Solution Non Linear Partial Differential Equations By ZMA Decomposition Method

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Abstract: - In this survey, viewed integral transformation (IT) combined with Adomian decomposition method (ADM) as ZMA- transform (ZMAT) coupled with (ADM) in which said ZMA decomposition method has been utilized to solve nonlinear partial differential equations (NPDE's).This work is very useful for finding the exact solution of (NPDE's) and this result is more accurate obtained with compared the exact solution obtained in the literature.

Key-Words: - ADM, DZMAT , Exact Solution , IT , ZMAT, NPDE's.

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1 Introduction

There are many powerful methods in vast diversity of scientific and engineering applications can be found to solve linear (PDE's) is an (IT). But to solve (NPDE's) applied (IT) with (ADM) .To find the solution for (PDE's) a lot of researchers gave attentiveness of (IT) like variational iterative method has been solved integro differential equations [1],[2],[13], orderly solve linear and nonlinear (PDE's) by homotopy perturbation method [3], differential transform method [4], (ADM) [5],[6].There are very much various (IT) identical LaplaceTransform, Sumudu transform, Natural transform, Elzaki transform, Aboodh transform, Kashuri and Fundo transform, ZZ transform Combined with Adomian Method introduce to solve (NPDE's) [7]. in [8],[12] proposed a new integral transform said the Sumudu transform and used for solving problems in control engineering. T. Elzaki proposed a new (IT) named Elzaki transform and stratified it for solving (PDE's) [9]. Shaikh Sadikali presented solution of some linear (PDE's) by Sadik Transform [10]. Offered the new (IT) of (ZMAT) with supporting theory and applications.

In this procedure has been studied new technique named ZMA- transform (ZMAT) coupled with (ADM) to find general solution of (NPDEs) and to more understand this method show some applications about it .Moreover prove the existence and uniqueness of (DZMAT).

2 ZMA- transform

ZMAT of the write as:

$$\Lambda\{\wp(t)\}=\{\wp(t): \exists M, \Omega_1, \Omega_2 > 0, |\wp(t)| < M e^{-\frac{t}{sv}}\} \quad ...(2.1)$$

$$\mathbb{Z}_{MA}(v;s) = \Lambda\{\wp(t)\} = \frac{1}{sv}\int_0^\infty \wp(t) e^{-\frac{t}{sv}} dt \quad ...(2.2)$$

In [11] given some properties for ZMAT, method of applying ZMAT to partial derivative presented as:

$$\Lambda_2\{\frac{\partial \wp(x,t)}{\partial t}\} = \frac{1}{sv}\mathbb{Z}_{MA}(u,v;s) - \frac{1}{sv}\mathbb{Z}_{MA}(u,0;s) \quad ...(2.3)$$

$$\Lambda_2\{\frac{\partial^2 \wp(x,t)}{\partial t^2}\} = \frac{1}{s^2v^2}\mathbb{Z}_{MA}(u,v;s) - \frac{1}{s^2v^2}\mathbb{Z}_{MA}(u,0;s) - \frac{1}{sv}\frac{\partial \mathbb{Z}_{MA}(u,0;s)}{\partial t} \quad ...(2.4)$$

Where $\mathbb{Z}_{MA}(u,v;s)$ is ZMAT of $\wp(x,t)$ and $\frac{\partial \wp(x,t)}{\partial t}$ is the 1-partial derivative of $\wp(x,t)$ with respect to variable t.

3 Derivation of ZMA Decomposition Method

Look the more general nonhomogeneous (NPDE's):

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = h(x,t) \quad ...(3.1)$$

With initial conditions (IC):

$$u(x,0) = \Upsilon(x), \quad u_t(x,0) = T(x) \quad ...(3.2)$$

Where

L : is the 2-order linear differential operator.

$L^{-1}(.) = \int_0^t \int_0^t (.)\, dt\, dt$: an inverse operator .

R : is the remaining linear operator less than L .

$Nu(x,t)$: is the nonlinear operator.

$h(x,t)$: is the nonhomogeneous term.

By (ZMAT) of (2.5) in (t) find:

$$\Lambda_2\{Lu(x,t)\} + \Lambda_2\{Ru(x,t)\} + \Lambda_2\{Nu(x,t)\} = \Lambda_2\{h(x,t)\} \quad ...(3.3)$$

By substituting (2.4) of (3.3) obtain:

$$\frac{1}{s^2 v^2}\Lambda_2\{u(x,t)\} - \frac{1}{s^2 v^2}u(x,0) - \frac{1}{sv}\frac{\partial u(x,0)}{\partial t} + \Lambda_2\{Ru(x,t)\} + \Lambda_2\{Nu(x,t)\} = \Lambda_2\{h(x,t)\} \quad ...(3.4)$$

$\Rightarrow$

$$\Lambda_2\{u(x,t)\} - u(x,0) - sv\frac{\partial u(x,0)}{\partial t} + s^2v^2\Lambda_2\{Ru(x,t)\} + s^2v^2\Lambda_2\{Nu(x,t)\} = s^2v^2\Lambda_2\{h(x,t)\} \quad ...(3.5)$$

Where

$$\Lambda_2^{-1}\{sv\} = t \quad ...(3.6)$$

Yet, exchange the unknown function $u(x,t)$ by any infinite series of u_m as:

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t) \quad ...(3.7)$$

And the nonlinear term by infinite series of Adomian polynomial A_m as:

$$Nu(x,t) = \sum_{m=0}^{\infty} A_m(u_0, u_1, u_2, \ldots), \quad m = 0,1,2,\ldots, \quad ...(3.8)$$

$$A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[F\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, \quad m = 0,1,2,\ldots, \quad ...(3.9)$$

Now, substituting (3.7) and (3.8)in (3.5) find:

$$\Lambda_2\left\{\sum_{m=0}^{\infty}u_m(x,t)\right\} - u(x,0) - sv\frac{\partial u(x,0)}{\partial t} + s^2v^2\Lambda_2\left\{\sum_{m=0}^{\infty}Ru_m(x,t)\right\} + s^2v^2\Lambda_2\left\{\sum_{m=0}^{\infty}A_m(x,t)\right\} = s^2v^2\Lambda_2\{h(x,t)\} \quad ...(3.10)$$

$\Rightarrow$

$$\Lambda_2\left\{\sum_{m=0}^{\infty}u_m(x,t)\right\} = u(x,0) + sv\frac{\partial u(x,0)}{\partial t} + s^2v^2\Lambda_2\{h(x,t)\} - s^2v^2\Lambda_2\left\{\sum_{m=0}^{\infty}Ru_m(x,t)\right\} - s^2v^2\Lambda_2\left\{\sum_{m=0}^{\infty}A_m(x,t)\right\} \quad ...(3.11)$$

At last, by comparing both sides of (3.11) and substituting (IC) give in (3.2) also taking inverse (ZMAT) get the general solution as:

$$u_0(x,t) = \Upsilon(x) + t\,T(x) + \Lambda_2^{-1}s^2v^2\Lambda_2\{h(x,t)\}, \quad m=0 \quad ...(3.12)$$

$$u_{m+1}(x,t) = -\Lambda_2^{-1}\left[s^2v^2\left\{\Lambda_2\left[\sum_{m=0}^{\infty}Ru_m(x,t) + \sum_{m=0}^{\infty}A_m(x,t)\right]\right\}\right], m>0 \quad ...(3.13)$$

4 Existence and Uniqueness of (DZMAT)

Theorem (4.1)

If $\wp(x,t)$ is piecewise continuous in all finite interval $(0,X),(0,T)$ and e^{ρ}, e^{γ} .

For $T > K, X > M$ then (DZMAT) exists $\forall s > \rho, v_1 > \rho, s > \gamma, v_2 > \gamma$.

Proof: $\forall X > 0, T > 0$ then

$$\frac{1}{s^2 v_1 v_2}\int_0^{\infty}\int_0^{\infty}\wp(x,t)\, e^{-(\frac{x}{sv_1}+\frac{t}{sv_2})}\, dx\, dt \quad ...(4.1)$$

$\Rightarrow$

$$\frac{1}{s^2 v_1 v_2}\int_0^{X}\int_0^{T}\wp(x,t)\, e^{-(\frac{x}{sv_1}+\frac{t}{sv_2})}\, dx\, dt + \frac{1}{s^2 v_1 v_2}\int_X^{\infty}\int_T^{\infty}\wp(x,t)\, e^{-(\frac{x}{sv_1}+\frac{t}{sv_2})}\, dx\, dt \quad ...(4.2)$$

From hypothesis $\wp(x,t)$ is piecewise continuous in all finite interval has been get:

$$|\mathbb{Z}_{MA}(v_1, v_2, s)| = \left|\frac{1}{s^2 v_1 v_2}\int_0^{\infty}\int_0^{\infty}\wp(x,t)\, e^{-(\frac{x}{sv_1}+\frac{t}{sv_2})}\, dx\, dt\right|$$

$$\leq \frac{1}{s^2 v_1 v_2}\int_0^{\infty}\int_0^{\infty}\left|\wp(x,t)\, e^{-(\frac{x}{sv_1}+\frac{t}{sv_2})}\right| dx\, dt$$

$$\leq \frac{1}{s^2 v_1 v_2}\int_0^{\infty}\int_0^{\infty}|\wp(x,t)|\, e^{-(\frac{x}{sv_1}+\frac{t}{sv_2})}\, dx\, dt \quad ...(4.3)$$

$$\leq \frac{\Theta}{sv_1}\int_0^{\infty} e^{-\frac{x}{sv_1}} e^{KX} \cdot \int_0^{\infty} e^{-\frac{t}{sv_2}} e^{Mt}\, dx\, dt$$

$$\leq \frac{\Theta}{sv_1}\int_0^{\infty} e^{-(\frac{1}{sv_1}-K)x}\, dx \cdot \int_0^{\infty} e^{-(\frac{1}{sv_2}-M)t}\, dt$$

$$= \frac{\Theta}{sv_1}\left|\frac{e^{-(\frac{1}{sv_1}-K)x}}{-(\frac{1}{sv_1}-K)}\right|_0^{\infty} \cdot \frac{1}{sv_2}\left|\frac{e^{-(\frac{1}{sv_2}-M)t}}{-(\frac{1}{sv_2}-M)}\right|_0^{\infty} \quad ...(4.4)$$

$$= \frac{\Theta}{sv_1v_2}\left(\frac{1}{sv_1}-K\right)\cdot\left(\frac{1}{sv_2}-M\right) \quad ...(4.5)$$

Theorem (4.2)

Let $q(x,t)$ and $p(x,t)$ be continuous function defined for $x, t \geq 0$ and having

(DZMAT) $Q(u,v)$ and $P(u,v)$,if $Q(u,v)=P(u,v)$ then $q(x,t)=p(x,t)$

Proof: if assume $\varepsilon_1, \varepsilon_2$ to be sufficiently large

$$\varpi(x,t)=\frac{1}{2\pi i}\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty} sue^{\frac{x}{su}}.\left(\frac{1}{2\pi i}\int_{\varepsilon_2-i\infty}^{\varepsilon_2+i\infty} sve^{\frac{t}{su}}\ \wp(u,v)\ dv\right)du$$
...(4.6)

Deduced that

$$q(x,t)=\frac{1}{2\pi i}\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty} sue^{\frac{x}{su}}.\left(\frac{1}{2\pi i}\int_{\varepsilon_2-i\infty}^{\varepsilon_2+i\infty} sve^{\frac{t}{su}}\ Q(u,v)\ dv\right)du$$

$$=q(x,t)=\frac{1}{2\pi i}\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty} sue^{\frac{x}{su}}.\left(\frac{1}{2\pi i}\int_{\varepsilon_2-i\infty}^{\varepsilon_2+i\infty} sve^{\frac{t}{su}}\ P(u,v)\ dv\right)du=p(x,t)$$
...(4.7)

5 Applications

(5.1)- Consider the (NPDE)

$$u_t + u u_x = u_{xx}$$ ...(5.1)

With (IC)

$$u(x,0) = 2x,\ t>0$$ ...(5.2)

Solution: by (ZMAT) with ADM ,obtain the solution of (5.1) when taking (ZMAT) on both sides for (5.1) as:

$$\Lambda_2[u_t] = \Lambda_2[u_{xx} - u u_x]$$ ...(5.3)

$\Rightarrow$

$$\frac{1}{sv}\Lambda_2[u(x,t)] - \frac{1}{sv}u(x,0) = \Lambda_2[u_{xx} - u u_x]$$
...(5.4)

$\Rightarrow$

$$\Lambda_2[u(x,t)] = u(x,0) + sv\Lambda_2[u_{xx} - u u_x]$$
...(5.5)

Now, suppose that the infinite series solution of the unknown function $u(x,t)$ and comparing both sides of (5.5) ,lastly taking inverse (ZMAT) get the general solution as:

$$u_0(x,t) = u(x,0),\ m=0$$
$$u_{m+1}(x,t)=\Lambda_2^{-1}\left[sv\Lambda_2\{u_{mxx} - A_m\right], m>0$$
...(5.6)

Where

$$A_m = (u.u_x)_m$$
...(5.7)

After substituting (5.7)in (5.6) find the following :

$$u_0(x,t) = 2x$$ ...(5.8)

$$u_1(x,t) = -4xt$$ ...(5.9)

$$u_2(x,t) = 8xt^2$$ ...(5.10)

$$u_3(x,t) = -\frac{96xt^3}{3\dot{\iota}}$$ ...(5.11)

From above equations (5.8)-(5.11) the general solution has been obtained:

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t)=2x-4xt+\frac{16xt^2}{2\dot{\iota}}-\frac{96xt^3}{3\dot{\iota}}+\dots$$
$$=2x\left[1+(-2t)+(-2t)^2+(-2t)^3+\dots\right]$$
$$=\frac{2x}{1+2t}$$
...(5.12)

(5.2)- Look the (NPDE)

$$u_{tt} + u u_x = -\sin t$$ ...(5.13)

With (IC)

$$u(x,0)=0 \text{ and } u_t(x,0)=1$$ ...(5.14)

Solution: by (ZMAT) with ADM ,obtain the solution of (5.13) when taking (ZMAT) on both sides for (5.13) as:

$$\Lambda_2[u_{tt}] = \Lambda_2[-u u_x - \sin t]$$
...(5.15)

$$\frac{1}{s^2v^2}\Lambda_2[u(x,t)]-\frac{1}{s^2v^2}u(x,0)-\frac{1}{sv}u_t(x,0)=\Lambda_2[-u u_x - \sin t]$$
...(5.16)

$\Rightarrow$

$$\Lambda_2[u(x,t)]=u(x,0)+svu_t(x,0)+s^2v^2\Lambda_2[-\sin t]=s^2v^2\Lambda_2[-u u_x]$$
...(5.17)

Now, suppose that the infinite series solution of the unknown function $u(x,t)$

and comparing both sides of (5.5) ,lastly taking inverse (ZMAT) get the general solution like:

$$u_0(x,t)= u(x,0)+t\,u_t(x,0)+\Lambda_2^{-1}\left[s^2 v^2 \Lambda_2\left[-\sin t\right]\right]$$
...(5.18)

$\Rightarrow$

$$u_0(x,t) = t + \Lambda_2^{-1}\left[s^2 v^2 \left[-\frac{s\,v}{1+s^2 v^2}\right]\right]$$
...(5.19)

$\Rightarrow$

$$u_0(x,t)=t+\Lambda_2^{-1}\left[s\,v-\frac{s\,v}{1+s^2 v^2}\right]=t-t+\sin t=t$$
...(5.20)

Where

$$A_m = (u\,.\,u_x)_m$$
...(5.21)

$\Rightarrow$

$$u_{m+1}(x,t)=-\Lambda_2^{-1}\left[s^2 v^2 \Lambda_2\left[u\,.\,u_x\right]_m\right]$$

...(5.22)

By solving (5.22) the following result obtained:

$$u_1(x,t) = 0$$
...(5.23)

$$u_2(x,t) = 0$$
...(5.24)

$$u_3(x,t) = u_4(x,t) = u_5(x,t) = \ldots = 0$$

...(5.25)

From above equations (5.20) and (5.23)-(5.25) the general solution has been obtained:

$$u(x,t)=\sum_{i=0}^{\infty} u_i(x,t)= \sin t + 0 + 0 + \ldots + 0$$
$$= \sin t$$

...(5.26)

(5.3)- Consider the (NPDE)

$$u_t = \frac{1}{2}x^2\,u\,u_{xx}$$
...(5.27)

With (IC)

$$u(x,0) = x^2$$
...(5.28)

By (ZMAT) of both sides (5.27) find as:

$$\Lambda_2\left[u(x,t)\right]= u(x,0)+s\,v\,\Lambda_2\left[\frac{1}{2}x^2\,u\,u_{xx}\right]$$
...(5.29)

Now , taking inverse (ZMAT) of (5.29) get:

$$u(x,t) = u(x,0)+\Lambda_2^{-1}\left[s\,v\,\Lambda_2\left[\frac{1}{2}x^2\,u\,u_{xx}\right]\right]$$
...(5.30)

From the above steps find the general solution as:

$$u_0(x,t)= u(x,0)\ ,\ m=0$$
$$u_{m+1}(x,t)=\Lambda_2^{-1}\left[s\,v\,\Lambda_2\{\frac{1}{2}\ x^2\,u_m\,u_{mxx}\}\right], m > 0$$

...(5.31)

Where

$$A_m = \frac{1}{2}x^2\,u_m\,.\,u_{mx}$$
...(5.32)

$\Rightarrow$

$$u_0(x,t) = x^2$$
...(5.33)

$$u_1(x,t) = t$$
...(5.34)

$$u_3(x,t) = u_4(x,t) = u_5(x,t) = \ldots = 0$$

...(5.35)

From above equations (5.33) and (5.33) the general solution has been obtained:

$$u(x,t)=\sum_{i=0}^{\infty} u_i(x,t)=x^2+t+0+\ldots+0$$
$$=x^2+t$$

...(5.36)

6 Conclusion

In this method taking(ZMAT) with (ADM) to find general solution for (NPDE's) is more powerful method to solve it. The results are given briefly applications which concludes as present solution taking improved technique is more accurate to find the exact solution . Hence this technique is simple to get accurate result .

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References:

[1] J. H. He, *Mech. Appl.* 20, 30-31(1998).
[2] J. H. He, *Appl. Math. Computation* 114, 115-123 (2000).
[3] A. Ghobani and J. S. Nadjafi, *Comp. Math. Appl.,* 56, 1032-1039 (2008).
[4] G. Methi, *Asian J. Math. Stat.,* 9, 1-5 (2016).
[5] S. Saha Ray, A. Patra, *Int. J. Nuc. Energy Sc. Tech.,* 6, 310-320 (2011).
[6] A. M. Wazwaz, *Appl. Math. Comp.,* 111, 33-51 (2000).
[7] R.I.Nuruddeen, L. Muhammad, A.M.Nass, T.A.Sulaiman, "A Review of the Integral Transforms- based Decomposition Methods and their Applications in Solving Nonlinear PDEs.", *P. J .Math.* Vol. 7(1), 262280, (2018).
[8] G. K. Watugula, A New Integral Transform to solve differential equations and control engineering problems, *International Journal of Mathematical Education in sxience 7 Technology,* 24, 409–421 (1993).
[9] M. E. Tariff and M. E. Salih, "Application of New Transform Elzaki Transform to Partial Differential Equations*", Global Journal of pure and applied Mathematics,* 7(1), pp. 65–70, (2011).
[10] S. latif Shaikh *"Some Applications of The New Integral Transform For Partial Differential Equations"*.
[11] Z. Mohammed Alwan, (ZMA)-transform method.*Journal of Interdisciplinary Mathematics,* DOI: 10.1080/09720502.2021.1963520,(2021).
[12] C.VIRIYAPONG and N. VIRIYAPONG, Modified Sumudu Decomposition Method for Solving,Lane-Emden-Fowler Type Systems,*WSEAS TRANSACTIONS on MATHEMATICS*,DOI: 10.37394/23206.2021.20.46,2021.
[13] D.Rubio, D.A.Tarzia, G.F.Umbricht,Heat Transfer Process with Solid-solid Interface: Analytical and Numerical Solutions, *TRANSACTIONS on MATHEMATICS*, DOI: 10.37394/23206.2021.20.42,2021.