Generalizations of S- Prime Ideals

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Abstract: Let R be a commutative ring with identity and S be a multiplicative subset of R . In this paper we introduce the concept of almost S-prime ideal as a new generalization of $S-$prime ideal. Let P be a proper ideal of R disjoint with S. Then P is said to be almost S- prime ideal if there exists $s \in S$ such that, for all $x, y \in R$ if $xy \in P - P^2$ then $sx \in P$ or $sy \in P$. Number of results concerning this concept and examples are given. Furthermore, we investigate an almost S- prime ideals of trivial ring extensions and amalgamation rings..

Key-Words: prime ideal, S-prime ideal, weakly S-prime ideal, almost S-prime ideal

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1 Introduction

Throughout, all rings are commutative with $1 \neq 0$ and modules are unitary. Consider a multiplicative set S that satisfies 0 $\notin$ S, 1 ∈ S, and xy ∈ S for all x, y ∈ S . Recall that, a proper ideal P of a ring R is said to be prime if whenever x, y ∈ R with xy ∈ P then either x ∈ P or y ∈ P; equivalently for ideals X and Y (if XY ⊆ P, then either X ⊆ P or Y ⊆ P). Prime ideals play a central role in commutative ring theory and as a result it has been generalized and studied in several aspects, the importance of some of these generalizations is not less importance than prime ideals (see [13). A proper ideal is said to be a weakly prime ideal, if for x, y ∈ R with 0xy ∈ P , then either x ∈ P or y ∈ P (see [5]). It follows that, every prime ideal is weakly prime ideal but the converse is not true (since {0} is always weakly prime, but not prime if R is an integral domain). A proper ideal is said to be almost prime ideal, if for x, y ∈ R with xy ∈ P − P^2 , then either x ∈ P or y ∈ P (see [1]). Let P be an ideal of R disjoint with S, then P is an S-prime ideal of R if there exists an s ∈ S such that for all a, b ∈ R if ab ∈ P, then sa ∈ P or sb ∈ P (see [10]). In a very recent paper [2], the authors introduced the notion of S-prime ideal which is also a generalization of prime ideals. An ideal P of R satisfying P ∩ S = ϕ is said to be weakly S-prime if there exists an element s ∈ S such that, whenever a, b ∈ R, 0 ≠ ab ∈ P implies sa ∈ P or sb ∈ P. In this case, we say that P is associated to s (see [2]).

2 Problem Formulation

The main purpose of this paper is introduce the concept of almost S-prime ideal as a new generalization of S-prime ideals. A proper ideal P of R is almost S-prime if there exists $s \in S$ where P is disjoint with S such that, whenever $a, b \in R$, $ab \in P - P^2$ implies $sa \in P$ or $sb \in P$. Note that every (weakly) prime ideal of R which is disjoint with S is an S-prime ideal. However, the converse is not true in general [10, Example 1]. Also, in this paper we present the properties and characteristics of almost S-prime ideals as well as it is relationship among other ideals. We investigate an almost S- prime ideals under localization, trivial ring extensions and amalgamation rings.

Definition 2.1. *A proper ideal P of a commutative ring R is almost S-prime if there exists $s \in S$ where P is disjoint with S such that for all $a, b \in R$ if $ab \in P - P^2$ then $sa \in P$ or $sb \in P$.*

Example 2.2. *Let $R = \mathbb{Z}/6\mathbb{Z}$. If we take $P = \langle \overline{3} \rangle$*

and $S = \{\overline{1}, \overline{2}, \overline{4}\}$, then P is prime deal, P is weakly prime deal and also P is almost S- prime deal.

Example 2.3. *Let $R = Z/16Z$. If we take $P = 8Z/16Z$ and $S = \{\overline{0}, \overline{1}, \overline{4}\}$, then P is not prime deal, P is not weakly prime deal and also P is not almost prime deal, but P is almost S-prime.*

Example 2.4. *Let $k[X^3, X^4, X^5]$, where k is a field. If we take $P = (X^3, X^4)$ and $S = X^5$, then P is not prime ideal, but P is almost prime and almost S-prime.*

Theorem 2.5. *Let P be an almost S-prime ideal in a commutative ring R , if I is an ideal of R with $I \in P$ then P/I is an S-almost prime ideal of R/I.*

Proof. Let $(a + I)(b + I) \in P/I - (P/I)^2$ which means $ab + I \in P/I - (P/I)^2$ or $ab \in P$ and $ab \notin P^2$which implies $ab \in P - P^2$ but P is an almost S-prime , so there exist $s \in S$ such that $s.a \in P$ or $s.b \in P$ so $s.a + I \in P + I$ or $s.b + I \in P + I$ which means P/I is S- almost prime of R/I. □

Theorem 2.6. *Let R be a commutative ring , then P is almost S-prime ideal of R if and only if P/P^2 is $S-$ weakly prime in R/P^2 .*

Proof. Assume that P is almost S-prime, let $\overline{0} \neq (a+P^2)(b + P^2) \in P/P^2$ in R/P^2 so $ab \in P$ and $ab \notin P^2$. Since P is almost S-prime ideal $\exists \in S$ such that $sa \in P$ or $sb \in P$ which means P/P^2 is $S-$ weakly prime ideal. On the other hand, assume that P/P^2 is $S-$ weakly prime in R/P^2 and let $ab \in P - P^2$, then $\overline{0} \neq (a + P^2)(b + P^2) \in P/P^2$. But P/P^2 is $S-$ weakly prime ideal, then either $sa + P^2 \in P/P^2$ or $sb + P^2 \in P/P^2$. Thus, $sa \in P$ or $sb \in P$ which means P is almost S-prime ideal. □

Theorem 2.7. *Let $R = R_1 \times R_2$ and $S = S_1 \times S_2$ where S_i is multiplicative set of R_i, where $i = 1, 2$. Let $P = P_1 \times P_2$ is an ideal in R. The following are equivalent:*

(i) P is almost S-prime.
(ii) P_1 is almost S_1-prime and $S_2 \cap P_2 \neq \phi$ or $P_1 \cap S_1 \neq \phi$ and P_2 is almost S_2-prime ideal.

Proof. $(i) \Rightarrow (ii)$ Let P is almost S-prime in R. Since $(1, 0).(0, 1) = (0, 0) \in P - P^2$, then there exists $s = (s_1, s_2$ in a multiplicative set S. Now $s(1, 0) = (s_1, s_2)(1, 0) = (s_1, 0) \in P$ or $s(0, 1) = (0, s_2) \in P$, so $P_1 \cap s_1 \neq \phi$ or $s_2 \cap P_2 \neq \phi$. Assume that $P_1 \cap s_1 \neq \phi$. Now, since $P \cap S = \phi$, then we have $s_2 \cap P_2 = \phi$. Consider $x, y \in R_2$ and $xy \in P_2 - P_2^2$, then $(0, x)(0, y) \in P - P^2$ and as P is almost S- prime in R, then $s(0, x) = (0, s_2x) \in P$ or $s(0, y) = (0, s_2y) \in P$ so s_2xP_2 or s_2yP_2. Which means P_2 is almost S_2-prime in R_2 and P_1 is almost S_1-prime in R_1 in the same way.
$(ii) \Rightarrow (i)$ Assume that P_2 is S_2-almost prime in R_2 and $P_1 \cap S_1 \neq \phi$ then there exists $s_1 \in P_1 \cap S_1$ and $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2) \in P - P^2$, where $a_1, b_1 \in R_1$ and $a_2, b_2 \in R_2$. Thus, $a_2b_2 \in P_2 - P_2^2$. But P_2 is S_2-almost prime in R_2, then there exists $s_2 \in S_2$ such that $s_2.a_2 \in P_2$ or $s_2.b_2 \in P_2$. Let $s = (s_1, s_2) \in S, s(a_1, a_2) = (s_1a_1, s_2a_2) \in P - P^2$ or $s(b_1, b_2) = (s_1.b_1, s_2.b_2 \in P$, then P is almost S-prime in R. Similarly, we can prove P is almost S-prime by using P_1 is S_1-almost prime in R_1 . □

Theorem 2.8. *A proper ideal P in a commutative ring R is weakly $S-$prime, where S is multiplicative set disjoint with P if and only if $SP = 0$ or P is almost S-prime.*

Proof. Assume that P is almost S- prime in R, where $R = R_1 \times R_2$. Let $(a, 1).(1, b) = (a, b) \in P$, where $P = P_1 \times P_2$ such that if $(a, b) \in P - P^2$, then there exists $s = (s_1, s_2)$ with $s(a, 1) \in P$ or $s(1, b) \in P$
if and only if $(s_1, s_2)(a, 1) \in P_1 \times P_2$ or $(s_1, s_2)(1, b) \in P_1 \times P_2$,
if and only if $(s_1a, s_2) \in P_1 \times P_2$ or $(s_1, s_2b) \in P_1 \times R_2$,
if and only if $s_1a \in P_1$ and $s_2 \in P_2$ or $s_1 \in P_1$ and $s_2b \in P_2$,
if and only if $s_2 \cap P_2 \neq \phi$ or $s_1 \cap P_1 \neq \phi$. Without loss of generality, to show P_1 is almost S_1-prime, consider $a.b \in P_1 - P_1^2$ with $a, b \in R_1$, then $(a, 0).(b, 0) \in P - P^2$. Since P is almost S-prime, then there exists $(s_1, s_2) \in S$ such that $(s_1a, 0) \in P$ or $(s_2b, 0) \in P$. Therefor, $s_1a \in P_1$ or $s_2b \in P_1$ which means P_1 is almost S_1 prime. The converse is straight forward.
Let R be a ring, I an ideal of R, S multiplicative subset of R disjoint of I and $\overline{S} = s + I$, where $s \in S$, is multiplicative subset of R I, then if $P \cap \overline{S} = \phi$, then $P \cap S = \phi$. □

Theorem 2.9. *Let R be a ring, $S \subseteq R$ a multiplicative set, P an ideal of R disjoint with S and I a proper ideal of R containing P such that $(I/P) \cap S = \phi$, then the following are equivalent.*

(i) I is an almost S- prime ideal of R.
(ii) I/P is almost $\overline{S}-$ prime ideal of R/P.

Proof. $(i) \Rightarrow (ii)$ Let $\overline{a}, \overline{b} \in R/P$ such that $\overline{a}.\overline{b} \in I/P - (I/P)^2$, then $ab \in I - I^2$. But I is an almost S- prime ideal of R, so there exists $s \in S$ such that $sa \in I$ or $sb \in I$ which means $\overline{sa} \in I/P$ or $\overline{s}\overline{b} \in I/P$. Thus, I/P is almost $\overline{S}-$ prime ideal.
$(ii) \Rightarrow (i)$ It is true if P is an almost S- prime ideal of R.
Let $a, b \in R$ with $ab \in I - I^2$. If $\overline{a}\overline{b} = \overline{0}$, then

$ab \in P$ and there exists $s \in S$ such that $sa \in P \subseteq I$ or $sb \in P \subseteq I$. Now assume that $\overline{0} \neq \overline{a}\overline{b}$, then $\overline{s} \in \overline{S}$ with $\overline{sa} \in I/P$ or $\overline{s}\overline{b} \in I/P$ and $sa \in I$ or $sb \in I$ which means I is almost S-prime ideal of R. ☐

Theorem 2.10. *Let R and Q be two commutative rings, if every proper ideal of R and Q is product of almost $S-$prime ideals, then every proper ideals of $R \times Q$ is product of almost S-prime.*

Proof. Let p and q be proper ideals of R and Q respectively, where $p = p_1.p_2.p_3.....p_n$ and $q = q_1.q_2.q_3.....q_m$, where p_i and q_j are almost S-prime ideals. Now, $p \times Q = (p_1.p_2.p_3.....p_n) \times Q = (p_1 \times Q)(p_2 \times Q).....(p_n \times Q)$ which is a product of almost S-prime ideals by (Lemma 2.10). Similarly, If the proper ideal is of the form $R \times q$. Now, if the proper ideal is of the form $p \times q$, then we can write it as $(p_1.p_2....p_n) \times (q_1.q_2....q_m) = (p_1.p_2.....p_n \times Q)(q_1.q_2.....q_m \times R) = (p_1 \times Q)(p_2 \times Q).....(p_n \times Q) \times (R \times q_1)(R \times q_2).....(R \times q_m)$. Thus, we get a product of almost S-prime ideals. ☐

also we can generalize this lemma to n-proper ideals from n-commutative rings.

Lemma 2.11. *Let P be an almost S-prime ideal in a ring R and G be any closed multiplicative subset of R which disjoint from P, then $G^{-1}P$ is an almost S-prime in $G^{-1}R$.*

Proof. Let $r_1, r_2 \in R$, $g_1, g_2 \in G$ and $r_1r_2/g_1g_2 \in G^{-1}P - G^{-1}P^2$, then there exists u_1 with $u_1r_1r_2 \in P$ and $u_2r_1r_2 \notin P^2$ for any $u_2 \in G$. Hence, $u_1r_1r_2 \in P - P^2$. But P is almost S- prime ideal, then there exists $s \in S$ where S is a multiplicative subset of R disjoint with P such that $su_1r_1 \in P$ or $sr_2 \in P$ which means $sr_1/g_1 \in G^{-1}P$ or $sr_2/g_2 \in G^{-1}P$. Thus, $G^{-1}P$ is almost S-prime ideal. ☐

Theorem 2.12. *Let R be a ring with multiplicative subset S of R which is disjoint with the proper ideal P. If P is almost S- prime of R, then $S^{-1}P$ is almost prime of $S^{-1}R$ and $S^{-1}P \cap R = (P : s) \cup (0_s \cap R)$ for some $s \in S$*

Proof. Let P be an almost S-prime ideal. To show $S^{-1}P$ is almost prime of $S^{-1}R$, let $\frac{x}{s}, \frac{y}{t}$ in $S^{-1}R$ such that $\frac{x}{s}\frac{y}{t} \in S^{-1}P - (S^{-1}P)^2$. Now, let $xy = p$ and $st = u \in S$, then $\frac{p}{u} \in S^{-1}P - (S^{-1}P)^2$ and hence there exists $w \in S$ such that $wuxy \in P - P^2$. But P is almost S- prime ideal, then there exists $s' \in S$ with $s'wux \in P$ or $s'y \in P$. But $\frac{x}{s} = \frac{wux}{wus} \in S^{-1}P$ or $\frac{y}{t} = \frac{s'y}{s't} \in S^{-1}P$, then $S^{-1}P$ is almost prime. Let $x \in S^{-1}P \cap R$, then $x \in S^{-1}P$ and there exists $s \in S$ such that if $sx \in P$ gives $x \in (P : s)$ and if $0 \neq sx$, then take $s' \in S$. If $xs' = 0$, then $x \in 0_s \cap R$, and if $xs' \neq 0$, then $x \in (0_s \cap R) \cup (P : s)$. On the other hand, consider $x \in (0_s \cap R) \cup (P : s)$. If $x \in (P : s)$, then $xs \in P$ so $x = \frac{xs}{s} \in S^{-1}P$, and if $x \in (0_s \cap R)$, then $xs = 0 \in P$ and $x = \frac{xs}{s} \in S^{-1}P$, as a result we get $x \in S^{-1}P \cap R$. Therefore, $S^{-1}P \cap R = (0_s \cap R) \cup (P : s)$. ☐

Theorem 2.13. *Let R be a ring, P is a proper ideal of R disjoint with the multiplicative set S of R such that $Z(P) \cap S = \phi$ then P is almost S- prime if and only if there exists $s \in S$ such that for all I, J are two ideals of R if $IJ \subseteq P - P^2$, then $sI \subseteq P$ or $sJ \subseteq P$.*

Proof. $(\Rightarrow)$ Let P be an almost S-prime and I, J two ideals of R satisfy that $IJ \subseteq P - P^2$. Assume that $sI \not\subset P$ and $sJ \not\subset P$ for each $s \in S$ and $x \in I$. If $sx \in I/P$, then $sxJ \subseteq P$ implies that $J \subseteq (P : sx)$. But $J \not\subset (P : s)$, then we get $sxJ = 0$ and hence $s^2IJ = 0$. Similarly, if $sy \in J/P$ where $y \in J$, then $syI = 0$ and hence $s^2IJ = 0$. Now, assume that $sx \in I \cap P$ and $sy \in J \cap P$, then $x \in (P : s)$ and $y \in (P : s)$. Since $(P : s)$ is weakly prime which is not prime by (Proposition 1, [10]), we get $xy \in (P : s)^2 = 0$ by (Theorem 1, [5]). Hence, $s^2IJ = 0 \in P$ which is contradiction since P is almost S- prime.
$(\Leftarrow)$ Suppose that $xy \in P - P^2$, then $(x)(y) \subseteq P - P^2$ and so $(sx) \subseteq P$ or $(sy) \subseteq P$ (i.e.,$sx \in P$ or $sy \in P$). ☐

Theorem 2.14. *If $< x >$ is an almost S-prime ideal in a domain R, then $< x >$ is almost prime ideal.*

Proof. Suppose that $< x >$ is not almost prime ideal, then there exists $a \notin < x >$ and $b \notin < x >$ such that $ab \in < x >$ but $ab \notin < x^2 >$. If $ab \notin < x^2 >$, then we are done. So assume that $ab \in < x^2 >$, then $a(b+x) \in < x >$ and $a, (b+x) \notin < x >$. If $a(b+x) \in < x^2 >$, then $ax \in < x^2 >$ and $a \in < x >$ which is a contradiction.

☐

Theorem 2.15. *Let x be a non zero and non unit element in R where R is an integral domain. If $< x >$ is not almost S-prime ideal, then there exist $a \notin Rx$ and $b \notin Rx$ such that $ab \in Rx$ and $ab \notin < x^2 >$.*

Proof. Suppose that $a, b \in R$ with $ab \in < x >$ and $a, b \notin < x >$, which is equivalent to say that there exists $s \in S$ such that $sa \notin < x >$ and $sb \notin < x >$. If $ab \notin < x^2 >$, then we are done. So assume that $ab \in < x^2 >$ which gives $a(b + x) \in < x >$ and $sa, s(b + x) \notin < x >$. If $a(b+x) \in < x^2 >$, then $sax \in < x^2 >$ and so $sa \in < x >$ which is a contradiction. ☐

Recall that, if R and G are any two commutative rings, then the prime ideals of $R \times G$ have the form $p \times g$ or $R \times g$ where p is a prime ideal of R and g is prime ideal of G. Next, we generalize this result to almost S-prime ideal.

Theorem 2.16. *Let R and G be any two commutative rings, then the ideal $R \times G$ is almost S-prime if and only if it has one of the following forms:*

(i) *$P \times G$, where P is almost S-prime of R.*
(ii) *$R \times g$, where g is almost S-prime of G.*
(iii) *$I \times g$, where I is an idempotent of R and g is an idempotent of G.*

Proof. Let C be an almost S-prime of $R \times G$. If C is of the form $p \times G$ or $R \times g$, with p and g are both proper ideals in R and G respectively, then by (Lemma 2.10) if we take $C = p \times g$, where p and g are proper ideals of R and G respectively, then we are done. Let $a \in p - p^2$, then $C - C^2 = (p - p^2) \times g \cup (I \times g - g^2)$ and so $(a, 0) \in C - C^2$. But $(a, 0) = (a, 1)(1, 0)$, then there exists $s \in S$ such that $s(a, 1) \in C$ or $s(1, 0) \in C$. If $s(a, 1) \in C$, then $s(1) \in g$ or if $s(1, 0) \in C$, then $s(1) \in p$ which is a contradiction. Since g, p are proper ideals, then $C - C^2 = \phi$, which implies that $C = C^2$. So C is idempotent. If p and g are idempotent and $C = p \times g$ then, C is idempotent. □

Observe that, if (R, M) is a local ring with M is maximal ideal where $M^2 = 0$. Then every proper ideal of R is almost S-prime ideal, where S is multiplicative set disjoint with any proper ideal P of R such that if $0 \neq ab \in P - P^2$, then $a \notin M$ or $b \notin M$. Without loss of generality we can assume that $a \notin M$, so $a^{-1}ab = b \in P$ and $sb \in P$ which means P is almost S-prime ideal, where s is an element in S.

Theorem 2.17. *Let R be a ring and S be a multiplicative subset of R, then the following holds.*

(i) *Let $f : X \to Y$ be a linear ring homomorphism, if Q is almost $f(S)-$ prime ideal in Y, then f^{-1} is almost S-prime ideal of X.*
(ii) *If $X \subseteq Y$ an extension and Q is almost S-prime ideal of Y, then $Q \cap R$ is almost S-prime ideal of R .*
(iii) *Let P be an almost S-prime ideal and I an ideal in R such that $I \cap S \neq \phi$, then IP is an almost S-prime ideal .*

Proof. (i) Let $xy \in f^{-1}(Q - Q^2)$ where $x, y \in R$, then $f(x)f(y) \in Q - Q^2$. But Q is almost $f(S)-$prime ideal, then there exists $s \in S$ such that $sf(x) \in Q$ or $sf(y) \in Q$. ($i.e$; $sx \in f^{-1}(Q)$ or $sy \in f^{-1}(Q)$).

(ii) Let $x, y \in R$ with $xy \in Q \cap R - (Q \cap R)^2$, since Q is almost S-prime of R, then there exists $s \in S$ such that $sx \in Q$ or $sy \in Q$ and so $sx \in Q \cap R$ or $sy \in Q \cap R$.

(iii) Let $x, y \in R$ with $xy \in IP - (IP)^2 \subseteq P - P^2$, then there is an $s \in S$ with $sx \in P$ or $sy \in P$. But $I \cap S \neq \phi$, then we can find $r \in I \cap S$ such that $rsx \in IP$ or $rsy \in IP$. □

Theorem 2.18. *Let R be a ring with $S \subseteq R$ a multiplicative set disjoint with the proper ideal P. Then the following are equivalent.*

(i) *P is almost S-prime ideal.*
(ii) *There exists $s \in S$ such that for any $x \in R$ and $x \in R \setminus (P : s)$ we have $(P : sx) \subseteq (P : s) \cup (P^2 : sx)$.*
(iii) *For all $x \in R$ with $sx \notin P$ for some $s \in S$, we have $(P : sx) = (P : s)$ or $(P : sx) \subseteq (P^2 : sx)$.*

Proof. $(i) \Rightarrow (ii)$ Assume that P is almost S-prime ideal, then there is an $s \in S$ such that for any $a, b \in R$ with $ab \in P - P^2$, we have $sa \in P$ or $sb \in P$. Now, let $x \in R$ with $x \in R \setminus (P : s)$, then if $y \in (P : sx)$ then $sxy \in P$ and if $sxy \in P - P^2$, then $s^2y \in P$. Since P is almost S-prime and $sx \notin P$ but $P \cap S = \phi$, then $sy \in P$. Hence, $y \in (P : s)$. Also, if $sxy \in P^2$, then $y \in (P^2 : sx)$.
$(ii) \Rightarrow (iii)$ It follows clearly.

$(iii) \Rightarrow (i)$ Let $xy \in R$ with $xy \in P - P^2$ and $sx \notin P$, where $s \in S$, then to prove that $sy \in P$ and so $sxy \in P$, then $y \in (P : sx)$. If $y \in (P^2 : sx)$, then $sxy \in P^2$ and we get a contradiction. If we take $s = 1$, then $y \in (P : s)$. Therefor, $sy \in P$ and hence P is almost S-prime. □

Theorem 2.19. *If P is almost S-prime ideal in a ring R, if $r \in R - P$ then $(P : r) = P \cup (P^2 : r)$.*

Proof. Let $r \in R - P$ and take $t \in (P : r)$, then $rt \in P$. if $rt \in P - P^2$, then there exists $s \in S$ such that $st \in P$ and if $rt \in P^2$, then $t \in (P^2 : r)$. □

Theorem 2.20. *Let $R = R_1 \times R_2$ be a ring and $S = S_1 \times S_2$, where S_i a multiplicative subset of a ring R_i, then every proper ideal of R is almost S-prime ideal if and only if R_i are fields.*

Proof. $(\Rightarrow)$ Let P_1 be a proper ideal of R_1, then $P_1 \times R_2$ is an almost S-prime ideal of $R_1 \times R_2$ and so almost S_1-prime ideal of R_1 by (Theorem 2.7). Also, for any ring R and S
a multiplicative subset of R, then R is field if and only if each proper ideal is almost S-prime, R_1 is field. Similarly, R_2 is a field.
$(\Leftarrow)$ Since R_1 and R_2 are fields, then we have only these proper ideal $\{0\} \times \{0\}$, $\{0\} \times R_2$ and $R_1 \times \{0\}$ which are almost S-prime ideals. □

Theorem 2.21. *A commutative ring R has every proper ideal almost S-prime if and only if either (R, M) is quasi local ring (possibly a field) with $M^2 = 0$ or $R = F_1 \times F_2$ where F_1 and F_2 are fields.*

Next theorem gives the relationship between S-prime and almost S-prime ideals

Remark 2.22. *Consider S_1 and S_2 be a multiplicative subsets of R where $S_1 \subseteq S_2$ and P an ideal of R, if P is almost S_1-prime then its almost S_2- prime of R, but the converse is not true in general. Let $R = \mathbb{Z}[\mathbb{X}]$, $P = 4X\mathbb{Z}[\mathbb{X}]$ and take $S_1 = \{1\}$ and $S_2 = \{2^n : n \in \mathbb{N}\}$.*

3 Some extensions of almost S-prime ideals

Let R be a ring and P be an ideal of R. The amalgamated duplication of R along P denoted by $R \bowtie P$ and its subring of $R \times R$ given by $R \bowtie P = \{r, r + p \mid r \in R, p \in P\}$. This construction was introduced and its basic properties were studies dy D'Anna and Fontana in [7, 8] and then it was investigated by D'Anna with the aim of applying it to curve singularities (over algebraic closed fields) where he proved that the amalgamated dupliction of an algebroid curve along a regular canonical ideal yields a Gorenstein algebroid curve. Recently Mahdou, Moutui and Y.Zahir studied some kind of weakly prime ideals in amalgamated algebras (a generalization of the duplication construction) [11]. In the next result of this paper, we initiate the study of almost S- prime ideals over amalgamated duplication of a ring along an ideal. Also, we transfer almost S- prime ideal property to some ideals of trivial ring extension and amalgamation rings. Take S as a multiplicative subset of a ring R and E be an $R-$module and the additive group $R \times E$ with $(a, e_1)(b, e_2) = (ab, ae_2 + be_1)$ is commutative ring with identity. If $S \propto E$ is a multiplicative subset of $R \propto E$, then we want to study the property for $P \propto E$ if its almost $S \propto E-$ prime ideal of $R \propto E$, where P is almost S-prime of R.

Theorem 3.1. *Let R be a ring with the proper ideal P and E be an $R-$module. If S is a multiplicative subset of R, then $P \propto E$ is almost $S \propto E-$ prime ideal if and only if P is almost S-prime ideal of R.*

Proof. $(\Rightarrow)$ Let $P \propto E$ be almost $S \propto E$-prime ideal and $x, y \in R$ such that $xy \in P - P^2$, then $(x, 0)(y, 0) = (xy, 0) \in P \propto E - P^2 \propto E$ and there exists $(s, e) \in S \propto E$ with $(s, e)(x, 0) \in P \propto E$ or $(s, e)(y, 0) \in P \propto E$.Hence, $sx \in P$ or $sy \in P$ and P is almost S-prime ideal in R.
$(\Leftarrow)$ If $(x, e_1)(y, e_2) \in P \propto E - (P \propto E)^2$, then $(xy, xe_2 + ye_1) \in P \propto E - P^2 \propto E$, where $(x, e_1), (y, e_2) \in R \propto E$ and so $xy \in P - P^2$. But P is almost S-prime, then there exists $s \in S$ with $sx \in P$ or $sy \in P$ (i.e;$(s, e)(x, e_1) \in P$ or $(s, e)(y, e_2) \in P$). □

Theorem 3.2. *Let R be a ring and P an ideal of R with E be an $R-$module. If S multiplicative subset of R, then $P \propto E$ is almost $S \propto E-$ prime ideal if and only if P is almost S- prime of R and for $x, y \in R$ with $xy = 0$ but $sx \notin P$ and $sy \notin P$ for each $s \in S$ and $x \in Ann(E)$ and $y \in Ann(E)$.*

Proof. $(\Rightarrow)$ Let $x, y \in R$ such that $xy = 0$, $sx \notin P$ and $sy \notin P$ for each $s \in S$. Suppose that $x \notin Ann(E)$, then there exists $e \in E$ with $xe \neq 0$ and $(x, 0)(y, e) \in P \propto E-(P \propto E)^2$. But $(s, r)(x, 0) \notin P \propto E$ and $(s, r)(y, e) \notin P \propto E$ for any $(s, r) \in S \propto E$, then it is a contradiction (since $P \propto E$ is almost $S \propto E-$ prime).
$(\Leftarrow)$ Let $(x, e)(y, r) = (xy, xr + ye) \in P \propto E - (P \propto E)^2$. If $xy \neq 0$, then there exists $s \in S$ with $sx \in P$ or $sy \in P$ and so $(s, 0)(x, e) \in P \propto E$ or $(s, 0)(y, r) \in P \propto E$. Now, assume that $xy = 0$, $sx \notin P$ and $sy \notin P$ for each $s \in S$, then by using $x \in Ann(E)$ and $y \in Ann(E)$, there exists $r \in E$ such that $xr = 0, ye = 0$, and so $(x, e)(y, r) = (xy, xr + ye) = (0, 0)$, which is not always true, so we get a contradiction. □

Corollary 3.3. *Let (R,M) be a local ring with multiplicative set S of R and if E is an $R-$module such that $ME = 0$, then $P \propto E$ is a almost $S \propto E-$prime ideal of $R \propto E$ if and only if P is almost S-prime ideal of R.*

Proof. $(\Rightarrow)$ Let $x, y \in R$ such that $xy = 0$ but $sx \notin P$ and $sy \notin P$, for some $s \in S$. With out loss of generality, take $x \notin M$ so there exists r/inR with $xr = 1$ and so $y = 0$, which is a contradiction. Hence, $x \in M = Ann(E)$ and $y \in M = Ann(E)$. By (Theorem 3.2), we get $P \propto E$ is $S \propto E-$almost prime ideal.
$(\Leftarrow)$ Follows Clearly. □

Theorem 3.4. *Let R be a ring with the proper ideal P and E be an $R-$module, if S multiplicative subset of R disjoint with P, then the following are equivalent.*

(i) $P \bowtie E$ is almost $S \bowtie 0-$prime ideal of $R \bowtie E$.
(ii) $P \bowtie E$ is almost $S \bowtie E-$prime ideal of $R \bowtie E$.
(iii) P is almost S-prime of R associated to $s \in S$ and if there exist $x, y \in R$ with $xy = 0$ but $sx \notin P$ and $sy \notin P$ then $x \in Ann(E), y \in Ann(E)$.

Proof. $(i) \Rightarrow (ii)$ is trivial since $S \bowtie 0 \subseteq S \bowtie E$.
$(ii) \Rightarrow (iii)$ suppose $P \bowtie E$ is almost $(S \bowtie$

$E)-$prime of $R \bowtie E$, let $x, y \in R$ such that $xy \in P - P^2$ then $(x, x)(y, y) \in P \bowtie E - (P \bowtie E)^2$ so $(s, s+e)(x, x) \in P \bowtie E$ or $(s, s+e)(y, y) \in P \bowtie E$ so $sx \in P$ or $sy \in P$ and hence $P$is almost S-prime of R associated to $s \in S$ now suppose there is $x, y \in R$ such that $xy = 0$ but $sx \notin P$ and $sy \notin P$ without loss of generality suppose $x \notin Ann(E)$ then there exist $e \in E$ such that $xe \neq 0$ and so we have $(x, x)(y, y + e) \in P \bowtie E - (P \bowtie E)^2$ hence $(s, s + r)(x, x) = (sx, sx + rx) \in P \bowtie E$ or $(s, s+r)(y, y+e) = (sy, sy+ry+se+re) \in P \bowtie E$ which is contradiction so must $x \in Ann(E)$ we can show $y \in Ann(E)$ similarly.
$(iii) \Rightarrow (i)$ Let $(x, x+e)(y, y+r) \in P \bowtie E - (P \bowtie E)^2$ where $(x, x + e)(y, y + r) \in R \bowtie E$ if $xy \neq 0$ then $sx \in P$ or $sy \in P$ and hence $(s, s)(x, x + e) \in P \bowtie E$ or $(s, s)(y, y + r) \in P \bowtie E$. assume $xy = 0$ but $sx \notin P$ and $sy \notin P$ then $x, y \in Ann(E)$ so we get $(x, x + e)(y, y + r) = (0, 0)$ which is contradiction. □

4 Conclusion

In this paper we introduce the concept of almost S-prime ideal as a new generalization of $S-$prime ideal. Also, Number of results concerning this concept and examples are given. Many theorems are proved using the definition and different methods.

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