Abstract: - In this paper, by making use of Borel distribution we introduce a new family $G_{\Sigma}(\delta, \gamma, \lambda, \tau, r)$ of normalized analytic and bi-univalent functions in the open unit disk $U$, which are associated with Horadam polynomials. We establish upper bounds for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions belonging to the analytic and bi-univalent function family which we have introduced here. Furthermore, we establish the Fekete-Szegö problem of functions in this new family.

Key-Words: - Bi-univalent function, Bazilevič function, $\lambda$-Pseudo-starlike function, Borel distribution, Horadam polynomials, Upper bounds, Fekete-Szegö problem.

1 Introduction

Indicate by $A$, the collection of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

Further, assume that $S$ stands for the sub-collection of the set $A$ containing of functions in $U$ satisfying (1) which are univalent in $U$.

A function $f \in A$ is called Bazilevič function in $U$ if (see [26])

$$Re \left\{ \frac{z f'(z)}{f(z)} \right\} > 0, \quad (z \in U, \gamma \geq 0).$$

A function $f \in A$ is called $\lambda$-pseudo-starlike function in $U$ if (see [26])

$$Re \left\{ \frac{z (f'(z))^\lambda}{f(z)} \right\} > 0, \quad (z \in U, \lambda \geq 1).$$

The elementary distributions such as the Poisson, the Pascal, the Logarithmic, the Binomial, the beta negative binomial have been partially studied in Geometric Function Theory from a theoretical point of view (see for example [6, 11, 22, 24, 40]).

Very recently, Wanas and Khuttar [42] introduced the following power series whose coefficients are probabilities of the Borel distribution:

$$M(\tau, z) = z + \sum_{n=2}^{\infty} \frac{(\tau(n-1))^{n-2} e^{-\tau(n-1)}}{(n-1)!} z^n$$

$$\quad (z \in U; \ 0 < \tau \leq 1).$$

We note by the familiar Ratio Test that the radius of convergence of the above series is infinity.
The linear operator $B_\tau : A \to A$ is defined as follows (see [12]):

$$B_\tau f(z) = M(\tau, z) \ast f(z) = z + \sum_{n=2}^{\infty} \frac{\tau(n-1)(n-2)e^{-\tau(n-1)}}{(n-1)!} a_n z^n \in U,$$

where ($\ast$) indicate the Hadamard product of two series.

According to the Koebe One-Quarter Theorem [10], every function $f \in S$ has an inverse $f^{-1}$ defined by $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{2})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \quad (2)$$

A function $f \in A$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. Let $\Sigma$ stands for the class of bi-univalent functions in $U$ given by (1).

Srivastava et al. [29] have actually revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by Bulut [8], Adegani et al. [2], G"uney et al. [12], Srivastava and Wanas [30] and others (see, for example [14, 16, 19, 23, 25, 27, 31, 34, 35, 36, 37, 38, 39]).

We notice that the class $\Sigma$ is not empty. For example, the functions $z, \frac{z}{1-z}, -\log(1-z)$ and $\frac{1}{2} \log \frac{1+z}{1-z}$ are members of $\Sigma$. However, the Koebe function is not a member of $\Sigma$. Until now, the coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$, $(n = 3, 4, \cdots)$ for functions $f \in \Sigma$ is still an open problem.

Let the functions $f$ and $g$ be analytic in $U$. We say that the function $f$ is subordinate to $g$, if there exists a Schwarz function $\omega$ analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$ such that $f(z) = g(\omega(z))$. This subordination is denoted by $f \prec g$ or $f(\omega) \prec g(z)$ $(z \in U)$. It is well known that (see [21]), if the function $g$ is univalent in $U$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f'(U) \subseteq g(U)$.

The Horadam polynomials $h_n(r)$ are defined by the following repetition relation (see [14]):

$$h_n(r) = prh_{n-1}(r) + qh_{n-2}(r) \quad (3)$$

$(r \in \mathbb{R}, n \in \mathbb{N} = \{1, 2, 3, \cdots\}),$

with $h_1(r) = a$ and $h_2(r) = br$, for some real constant $a, b, p$ and $q$. The characteristic equation of repetition relation (3) is $t^2 - prt - q = 0$. This equation has two real roots $x = \frac{pr + \sqrt{p^2r^2 + 4q}}{2}$ and $y = \frac{pr - \sqrt{p^2r^2 + 4q}}{2}$.

**Remark 2.1.** By selecting the particular values of $a, b, p$ and $q$, the Horadam polynomial $h_n(r)$ reduces to several polynomials. Some of them are illustrated below:

1. Taking $a = b = p = q = 1$, we obtain the Fibonacci polynomials $F_n(r)$;
2. Taking $a = 2$ and $b = p = q = 1$, we attain the Lucas polynomials $L_n(r)$;
3. Taking $a = q = 1$ and $b = p = 2$, we have the Pell polynomials $P_n(r)$;
4. Taking $a = b = p = 2$ and $q = 1$, we get the Pell-Lucas polynomials $Q_n(r)$;
5. Taking $a = b = 1, p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $T_n(r)$ of the first kind;
6. Taking $a = 1, b = p = 2$ and $q = -1$, we have the Chebyshev polynomials $U_n(r)$ of the second kind.

These polynomials, the families of orthogonal polynomials and other special polynomials, as well as their generalizations, are potentially important in a variety of disciplines in many of sciences, especially in the mathematics, statistics and physics. For more information associated with these polynomials see [13, 14, 17, 18].

The generating function of the Horadam polynomials $h_n(r)$ (see [15]) is given by

$$\Pi(r, z) = \sum_{n=1}^{\infty} h_n(r)z^{n-1} = \frac{a + (b - ap)r z}{1 - prz - qz^2}. \quad (4)$$

Srivastava et al. [28] have studied the Horadam polynomials in a similar context involving analytic and bi-univalent functions, it was followed by such works as those by Al-Amoush [3], Wanas and Alb Lupaş [43], Abramovich et al. [1] and others (see, for example, [14, 28, 31, 34, 35, 36, 37, 38, 39, 43]).

In this paper we define a subclass $G_\Sigma(\delta, \gamma, \lambda, \tau, r)$ of normalized analytic and bi-univalent function using Borel distribution and Horadam polynomial $h_n(r)$. We obtain Taylor-Maclaurin coefficient inequalities for functions belonging to the defined subclass $G_\Sigma(\delta, \gamma, \lambda, \tau, r)$ and study the famous Fejèke Szegő problem.

## 2 Main Results

We begin this section by defining the family $G_\Sigma(\delta, \gamma, \lambda, \tau, r)$ as follows:

**Definition 2.1.** For $0 \leq \delta \leq 1$, $\gamma \geq 0$, $\lambda \geq 1$, $0 < \tau \leq 1$ and $r \in \mathbb{R}$, a function $f \in \Sigma$ is said to be in the family $G_\Sigma(\delta, \gamma, \lambda, \tau, r)$ if it satisfies the
subordinations:

\[(1 - \delta) \frac{z^{1-\gamma} (B_{\gamma} f(z))' + \delta z (B_{\gamma} f(z))'}{(B_{\gamma} f(z))^{1-\gamma}} < \Pi(r, z) + 1 - a\]

and

\[(1 - \delta) \frac{w^{1-\gamma} (B_{\gamma} g(w))' + \delta w (B_{\gamma} g(w))'}{(B_{\gamma} g(w))^{1-\gamma}} < \Pi(r, w) + 1 - a\]

where \(a\) is a real constant and the function \(g = f^{-1}\) is given by \((3)\).

Note: \(\theta = (1 - \delta)(\gamma + 1) + \delta(2\lambda - 1)\) is used throughout the paper unless otherwise mentioned.

**Theorem 2.1.** For \(0 \leq \delta \leq 1, \gamma \geq 0, \lambda \geq 1, 0 < \tau \leq 1\) and \(r \in \mathbb{R}\), let \(f \in \mathcal{A}\) be in the family \(\mathcal{G}_{\Sigma}(\delta, \gamma, \lambda, \tau, r)\). Then

\[|a_2| \leq \frac{e^\delta |br| \sqrt{2|br|}}{\sqrt{|\varphi(\delta, \gamma, \lambda, \tau)b - 2p|}}, \quad |a_3| \leq \frac{e^{2\tau} |br|}{\tau [(1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)] + \frac{e^{2\tau} b^2}{a^2}},\]

where

\[\varphi(\delta, \gamma, \lambda, \tau) = (1 - \delta)(\gamma + 2)(4\tau + \gamma - 1) + 2\delta (2\tau (3\lambda - 1) + 2\lambda(\lambda - 2) + 1)\]  

**Proof** Let \(f \in \mathcal{G}_{\Sigma}(\delta, \gamma, \lambda, \tau, r)\). Then there are two analytic functions \(u, v : U \rightarrow U\) given by

\[u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \quad (z \in U)\]  

and

\[v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \quad (w \in U),\]

with \(u(0) = v(0) = 0, |u(z)| < 1, |v(w)| < 1, z, w \in U\) such that

\[(1 - \delta) \frac{z^{1-\gamma} (B_{\gamma} f(z))' + \delta z (B_{\gamma} f(z))'}{(B_{\gamma} f(z))^{1-\gamma}} = \Pi(r, u(z)) + 1 - a\]

and

\[(1 - \delta) \frac{w^{1-\gamma} (B_{\gamma} g(w))' + \delta w (B_{\gamma} g(w))'}{(B_{\gamma} g(w))^{1-\gamma}} = \Pi(r, v(w)) + 1 - a\]

Or, equivalently

\[(1 - \delta) \frac{z^{1-\gamma} (B_{\gamma} f(z))' + \delta z (B_{\gamma} f(z))'}{(B_{\gamma} f(z))^{1-\gamma}} = 1 + h_1(r) + h_2(r)u(z) + h_3(r)u^2(z) + \cdots \quad (8)\]

and

\[(1 - \delta) \frac{w^{1-\gamma} (B_{\gamma} g(w))' + \delta w (B_{\gamma} g(w))'}{(B_{\gamma} g(w))^{1-\gamma}} = 1 + h_1(r) + h_2(r)v(w) + h_3(r)v^2(w) + \cdots \quad (9)\]

Combining \((3), (7), (8)\) and \((9)\) yields

\[(1 - \delta) \frac{z^{1-\gamma} (B_{\gamma} f(z))' + \delta z (B_{\gamma} f(z))'}{(B_{\gamma} f(z))^{1-\gamma}} = 1 + h_2(r)u_1 z + \left[ h_2(r)u_2 + h_3(r)u_1^2 \right] z^2 + \cdots \quad (10)\]

and

\[(1 - \delta) \frac{w^{1-\gamma} (B_{\gamma} g(w))' + \delta w (B_{\gamma} g(w))'}{(B_{\gamma} g(w))^{1-\gamma}} = 1 + h_2(r)v_1 w + \left[ h_2(r)v_2 + h_3(r)v_1^2 \right] w^2 + \cdots \quad (11)\]

It is quite well-known that if \(|u(z)| < 1\) and \(|v(w)| < 1, z, w \in U, then

\[|u_i| \leq 1 \quad \text{and} \quad |v_i| \leq 1 \text{ for all } i \in \mathbb{N}. \quad (12)\]

Comparing the corresponding coefficients in \((10)\) and \((11)\), after simplifying, we have

\[[(1 - \delta)(\gamma + 1) + \delta(2\lambda - 1)] e^{-\tau} a_2 = h_2(r)u_1, \quad (13)\]

\[2\tau [(1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)] e^{-2\tau} a_3 + \left\{ \frac{1}{2} (1 - \delta)(\gamma + 2)(\gamma - 1) + \delta(2\lambda(\lambda - 2) + 1) \right\} e^{-2\tau} a_2^2 = h_2(r)u_2 + h_3(r)u_1^2, \quad (14)\]

\[ - [(1 - \delta)(\gamma + 1) + \delta(2\lambda - 1)] e^{-\tau} a_2 = h_2(r)v_1 \quad (15)\]
In view of (17) and (18), we get from (21)
and (19) that
\[ u_1 = -v_1 \]  
(17)
and
\[ 2\theta^2 e^{-2\tau} a_2^2 = h_2^2(r)(u_1^2 + v_1^2). \]  
(18)
If we add (14) to (16), we find that
\[ \phi(\delta, \gamma, \lambda, \tau)e^{-2\tau} a_2^2 = h_2(r)(u_2 + v_2) + h_3(r)(u_2^2 + v_2^2), \]
where \( \phi(\delta, \gamma, \lambda, \tau) \) is given by (3).

Substituting the value of \( u_1^2 + v_1^2 \) from (18) in the right hand side of (19), we deduce that
\[ a_2^2 = \frac{e^{2\tau} h_3^2(r)(u_2 + v_2)}{h_2^2(r)\phi(\delta, \gamma, \lambda, \tau) - 2h_3(r)\theta^2}. \]  
(20)

Further computations using (3), (12) and (20), we obtain
\[ |a_2| \leq \frac{e^{\tau} |br| \sqrt{2|br|}}{\sqrt{|\phi(\delta, \gamma, \lambda, \tau)b - 2\theta^2|^{1/2}|br| - 2qa^2^2|}.} \]

Next, if we subtract (16) from (14), we can easily see that
\[ 2\tau [(1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)] e^{-2\tau} (a_3 - a_2^2) = h_2(r)(u_2 - v_2) + h_3(r)(u_2^2 - v_2^2). \]  
(21)
In view of (17) and (18), we get from (21)
\[ a_3 = \frac{e^{2\tau} h_2(r)(u_2 - v_2)}{2\tau [(1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)] + e^{2\tau} h_3^2(r)(u_1^2 + v_1^2).} \]

Thus applying (3), we obtain
\[ |a_3| \leq \frac{e^{2\tau} |br|}{\tau [(1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)] + e^{2\theta^2} \theta^2}. \]

This completes the proof of Theorem 2.1.

In the next theorem, we discuss the Fekete-Szegő problem for the family \( G_{\Sigma}(\delta, \gamma, \lambda, \tau, r) \).

**Theorem 2.2.** For \( 0 \leq \delta \leq 1, \gamma \geq 0, \lambda \geq 1, 0 < \tau \leq 1 \) and \( r, \mu \in \mathbb{R} \), let \( f \in A \) be in the family \( G_{\Sigma}(\delta, \gamma, \lambda, \tau, r) \). Then
\[ |a_3 - \mu a_2^2| \leq \frac{e^{2\tau} |br|}{\tau [(1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)]}; \]
\[ 2\tau |br| |\psi(\mu, r)|; \]
\[ 2\tau |br| |\psi(\mu, r)|; \]
After some computations, we obtain
\[ |a_3 - \mu a_2^2| \leq \frac{e^{2\tau} |br|}{\tau [(1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)]}. \]

**Proof.** It follows from (20) and (21) that
\[ a_3 - \mu a_2^2 = \frac{e^{2\tau} h_2(r)(u_2 - v_2)}{2\tau [(1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)]} + (1 - \mu) a_2^2 \]
\[ = \frac{2\tau [(1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)]}{2\tau [(1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)]} + \psi(\mu, r) + \frac{e^{2\tau}}{2\tau [(1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)]} v_2, \]
where
\[ \psi(\mu, r) = \frac{e^{2\tau} h_2(r)(1 - \mu)}{h_2^2(r)\phi(\delta, \gamma, \lambda, \tau) - 2h_3(r)\theta^2}. \]

According to (3), we find that
\[ |a_3 - \mu a_2^2| \leq \frac{e^{2\tau} |br|}{\tau [(1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)]}; \]
\[ 2\tau |br| |\psi(\mu, r)|; \]
\[ 2\tau |br| |\psi(\mu, r)|; \]
After some computations, we obtain
\[ |a_3 - \mu a_2^2| \leq \frac{e^{2\tau} |br|}{\tau [(1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)]}. \]
Putting $\mu = 1$ in Theorem 2, we obtain the following result:

**Corollary 2.1.** For $0 \leq \delta \leq 1$, $\gamma \geq 0$, $\lambda \geq 1$, $0 < \tau \leq 1$ and $r \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the family $\mathcal{G}_2(\delta, \gamma, \lambda, \tau, r)$. Then
\[
|a_2 - a_3| \leq e^{2\tau |br|} \frac{(1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)}{\tau (1 - \delta)(\gamma + 2) + \delta(3\lambda - 1)}.
\]

**3 Conclusion**

The fact that we can find many unique and effective uses of a large variety of interesting functions and specific polynomial in Geometric Function Theory provided the primary inspiration for our analysis in this article. The primary objective was to create a new family $\mathcal{G}_2(\delta, \gamma, \lambda, \tau, r)$ of normalized analytic and bi-univalent function defined by Borel distribution and also using the Horadam polynomial $h_n(r)$, which are given by the recurrence relation [3] and generating function $P(r, z)$ in [4]. We generate Taylor-Maclaurin coefficient inequalities for functions belonging to this newly introduced bi-univalent function family $\mathcal{G}_2(\delta, \gamma, \lambda, \tau, r)$ and viewed the famous Fekete-Szegő problem.

**References:**


**Contribution of individual authors to the creation of a scientific article**

**ghostwriting policy**

S. R. Swamy - conceptualization, methodology, formal analysis.

Alina Alb Lupas - writing—review and editing, supervision, funding acquisition.

Abbas Kareem Wanas - software, validation, data curation, writing—original draft preparation, project administration.

J. Nirmala - investigation, resources, visualization.

All authors have read and agreed to the published version of the manuscript.


**Sources of funding for research**

**presented in a scientific article or scientific article itself**

Report potential sources of funding if there is any

**Creative Commons Attribution License 4.0 (Attribution 4.0 International, CC BY 4.0)**

This article is published under the terms of the Creative Commons Attribution License 4.0

[https://creativecommons.org/licenses/by/4.0/deed.en_US](https://creativecommons.org/licenses/by/4.0/deed.en_US)