New Results for Arithmetic–Geometric Mean Inequality and Singular Values of Matrices

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Abstract: Matrix theory is very popular in different kind of sciences such as engineering, architecture, physics, chemistry, computer science, IT, so on as well as mathematics many theoretical results dealing with the structure of the matrices even this topic seems easy to work. That is why many scientists still consider some open problem in matrix theory.

In this paper, generalizations of the arithmetic-geometric mean inequality is presented for singular values related to block matrices. Singular values are also given for sums, products and direct sums of the matrices.

Key-words: Arithmetic-Geometric Mean, Hermitian Matrix, Singular Values, Positive definite Matrix, Block Matrix.

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1. Introduction and Preliminaries

In recent years, the Arithmetic-Geometric (A-G) mean inequality for singular values was introduced by Bhatia and Kittaneh [3]. On the other hand, Zhan [11] and Tao [9] have presented equivalent inequalities; also Hirzallah [7] described a lower bound of singular values of block matrices and authors in [4] proved an interesting singular value inequality.

Additionally, improvements and generalizations of the A-G mean inequality for unitarily invariant norms were presented in [10]. Several inequalities for singular values related to block positive semidefinite matrices were proved by Burqan and Kittaneh [5].

For readers, the notions given in this paper can be found in almost every book ([2, 8, 11]).

Definition 1.1. Let A be a complex matrix with degree n. If A is symmetric and $\forall\, u \in ₵^n$, $u^tAu > 0$, then it is called a positive definite. It is called by positive semidefinite if $\forall\, u \in ₵^n$, $u^tAu \geq 0$.

If A and B are Hermitian complex matrices with degree n, we use the notation $A \geq B$ to mean $A - B$ is positive semidefinite.

Definition 1.2. Let A be a complex matrix with degree n. The modulus of a matrix A is defined by $|A| = \sqrt{A^*A}$, where A^* is a complex conjugate of A.

As consequence of the Fundamental Theorem of Algebra for application to the characteristic polynomial, we obtain every $n \times n$ matrix has exactly n complex eigenvalues, counted with multiplicity. So, we can define followings:

Definition 1.3. Let A be a complex matrix with degree n. The eigenvalues of the modulus of A are named by the singular values of A and denoted by $s_1(A), s_2(A), \dots, s_n(A)$. They are also arranged as $s_1(A) \geq s_2(A) \geq \dots \geq s_n(A)$.

There are some properties of the singular values and eigenvalues of A and we give some of them as follows:

1. If A is a complex matrix with degree n, then
$$s_j(A) = s_j(A^*) = s_j(|A|), \quad j = 1, \dots, n.$$
2. If A is Hermitian complex matrix with degree n, then the eigenvalues of A satisfy
$$\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A).$$

By the way, Weyl's monotonicity principle introduced motivating relations for eigenvalues of Hermitian matrices, which says that if A, B are Hermitian complex matrices with degree n and $A \geq B$, then
$$\lambda_j(A) \geq \lambda_j(B), \quad j = 1, \dots, n.$$
Also, if A have singular values $s_1(A) \geq s_2(A) \geq \dots \geq s_n(A) \geq 0$, and eigenvalues ordered so that $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$, then
$$|\lambda_1(A)\lambda_2(A) \dots \lambda_k(A)| \leq s_1(A)s_2(A) \dots s_k(A)$$
for $k = 1, \dots, n$ with equality for $k = n$.

Definition 1.4. A block diagonal matrix A is a square diagonal matrix where the diagonal elements are square matrices of any size and the off diagonal elements are zero. If we summarize this definition, we can say following item; assume that A, B are complex matrices with degree n, the direct sums of A and B is denoted by $A \oplus B$ and defined as $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

It is well known that

$s_j(A) \leq s_j(B)$ if and only if $s_j(A \oplus A) \leq s_j(B \oplus B)$,

$j = 1, \dots, 2n$.

Moreover, if $s_j(A_1) \leq s_j(B_1)$ and $s_j(A_2) \leq s_j(B_2)$, then $s_j(A_1 \oplus A_2) \leq s_j(B_1 \oplus B_2)$, $\quad j = 1, \dots, 2n$.

In the last years, mathematicians worked on several special inequalities for eigenvalues and singular values of the complex matrices.

Bhatia and Kittaneh [3] presented the A-G mean inequality for singular values as follows:

If A, B are complex matrices with degree n, then
$$2s_j(AB^*) \leq s_j(A^*A + B^*B), j = 1, \dots, n. \tag{1.1}$$

For positive semidefinite complex matrices A, B with degree n, Zhan [12], has proved
$$s_j(A - B) \leq s_j(A \oplus B), j = 1, \dots, n. \tag{1.2}$$

Tao [9] proved that if A, B, C are complex matrices with degree n such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then
$$2s_j(B) \leq s_j \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}, j = 1, \dots, n \tag{1.3}$$

furthermore, he pointed out that the previous three inequalities are equivalent.

Hirzallah [7] gave a lower bound for singular values of 2×2 block matrices as follows:

If A, B, C, D are complex matrices with degree n, then
$$2s_j(AB^* + CD^*) \leq s_j^2 \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \; j = 1, \dots, n. \tag{1.4}$$

On other hand, Authors in [4] obtained that if A, B are complex matrices with degree n such that A is Hermitian, $B \geq 0$ and $\pm A \leq B$, then
$$s_j(A) \leq s_j(B \oplus B), \tag{1.5}$$
$$s_j(AB^* + BA^*) \leq s_j\big((AA^* + BB^*) \oplus (AA^* + BB^*)\big) \tag{1.6}$$

Some equivalent inequalities of (1.1) were obtained by researchers such as [1] if A, B, C are complex matrices with degree n such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then
$$s_j(B) \leq s_j(A \oplus C), \qquad j = 1, \dots, n. \tag{1.7}$$

The following singular value inequality for sums and direct sums of matrices was given by Buraqan and Kittaneh [5], If A, B, C, X, Y are complex matrices with degree n such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then

$$s_j(X^*BY + Y^*B^*X) \leq s_j\big((X^*AX + Y^*CY) \oplus (X^*AX + Y^*CY)\big), \quad j = 1, \dots, n. \tag{1.8}$$

In this research, Generalizations of the A-G mean inequality for singular values (1.1) are established. Also, other related inequalities to sums, direct sums and products of matrices are considered.

2. Main Results:

The following lemma is essential in our analysis, relates the singular values of a matrix K with the eigenvalues of $\begin{bmatrix} 0 & K \\ K^* & 0 \end{bmatrix}$.

Lemma 2.1 [2]**:** If K is a complex matrix with degree n and rank r, then the eigenvalues of $\begin{bmatrix} 0 & K \\ K^* & 0 \end{bmatrix}$ are

$s_1(K), \dots, s_r(K), 0, \dots, 0, -s_r(K), \dots, -s_1(K)$.

Theorem 2.1: Let A, B, C, D, X, Y be complex matrices with degree n. Then

$$2s_j(XA^*BY^* \oplus YC^*DX^*) \leq s_j\big((AX^*XA^* + BY^*YB^*) \oplus (CY^*YC^* + DX^*XD^*)\big),$$

$j = 1, \dots, 2n$.

Proof. Let $W = \begin{bmatrix} AX^* & 0 & BY^* & 0 \\ 0 & CY^* & 0 & DX^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$$Z = \begin{bmatrix} 0 & 0 & XA^*BY^* & 0 \\ 0 & 0 & 0 & YC^*DX^* \\ YB^*AX^* & 0 & 0 & 0 \\ 0 & XD^*CY^* & 0 & 0 \end{bmatrix}.$$

Then

$$WW^* = \begin{bmatrix} AX^*XA^* + BY^*YB^* & 0 & 0 & 0 \\ 0 & CY^*YC^* + DX^*XD^* & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$W^*W = \begin{bmatrix} XA^*AX^* & 0 & XA^*BY^* & 0 \\ 0 & YC^*CY^* & 0 & YC^*DX^* \\ YB^*AX^* & 0 & YB^*BY^* & 0 \\ 0 & XD^*CY^* & 0 & XD^*DX^* \end{bmatrix}$$

and

$$\begin{aligned} W^*W - 2Z &= \begin{bmatrix} XA^*AX^* & 0 & -XA^*BY^* & 0 \\ 0 & YC^*CY^* & 0 & -YC^*DX^* \\ -YB^*AX^* & 0 & YB^*BY^* & 0 \\ 0 & -XD^*CY^* & 0 & XD^*DX^* \end{bmatrix} \\ &= \begin{bmatrix} AX^* & 0 & -BY^* & 0 \\ 0 & CY^* & 0 & -DX^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^* \begin{bmatrix} AX^* & 0 & -BY^* & 0 \\ 0 & CY^* & 0 & -DX^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\geq 0. \end{aligned}$$

Weyl's monotonicity principle yields

$$2\lambda_j(Z) \leq \lambda_j(W^*W), \quad j = 1, \dots, 2n.$$

The eigenvalues of W^*W and WW^* are

$s_j\big((AX^*XA^* + BY^*YB^*) \oplus (CY^*YC^* + DX^*XD^*)\big)$,

$j = 1, \dots, 2n$.

By Lemma 2.1, the 2n eigenvalues of Z are

$s_j(XA^*BY^* \oplus YC^*DX^*)$, $j = 1, \dots, 2n$.

Therefore,

$$2s_j(XA^*BY^* \oplus YC^*DX^*) \leq s_j\big((AX^*XA^* + BY^*YB^*) \oplus (CY^*YC^* + DX^*XD^*)\big),$$

$j = 1, \dots, 2n$.■

Let $C = D = 0$, $X = Y = I$ in Theorem 2.1, we get inequality (1.1).

Another version of A-G mean inequality for block matrices is established in the following result.

Theorem 2.2: Let A, B, C, D, X, Y be complex matrices with degree n. Then

$$\begin{aligned} &s_j(XA^*BY^* \oplus YC^*DX^*) \\ &\leq s_j(X|A|^2X^* \oplus Y|B|^2Y^* \oplus Y|C|^2Y^* \oplus X|D|^2X^*), \end{aligned}$$

$j = 1, \dots, 2n$.

Proof. Let $W = \begin{bmatrix} AX^* & 0 & BY^* & 0 \\ 0 & CY^* & 0 & DX^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and

$$Z = \begin{bmatrix} 0 & 0 & XA^*BY^* & 0 \\ 0 & 0 & 0 & YC^*DX^* \\ YB^*AX^* & 0 & 0 & 0 \\ 0 & XD^*CY^* & 0 & 0 \end{bmatrix}.$$

Then

$$W^*W = \begin{bmatrix} XA^*AX^* & 0 & XA^*BY^* & 0 \\ 0 & YC^*CY^* & 0 & YC^*DX^* \\ YB^*AX^* & 0 & YB^*BY^* & 0 \\ 0 & XD^*CY^* & 0 & XD^*DX^* \end{bmatrix} \geq 0$$

and

$$W^*W - 2Z = \begin{bmatrix} XA^*AX^* & 0 & -XA^*BY^* & 0 \\ 0 & YC^*CY^* & 0 & -YC^*DX^* \\ -YB^*AX^* & 0 & YB^*BY^* & 0 \\ 0 & -XD^*CY^* & 0 & XD^*DX^* \end{bmatrix}$$

$$= \begin{bmatrix} AX^* & 0 & -BY^* & 0 \\ 0 & CY^* & 0 & -DX^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^* \begin{bmatrix} AX^* & 0 & -BY^* & 0 \\ 0 & CY^* & 0 & -DX^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0.$$

From the previous two inequalities, we get

$$\begin{bmatrix} XA^*AX^* & 0 & 0 & 0 \\ 0 & YC^*CY^* & 0 & 0 \\ 0 & 0 & YB^*BY^* & 0 \\ 0 & 0 & 0 & XD^*DX^* \end{bmatrix} \geq \pm \begin{bmatrix} 0 & 0 & XA^*BY^* & 0 \\ 0 & 0 & 0 & YC^*DX^* \\ YB^*AX^* & 0 & 0 & 0 \\ 0 & XD^*CY^* & 0 & 0 \end{bmatrix}.$$

By applying inequalities (1.5), we get

$$s_j\big((XA^*BY^* \oplus YC^*DX^*) \oplus (XA^*BY^* \oplus YC^*DX^*)^*\big) \leq s_j\begin{pmatrix} (XA^*AX^* \oplus YB^*BY^* \oplus YC^*CY^* \oplus XD^*DX^*) \oplus \\ (XA^*AX^* \oplus YB^*BY^* \oplus YC^*CY^* \oplus XD^*DX^*) \end{pmatrix}.$$

Thus,

$$s_j(XA^*BY^* \oplus YC^*DX^*) \leq s_j(XA^*AX^* \oplus YB^*BY^* \oplus YC^*CY^* \oplus XD^*DX^*). \blacksquare$$

Several inequalities of singular values for direct sums and products of matrices are presented in the following theorems.

Theorem 2.3: Let A, B, C, X, Y be complex matrices with degree n such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$. Then

$$s_j(XBY^*) \leq s_j(XAX^* \oplus YCY^*), \qquad j = 1 \dots, n. \quad (2.1)$$

Proof: Consider $T = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$ and $D = \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix}$. Then

$$T\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}T^* = \begin{bmatrix} XAX^* & XBY^* \\ YB^*X^* & YCY^* \end{bmatrix} \geq 0$$

and

$$D\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}D^* = \begin{bmatrix} XAX^* & -XBY^* \\ -YB^*X^* & YCY^* \end{bmatrix} \geq 0.$$

Thus,

$$\begin{bmatrix} XAX^* & 0 \\ 0 & YCY^* \end{bmatrix} \geq \pm \begin{bmatrix} 0 & -XBY^* \\ -YB^*X^* & 0 \end{bmatrix}.$$

By applying inequality (1.5), we get

$$s_j(XBY^* \oplus YB^*X^*) \leq s_j\big((XAX^* \oplus YCY^*) \oplus (XAX^* \oplus YCY^*)\big)$$

This is equivalent to saying that

$$s_j(XBY^*) \leq s_j(XAX^* \oplus YCY^*). \blacksquare$$

Let $X = Y = I$ in inequality (2.1), we get inequality (1.7).

Since $\begin{bmatrix} AA^* + CC^* & AB^* + CD^* \\ BA^* + DC^* & BB^* + DD^* \end{bmatrix} \geq 0$ for any complex matrices A, B, C, D, we have the following theorem.

Theorem 2.4: Let A, B, C, D be complex matrices with degree n. Then

$$s_j(AB^* + CD^*) \leq s_j\big((AA^* + CC^*) \oplus (BB^* + DD^*)\big), \qquad j = 1,2, \dots n.$$

Let $C = B$ and $D = A$, in Theorem 2.4, to get inequality (1.6),

$$s_j(AB^* + BA^*) \leq s_j\big((AA^* + BB^*) \oplus (AA^* + BB^*)\big).$$

Let $C = D = 0$, in Theorem 2.4, we get

$$s_j(AB^*) \leq s_j\big((AA^*) \oplus (BB^*)\big).$$

Note that for any matrix $T = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$, where A, B, C, D are complex matrices with degree n, we have

$$s_j(T) = s_j(T^*) = \lambda_j^{\frac{1}{2}}(TT^*) = \lambda_j^{\frac{1}{2}}\begin{pmatrix} AA^* + CC^* & AB^* + CD^* \\ BA^* + DC^* & BB^* + DD^* \end{pmatrix}$$

$$= s_j^{\frac{1}{2}} \begin{pmatrix} AA^* + CC^* & AB^* + CD^* \\ BA^* + DC^* & BB^* + DD^* \end{pmatrix}.$$

So by inequality (1.3), we get Hirzallah inequality (1.4),

$$2s_j(AB^* + CD^*) \leq s_j{}^2 \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

3. Conclusion

In 1990, arithmetic geometric mean inequality was proven for a version of the matrix. Since then, some new demonstrations, developments and related issues have been worked actively. In this work, a generalization of the Arithmetic and Geometric mean inequality is found out and proved. Moreover, various related inequalities to sums and products of matrices are investigated. One may consider our useful results and apply to some open problems of the advancing subject of matrix inequalities.

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