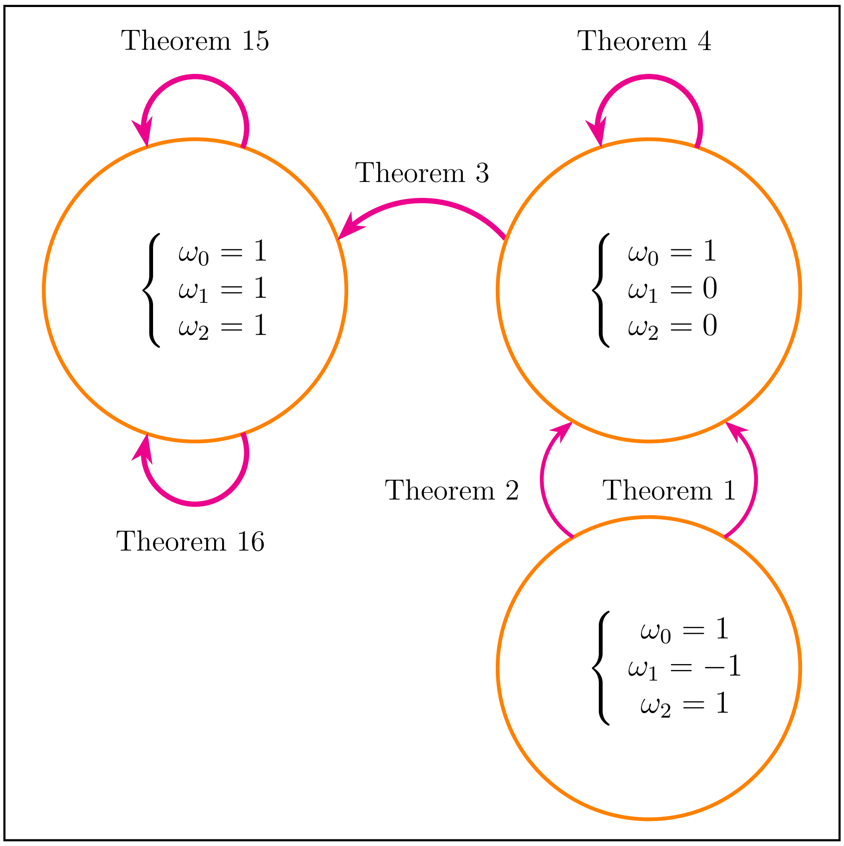


Figure 11: Graph which synthesizes the subdivisions of a parabola isometric to a line.

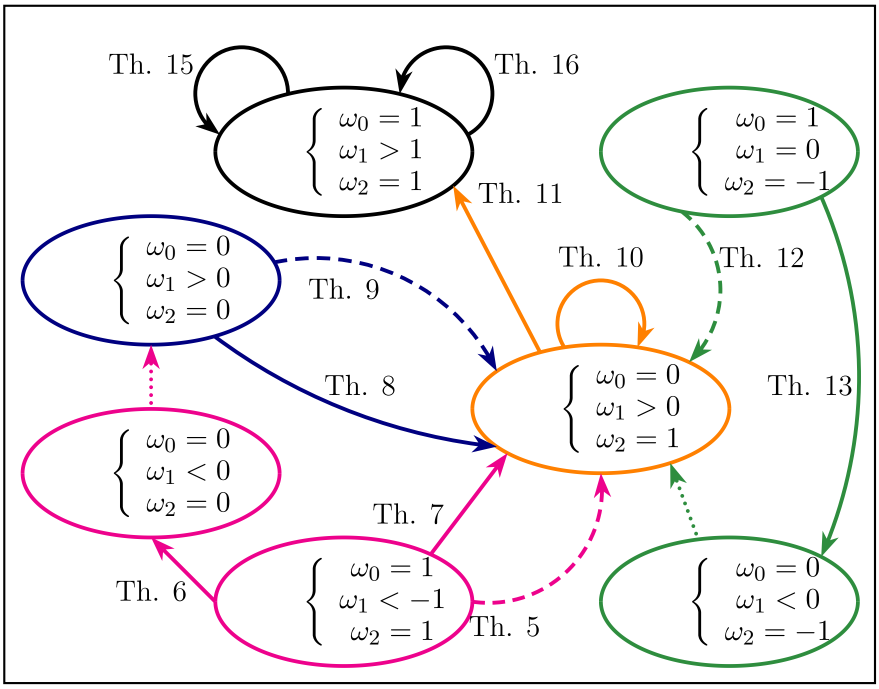


Figure 12: Graph which synthesizes the subdivisions of a non-connected circle (a hyperbola with an Euclidean point of view).

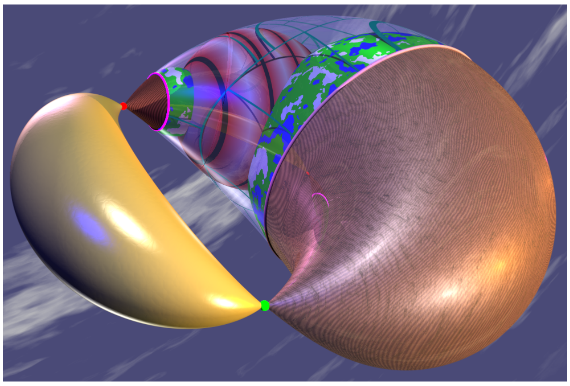


Figure 13: One iteration of an iterative making of a circle arc γ getting two branches on Λ^4 with σ_0 and σ_2 as edges and the straight lines $(\sigma_0 P_1)$ and $(\sigma_2 P_1)$ with $(P_1;\omega_1)$ and $\omega_1 < -1$ as tangents.

The Dupin cyclide bit between both singular points is modeled by the B curve with control mass points $(\overrightarrow{m_{01}};0)$, $(\Omega_{11};\omega_{11})$ and $(\overrightarrow{m_{21}};0)$. The following subdivisions will carried out by the use of Theorems 8 and 9 taking $(-\overrightarrow{m_{01}};0)$, $(\Omega_{11};-\omega_{11})$ and $(-\overrightarrow{m_{21}};0)$ control mass points such that the circle center gets a positive weight.

- Theorem 7

  We have $\sigma_{22}=\sigma_2$, $c_2=d_1$ and $(\overrightarrow{m_{02}};0)=(\overrightarrow{m_{21}};0)$ where

$$(P_{12};\omega_{12})=\left(\left(\frac{1}{4};-\frac{5}{4};-\sqrt{2};0;\frac{9}{8}\right);\frac{2\sqrt{3}}{3}\right)$$

  and the Dupin cyclide bit between the singular point in green color and the characteristic circle in pink color is modeled by the Bézier curve with control mass points $(\overrightarrow{m_{21}};0)$, $(P_{12};\omega_{12})$ and $(\sigma_{22};1)$. The following subdivisions will carried out by the use of Theorems 10 and 11.

The Figure 13 shows an iteration of subdivision based on Theorems 5, 6 and 7. The piece in "gold" texture matches with a circle branch. both other circle arcs offers to see the dark and light wooden parts.

3.6 Second example with a Horned Dupin cyclide

We choose a horned Dupin cyclide defined by a sphere and two singular points. This Dupin cyclide is represented by a circle C_1 containing two branches in a time-like 2-plane $\mathcal{P}$. That allows the use of Theorems 8 and 11.

The Dupin cyclide is defined by two singulars points M_0 and M_2 and a sphere S_0. On Λ^4, the non connected circle equals the intersection between Λ^4 and the plane defined by the light-like vectors $(\overrightarrow{m_0};0)$ and $(\overrightarrow{m_2};0)$ representing the points M_0 and M_2 and the point σ_0 representing the sphere S_0. The 2-plane $\mathcal{P}$ is defined by σ_0, $\overrightarrow{m_0}$ and $\overrightarrow{m_2}$. The point Ω_1 is the orthogonal projection of O_5 on the plane $\mathcal{P}$.

The first step of the iterative subdivision thanks to Theorem 8, consists in the transformation of the Bézier curve with control mass points $(\overrightarrow{m_0};0)$, $(\Omega_1;\omega_1)$ and $(\overrightarrow{m_2};0)$ into two curves with control mass points

$$\begin{cases} (\overrightarrow{m_{00}};0) \\ (P_{10};\omega_{10})=\left(\mathcal{T}_{\frac{1}{\omega_1}\overrightarrow{m_0}}(\Omega_1);\sqrt{\frac{\omega_1}{2}}\right) \\ (\sigma_{20};1) \end{cases}$$

for the first one and with control mass points

$$\begin{cases} (\sigma_{01};1) \\ (P_{11};\omega_{11})=\left(\mathcal{T}_{\frac{1}{\omega_1}\overrightarrow{m_2}}(\Omega_1);\sqrt{\frac{\omega_1}{2}}\right) \\ (\overrightarrow{m_{21}};0) \end{cases}$$

for the second one. Thus we obtain in $\mathcal{E}_3$ the red circle in Figure 14.

For symmetry reasons, only the iterations for $(\overrightarrow{m_{00}};0)$, $(P_{10};\omega_{10})$ et $(\sigma_{20};1)$ are detailed. We thus obtained two Bézier curves with control mass points $(\overrightarrow{m_{000}};0)$, $(P_{100};\omega_{100})$ and $(\sigma_{200};1)$ on one hand and $(\sigma_{001};1)$, $(P_{101};\omega_{101})$ and $(\sigma_{201};1)$ on the other hand $\sigma_{20}=\sigma_{201}$. We obtain the blue circles on Figure 14.

The Figure 14 shows three iterations based on Theorems 15 and 16 and the Theorems 8 to 10. The magenta circles are obtained after three iterations.

4 Subdivision of a Dupin cyclide patch

In this section, the algorithms given in [10] are simplified using the representation of the brother circles on Λ^4 [1]. These curves are modeled using Bézier curves. The same method is applied to all Dupin cyclides (non-degenerate, circular cone, circular cylinder or torus), just the number of singular point(s) is taking into consideration via the previous sections. Let σ_0 and τ_0 be two representations of spheres which define the Dupin cyclide, if they do not belong to the same circle on Λ^4, then the light-like vector $\overrightarrow{\sigma_0\tau_0}$ defines the Dupin

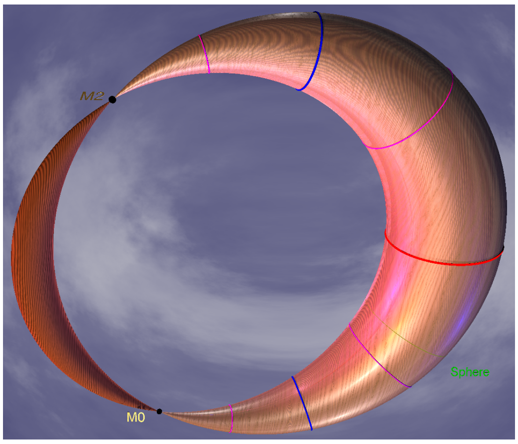


Figure 14: 3 iterations of subdivision of a Bézier curves defined by control mass points laying on the spheres space Λ^4 giving in $\mathcal{E}_3$ characteristic circles of a Dupin cyclide with a non null crescent.

cyclide point. Figure 15 shows a subdivision of a Dupin cyclide patch.

In Figure 16, the patch of vertices $P_{00}P_{20}P_{22}P_{02}$ is replaced by the four patches of vertices

- P_{00}, P_{010}, P_{0101} and P_{001},
- P_{010}, P_{20}, P_{201} and P_{0101},
- P_{0101}, P_{201}, P_{22} and P_{012},
- P_{012}, P_{02}, P_{001} and P_{0101}.

The original spheres S_0 and S_2 are defined by the points σ_0 and σ_2 whereas the sphere S_{01} is defined by the construction of the point σ_{01}.

Compared to the Figure 16, the Figure 17 shows the sphere S_{01} defined by the point τ_{01}.

Moreover, using the sections 3.3 and 3.2.2, a degenerate patch (i.e. a triangle, two vertices are confused) of a Dupin cyclide can be subdivides, Figure 18. In this example, the first computed sphere, defined by the point σ_{01} of the Figure 9, is in blue. The degenerate patch is replaced by a non-degenerate patch and a degenerate patch. The pink sphere defines points on the non-degenerate patch whereas the magenta sphere defines points on the new degenerate patch.

5 Conclusion and future works

In this paper and in [1], some methods to subdivide rational quadratic Bézier curves with a mass points representation have been presented. The rational quadratic Bézier curves represent arcs of conics that are ellipse, parabola or hyperbola in the usual Euclidean plane and patches of Dupin cyclides in the Minkowski-Lorentz space. Moreover, examples of subdivision of Dupin cyclides patches have been given. The same methods have been applied to subdivide 3D degenerated triangles on a horned or a spindle Dupin cyclide.

On a ring Dupin cyclides, there are three kinds of circle: meridian, parallel and Yvon-Villarceau circles. Next investigation will point out methods to subdivide 3D triangles on a ring Dupin cyclides bounded by three of the aforementioned circles. SA mix of 3D triangles and rectangular patches along curvature circles of Dupin cyclides is expected.

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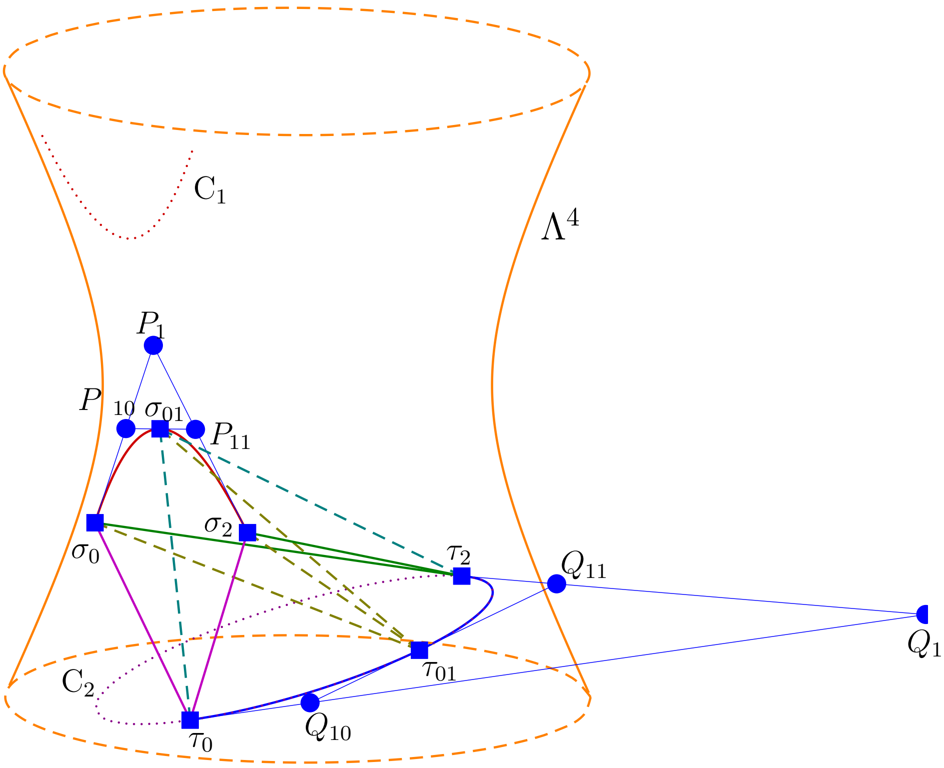

Figure 15: One iteration of a subdivision of a Dupin cyclide patch. Each Bézier curves represents the spheres whose Dupin cyclide is the envelope.

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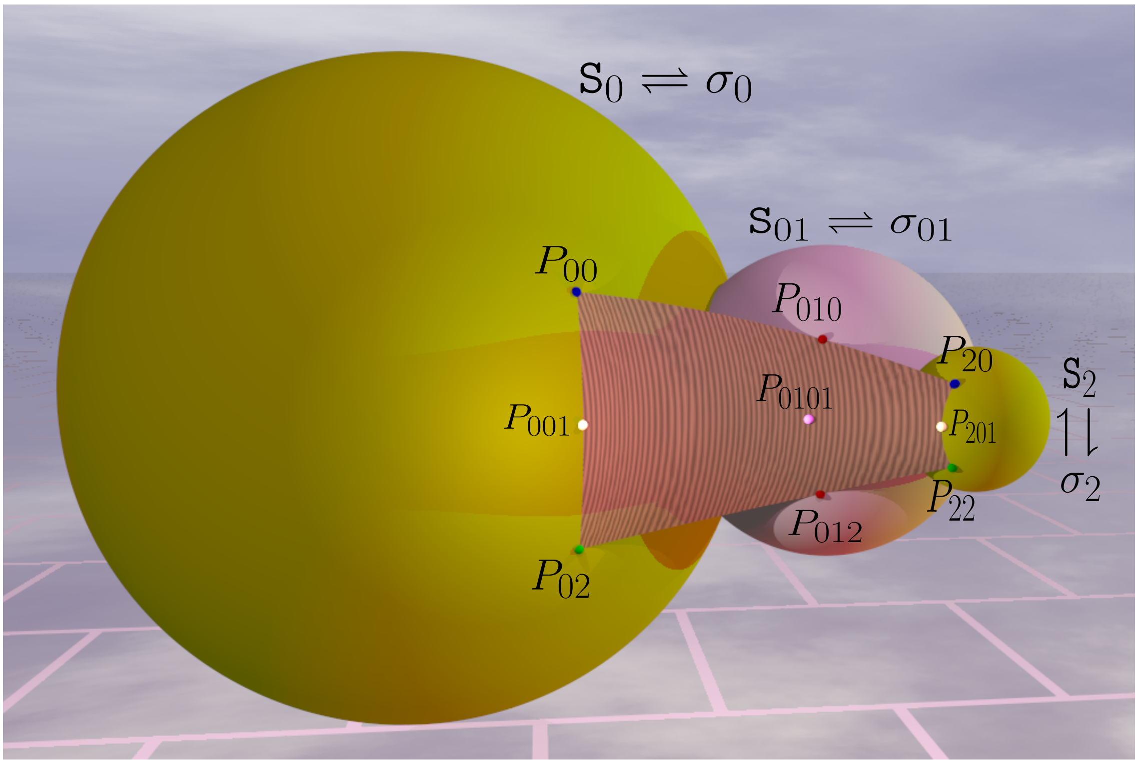

Figure 16: Subdivision of a Dupin cyclide patch: the patch of vertices $P_{00}P_{20}P_{22}P_{02}$ is replaced by the four patches of vertices $P_{00}P_{010}P_{0101}P_{001}$, $P_{010}P_{20}P_{201}P_{0101}$, $P_{0101}P_{201}P_{22}P_{012}$ and $P_{012}P_{02}P_{001}P_{0101}$.

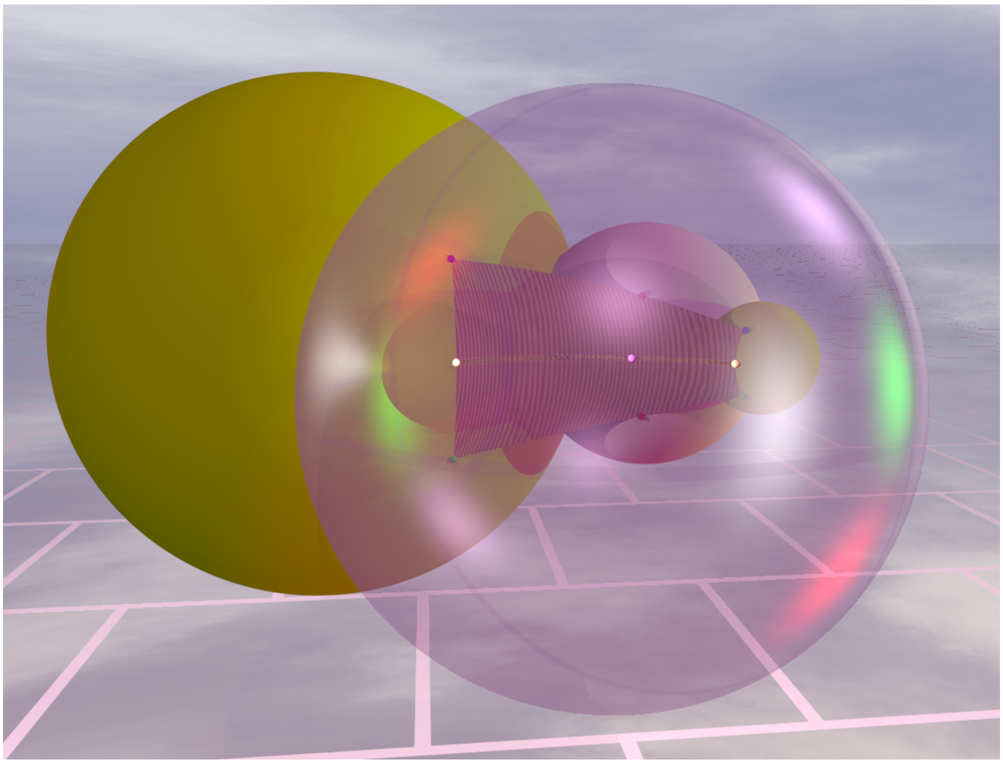

Figure 17: Subdivision of a Dupin cyclide patch: visualization of the spheres defined by σ_0, σ_{01}, σ_2 and τ_{01} (in transparency).

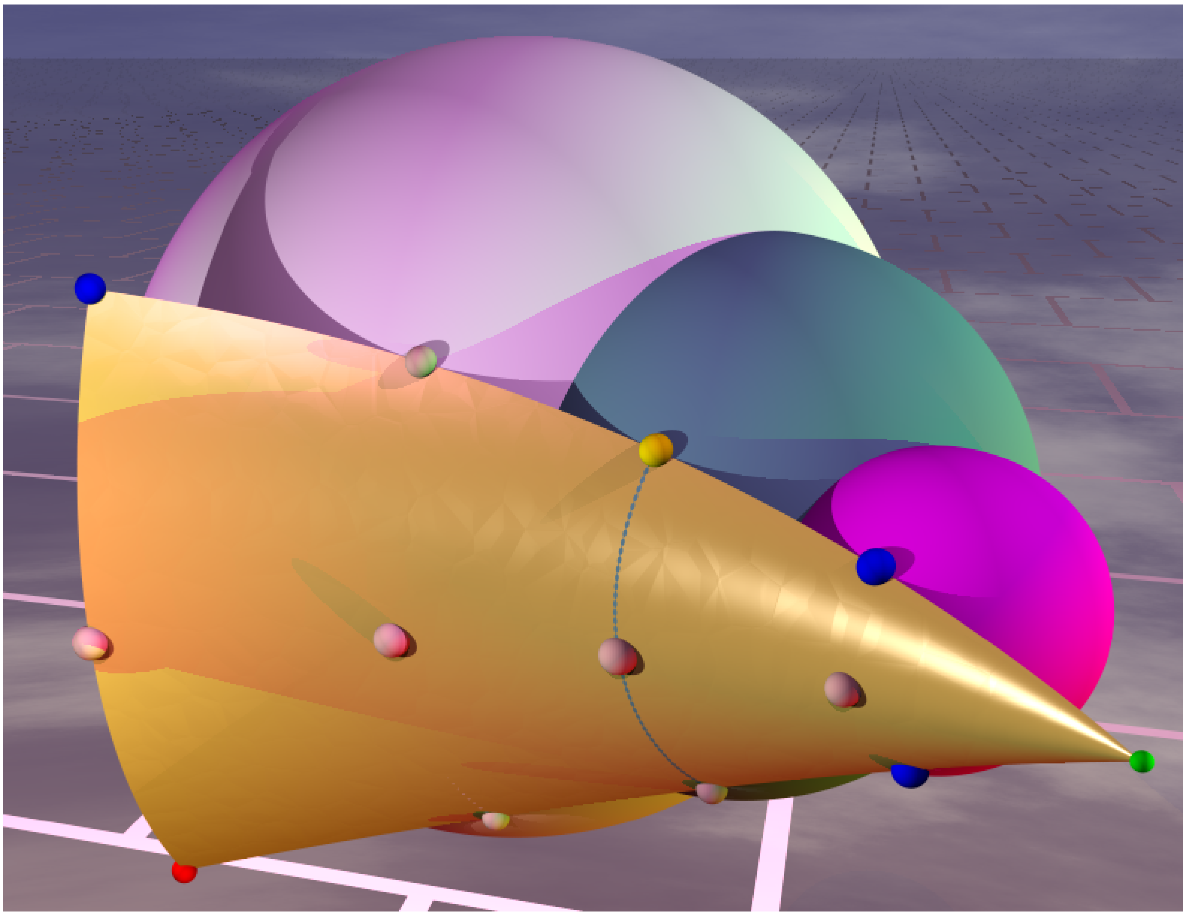

Figure 18: Subdivision of a 3D triangle as a degenerate patch of a horned Dupin cyclide. This 3D triangle leads to a 3D triangle and a non-degenerate rectangular patch.

A Kinds of Dupin cyclides

The Table 1 shows the kinds of circles, for the Minkowski-Lorentz quadratic form, witch represent Dupin cyclides. A singular point of a Dupin cyclide is a light-like vector in the Minkowski-Lorentz space.

B Recall of Theorems in [1]

B.1 Homographic Parameter Change

The Bézier curve and the Bézier curve obtained by this homographic parameter change model two different arcs of the same given conic.

Theorem 14 : Homographic Parameter Change

Let γ be a Bézier curve with control mass points $(P_0;\omega_0)$, $(P_1;\omega_1)$ and $(P_2;\omega_2)$ laying on the conic C. Let a, b, c and d be four real numbers satisfying

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \tag{40}$$

Let h be defined by

$$\begin{array}{rccc} h: & \overline{\mathbb{R}} & \longrightarrow & \overline{\mathbb{R}} \\ & u & \longmapsto & \dfrac{a\,(1-u)+b\,u}{c\,(1-u)+d\,u} \end{array} \tag{41}$$

then $\gamma \circ h$ is a Bézier curve of control mass points $(Q_0;\varpi_0)$, $(Q_1;\varpi_1)$ and $(Q_2;\varpi_2)$ with

$$\left\{\begin{array}{rc} (Q_0;\varpi_0) & = \\ (c-a)^2 \odot (P_0;\omega_0) & \oplus \\ 2\,a\,(c-a) \odot (P_1;\omega_1) & \oplus \\ a^2 \odot (P_2;\omega_2) & \\ (Q_1;\varpi_1) & = \\ (c-a)\,(d-b) \odot (P_0;\omega_0) & \oplus \\ (b\,c-2\,a\,b+a\,d) \odot (P_1;\omega_1) & \oplus \\ a\,b \odot (P_2;\omega_2) & \\ (Q_2;\varpi_2) & = \\ (d-b)^2 \odot (P_0;\omega_0) & \oplus \\ 2\,b\,(d-b) \odot (P_1;\omega_1) & \oplus \\ b^2 \odot (P_2;\omega_2) & \end{array}\right. \tag{42}$$

Proof: see [11, 12]. ■

The following corollary of Theorem 14 offers to keep the endpoints.

Corollary 1 : Homographic Parameter Change with 0 and 1 unmodified.

Let γ be a Bézier curve with mass control points $(P_0;\omega_0)$, $(P_1;\omega_1)$ and $(P_2;\omega_2)$ laying on the conic C. Let b and c be two non-zero numbers. Let h be defined by :

$$\begin{array}{rccc} h: & \overline{\mathbb{R}} & \longrightarrow & \overline{\mathbb{R}} \\ & u & \longmapsto & \dfrac{b\,u}{c\,(1-u)+b\,u} \end{array} \tag{43}$$

then $\gamma \circ h$ is a Bézier curve with mass control points $(Q_0;\varpi_0)$, $(Q_1;\varpi_1)$ and $(Q_2;\varpi_2)$ on the same conic C with

$$\left\{\begin{array}{rcl} (Q_0;\varpi_0) & = & c^2 \odot (P_0;\omega_0) \\ (Q_1;\varpi_1) & = & b\,c \odot (P_1;\omega_1) \\ (Q_2;\varpi_2) & = & b^2 \odot (P_2;\omega_2) \end{array}\right. \tag{44}$$

Table 1: Kinds of Dupin cyclides and their representations in the Minkowski-Lorentz space.

Name of Dupin cyclide	Number of singular point(s)	Lorentz property	Euclidean point of view
Ring	0	Two circles	Two ellipses
Horned	2	Two circles	An ellipse and a hyperbola
Spindle			
One-singularity spindle	1	A circle and a parabola	An ellipse and a parabola
Singly horned		a parabola isometric to a line	

Proof: see [11].

The denominator of a rational quadratic Bézier curve defined by the mass points $(P_0;\omega_0)$, $(P_1;\omega_1)$ and $(P_2;\omega_2)$ is

$$(\omega_0 - 2\,\omega_1 + \omega_2)\,t^2 + 2\,(\omega_1 - \omega_0)\,t + \omega_0 \qquad (45)$$

and the sign of the discriminant of this polynomial is

$$\omega_1^2 - \omega_2\,\omega_0 \qquad (46)$$

B.2 Subdivision

B.2.1 Homography h_0

We have

$$h_0\,(u) = \frac{a_0\,(1-u) + b_0\,u}{c_0\,(1-u) + d_0\,u} \qquad (47)$$

and we have to solve

$$\left\{ \begin{array}{rcl} h_0\,(0) & = & 0 \\ h_0\,(1) & = & \frac{1}{2} \end{array} \right. \qquad (48)$$

which leads to

$$\left\{ \begin{array}{rcl} a_0 & = & 0 \\ d_0 & = & 2\,b_0 \end{array} \right. \qquad (49)$$

and the homography becomes

$$h_0\,(u) = \frac{b_0\,u}{c_0\,(1-u) + 2\,b_0\,u} \qquad (50)$$

with $(b_0, c_0) \in (\mathbb{R}^+)^2$.

Let $(P_0;\omega_0)$, $(P_1;\omega_1)$ and $(P_2;\omega_2)$ be the control mass points of the Bézier curve γ. Using the Theorem 14, the control mass points of the Bézier curve $\gamma \circ h_0$ are $(Q_0;\varpi_0)$, $(Q_1;\varpi_1)$ and $(Q_2;\varpi_2)$ with

$$\left\{ \begin{array}{rcl} (Q_0;\varpi_0) & = & c_0^2 \odot (P_0;\omega_0) \\ (Q_1;\varpi_1) & = & c_0\,b_0 \odot (P_0;\omega_0) \\ & \oplus & b_0\,c_0 \odot (P_1;\omega_1) \\ (Q_2;\varpi_2) & = & b_0^2 \odot (P_0;\omega_0) \\ & \oplus & 2\,b_0^2 \odot (P_1;\omega_1) \\ & \oplus & b_0^2 \odot (P_2;\omega_2) \end{array} \right. \qquad (51)$$

Since we want that the first control mass point of the two curves γ and $\gamma \circ h_0$ is the same, we have $c_0 = 1$. If $\omega_0 + 2\,\omega_1 + \omega_2 \neq 0$, the last control mass point is a weighted point, in order to have $\varpi_2 = 1$, we choose

$$b_0 = \frac{1}{\sqrt{\omega_0 + 2\,\omega_1 + \omega_2}} \qquad (52)$$

else the computation of b_0 depends on the vector $\overrightarrow{Q_2}$: either $\overrightarrow{Q_2}$ is $\overrightarrow{e_\infty}$ or its first component equals 1.

B.2.2 Homography h_1

In the same way, we have

$$h_1\,(u) = \frac{a_1\,(1-u) + b_1\,u}{c_1\,(1-u) + d_1\,u} \qquad (53)$$

and we have to solve

$$\left\{\begin{array}{rcl} h_1(0) &=& \dfrac{1}{2} \\ h_1(1) &=& 1 \end{array}\right. \quad (54)$$

which leads to

$$\left\{\begin{array}{rcl} c_1 &=& 2\,a_1 \\ d_1 &=& b_1 \end{array}\right. \quad (55)$$

and the homography becomes

$$h_1(u) = \frac{a_1\,(1-u) + b_1\,u}{2\,a_1\,(1-u) + b_1\,u} \quad (56)$$

with $(a_1, b_1) \in (\mathbb{R}^+)^2$.

Let $(P_0;\omega_0)$, $(P_1;\omega_1)$ and $(P_2;\omega_2)$ be the control mass points of the Bézier curve γ. Using the Theorem 14, the control mass points of the Bézier curve $\gamma \circ h_1$ are $(Q_0;\varpi_0)$, $(Q_1;\varpi_1)$ and $(Q_2;\varpi_2)$ with

$$\left\{\begin{array}{rcl} (Q_0;\varpi_0) &=& a_1^2 \odot (P_0;\omega_0) \\ && \oplus\ 2\,a_1^2 \odot (P_1;\omega_1) \\ && \oplus\ a_1^2 \odot (P_2;\omega_2) \\ (Q_1;\varpi_1) &=& a_1\,b_1 \odot (P_1;\omega_1) \\ && \oplus\ a_1\,b_1 \odot (P_2;\omega_2) \\ (Q_2;\varpi_2) &=& b_1^2 \odot (P_2;\omega_2) \end{array}\right. \quad (57)$$

Since we want that the last control mass point of the two curves γ and $\gamma \circ h_1$ is the same, we have $b_1 = 1$. If $\omega_0 + 2\,\omega_1 + \omega_2 \neq 0$, the first control mass point is a weighted point, in order to have $\varpi_0 = 1$, we choose

$$a_1 = \frac{1}{\sqrt{\omega_0 + 2\,\omega_1 + \omega_2}} \quad (58)$$

else the computation of a_1 depends on the vector $\overrightarrow{Q_0}$: either $\overrightarrow{Q_0}$ is $\overrightarrow{e_\infty}$ or its first component is 1.

Theorem 15 Let γ be a Bézier curve with control mass points $(\sigma_0;1)$, $(P_1;\omega_1)$ and $(\sigma_2;1)$ laying on the conic C.

Let $h_0 : [0,1] \to \left[0, \frac{1}{2}\right]$ defined by :

$$h_0 : u \longmapsto \frac{\dfrac{1}{\sqrt{2+2\omega_1}}\,u}{(1-u) + 2\,\dfrac{1}{\sqrt{2+2\omega_1}}\,u} \quad (59)$$

then $\gamma \circ h_0$ equals a Bézier curve with control mass points $(\sigma_{00};1)$, $(P_{10};\varpi_{10})$ and $(\sigma_{20};1)$ laying on the conic C with :

$$\left\{\begin{array}{c} (\sigma_{00};1) = (\sigma_0;1) \\ P_{10} = Bar\left\{(\sigma_0;1)\,;(P_1;\omega_1)\right\} \\ \varpi_{10} = \sqrt{\dfrac{1+\omega_1}{2}} \\ \sigma_{20} = Bar\left\{(\sigma_0;1)\,;(P_1;2\,\omega_1)\,;(\sigma_2;1)\right\} \\ \varpi_{10} = 1 \end{array}\right. \quad (60)$$

Theorem 16 Let γ be a Bézier curve with control mass points $(\sigma_0;1)$, $(P_1;\omega_1)$ and $(\sigma_2;1)$ laying on the conic C.

Let $h_1 : [0,1] \to \left[\frac{1}{2}, 1\right]$ defined by

$$h_1 : u \longmapsto \frac{\dfrac{1}{\sqrt{2+2\omega_1}}\,(1-u) + u}{2\,\dfrac{1}{\sqrt{2+2\omega_1}}\,(1-u) + u} \quad (61)$$

then $\gamma \circ h_1$ equals a Bézier curve with control mass points $(\sigma_{01};1)$, $(P_{11};\varpi_{11})$ et $(\sigma_{21};1)$ laying on the conic C with :

$$\left\{\begin{array}{c} \sigma_{01} = Bar\left\{(\sigma_0;1)\,;(P_1;2\,\omega_1)\,;(\sigma_2;1)\right\} \\ \varpi_{01} = 1 \\ P_{11} = Bar\left\{(P_1;\omega_1)\,;(\sigma_2;1)\right\} \\ \varpi_{11} = \sqrt{\dfrac{1+\omega_1}{2}} \\ (\sigma_{21};1) = (\sigma_2;1) \end{array}\right. \quad (62)$$
