Characterizations of graded rings over which every graded semi-primary ideal is graded 1-absorbing primary

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Abstract: Let G be a group with identity e and R be a commutative G-graded ring with nonzero unity 1. Graded semi-primary and graded 1-absorbing primary ideals have been investigated and examined by several authors as generalizations of graded primary ideals. However, these three concepts are different. In this article, we characterize graded rings over which every graded semi-primary ideal is graded 1-absorbing primary and graded rings over which every graded 1-absorbing primary ideal is graded primary.

Key–Words: Graded 1-absorbing primary ideals; graded primary ideals; graded semi-primary ideals.

1 Introduction

The study of graded rings comes into being normally out of the study of affine schemes and admits them to establish and consolidate arguments by induction. However, this is not just an algebraic deception. The concept of grading in algebra, in particular graded modules is fundamental in the study of homological aspect of rings. Plenty of the contemporary growth of the commutative algebra give preference to graded rings. Graded rings play a principal role in algebraic geometry and commutative algebra. Gradings come into sight in many circumstances, both in elementary and advanced level. In recent years, rings with a group-graded structure have become progressively substantial and as a consequence, the graded analogues of various concepts are extensively studied.

Graded primary ideals have been introduced by Refai and Al-Zoubi in [1]. A proper graded ideal Q of R is said to be graded primary if whenever $x, y \in h(R)$ with $xy \in Q$, then $x \in Q$ or $y \in Grad(Q)$. In this case $P = Grad(Q)$ is a graded prime ideal of R, and Q is said to be graded P-primary. In [2], Al-Zoubi and Sharafat introduced a generalization of graded primary ideals called graded 2-absorbing primary ideals. A proper graded ideal I of R is called a graded 2-absorbing primary ideal of R if whenever $x, y, z \in h(R)$ and $xyz \in I$, then $xy \in I$ or $xz \in Grad(I)$ or $yz \in Grad(I)$. The concept of graded 2-absorbing primary ideals is generalized in many ways, for example, see [3, 4]. Recently in [5], we consider a new class of graded idea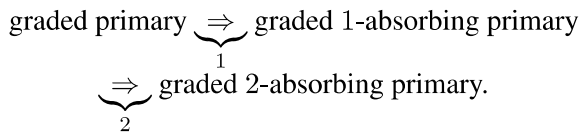ls called the class of graded 1-absorbing primary ideals. A proper graded ideal P of R is said to be a graded 1-absorbing primary ideal of R if whenever nonunit elements $x, y, z \in h(R)$ such that $xyz \in P$, then $xy \in P$ or $z \in Grad(P)$. Clearly, every graded primary ideal is graded 1-absorbing primary ideal. The next example shows that the converse is not true in general.

Example 1. *Consider $R = K[X, Y]$, where K is a field, and $G = \mathbb{Z}$. Then R is G-graded by $R_n = \bigoplus_{i+j=n, i,j \geq 0} KX^iY^j$ for all $n \in \mathbb{Z}$. Note that $deg(X) = deg(Y) = 1$. Consider the graded ideal $P = \langle X^2, XY \rangle$ of R. Then $Grad(P) = \langle X \rangle$. Since for $X.Y.X \in P$, either $X.Y \in P$ or $X \in Grad(P)$, P is a graded 1-absorbing primary ideal of R. On the other hand, P is not graded primary ideal of R by ([6], Example 2.11).*

Also, it is clear that every graded 1-absorbing primary ideal is graded 2-absorbing primary ideal. The next example shows that the converse is not true in general.

Example 2. *Let $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Consider $P = 12R$. Then as $12 \in h(R)$, P is a graded ideal of R. By ([2], Example 2.2 (ii)), P is a graded 2-absorbing primary ideal of R. On the other hand, $2, 3 \in h(R)$ such that $2.2.3 \in P$, but neither $2.2 \in P$ nor $3 \in Grad(P)$. So, P is not graded 1-absorbing primary ideal of R.*

So, we can state the following:

graded primary $\underbrace{\Rightarrow}_{1}$ graded 1-absorbing primary

$\underbrace{\Rightarrow}_{2}$ graded 2-absorbing primary.

And the arrows (1) and (2) are irreversible by Example 1 and Example 2.

In [7], a proper graded ideal P of R is said to be a graded semi-primary ideal of R if whenever $x, y \in h(R)$ such that $xy \in P$, then either $x \in Grad(P)$ or $y \in Grad(P)$. In this article, we prove that P is a graded semi-primary ideal of R if and only if $Grad(P)$ is a graded prime ideal of R (Proposition 4). Then we show that if P is a graded 1-absorbing primary ideal of R, then P is a graded semi-primary ideal of R (Proposition 5). After that, we prove that if R is a principal ideal domain, then the three concepts; graded primary, graded semi-primary and graded 1-absorbing primary ideals will be equivalent (Proposition 7).

Motivated by Proposition 7, our goal in this article is following [8] to characterize graded rings over which every graded semi-primary ideal is graded 1-absorbing primary and graded rings over which every graded 1-absorbing primary ideal is graded primary.

1.1 Preliminaries

Throughout this article, all rings are commutative with nonzero unity 1. Let G be a group with identity e. Then a ring R is said to be G-graded if $R = \bigoplus_{g \in G} R_g$ with $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written uniquely as $\sum_{g \in G} x_g$, where x_g is the component of x in R_g. Also, we set $h(R) = \bigcup_{g \in G} R_g$. Moreover, it has been proved in [9] that R_e is a subring of R and $1 \in R_e$. Let I be an ideal of a graded ring R. Then I is said to be a graded ideal if $I = \bigoplus_{g \in G}(I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g \in G} x_g$, where $x_g \in I$ for all $g \in G$. An ideal of a graded ring need not be graded (see [9]).

Let I be a proper graded ideal of R. Then the graded radical of I is $Grad(I)$ and is defined by:

$$x = \sum_{g \in G} x_g \in Grad(I) \text{ if for every } g \in G, \text{ there}$$

exists $n_g \in \mathbb{N}$ such that $x_g^{n_g} \in I$. Note that $Grad(I)$ is always a graded ideal of R (see [10]).

Proposition 3. ([11]) Let R be a G-graded ring.

1. If I and J are graded ideals of R, then $I + J$, IJ and $I \bigcap J$ are graded ideals of R.

2. If $a \in h(R)$, then Ra is a graded ideal of R.

Let R be a G-graded ring and I be a graded ideal of R. Then R/I is a G-graded ring by $(R/I)_g = (R_g + I)/I$ for all $g \in G$. Moreover, if P is an ideal of R containing I, then P is a graded ideal of R if and only if P/I is a graded ideal of R/I, see ([12], Lemma 3.2).

Let R and S be two G-graded rings. Then $T = R \times S$ is a G-graded ring by $T_g = R_g \times S_g$ for all $g \in G$. Furthermore, $P \times K$ is a graded ideal of $R \times S$ if and only if P is a graded ideal of R and K is a graded ideal of S ([7], Lemma 4).

If R is a G-graded ring and $S \subseteq h(R)$ is a multiplicative set, then $S^{-1}R$ is a G-graded ring with $(S^{-1}R)_g = \left\{ \frac{a}{s}, a \in R_h, s \in S \cap R_{hg^{-1}} \right\}$ for all $g \in G$. Moreover, if I is a graded ideal of R, then $S^{-1}I$ is a graded ideal of $S^{-1}R$ [9].

2 Graded Rings over which every Graded Semi-Primary Ideal is Graded 1-Absorbing Primary

In this section, we characterize graded ring R over which every graded semi-primary ideal is graded 1-absorbing primary. The next proposition has been proved in ([7], Lemma 1) for $\mathbb{Z}$-graded rings. In fact, it is true for any G-graded ring.

Proposition 4. *Let R be a graded ring and P be a graded ideal of R. Then P is a graded semi-primary ideal of R if and only if $Grad(P)$ is a graded prime ideal of R.*

Proof: Suppose that P is a graded semi-primary ideal of R. Let $x, y \in h(R)$ such that $xy \in Grad(P)$. Then $(xy)^n = x^n y^n \in P$ for some positive integer n. Since P is graded semi-primary, we have $x^n \in Grad(P)$ or $y^n \in Grad(P)$, and then $x^{nk} \in P$ or $y^{nk} \in P$ for some positive integer k, which implies that $x \in Grad(P)$ or $y \in Grad(P)$. Hence, $Grad(P)$ is a graded prime ideal of R. Conversely, let $x, y \in h(R)$ such that $xy \in P$. Then as $P \subseteq Grad(P)$, $xy \in Grad(P)$. Since $Grad(P)$ is graded prime, we have $x \in Grad(P)$ or $y \in Grad(P)$. Hence, P is a graded semi-primary ideal of R. $\square$

Proposition 5. *Let R be a graded ring and P be a graded ideal of R. If P is a graded 1-absorbing primary ideal of R, then P is a graded semi-primary ideal of R.*

Proof: Let $a, b \in h(R)$ such that $ab \in Grad(P)$. We may assume that a, b are nonunit elements of R. Let $k \geq 2$ be an even positive integer such that $(ab)^k \in P$. Then $k = 2s$ for some positive integer $s \geq 1$. Since $(ab)^k = a^k b^k = a^s a^s b^k \in P$ and P is a graded 1-absorbing primary ideal of R, we conclude that $a^s a^s = a^k \in P$ or $b^k \in P$. Hence, $a \in Grad(P)$ or $b \in Grad(P)$. Thus $Grad(P)$ is a graded prime ideal of R, and then P is a graded semi-primary ideal of R by Proposition 4. $\qquad\square$

Corollary 6. *Let R be a graded ring and P be a graded ideal of R. If P is a graded 1-absorbing primary ideal of R, then $Grad(P)$ is a graded prime ideal of R.*

Proof: Apply Proposition 4 and Proposition 5. $\quad\square$

Totally, we can state the following:

$$\text{graded primary} \underset{3}{\Rightarrow} \text{graded 1-absorbing primary}$$
$$\underset{4}{\Rightarrow} \text{graded semi-primary.}$$
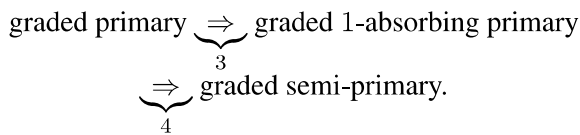
By Example 1, arrow (3) is irreversible. Now, let R be a graded ring and P be a graded 1-absorbing primary ideal of R which is not graded primary. Set $T = R \times R$ and $Q = P \times R$. Then Q is a graded ideal of T by ([7], Lemma 4) with $Grad(Q) = Grad(P \times R) = Grad(P) \times R$ is a graded prime ideal of T since $Grad(P)$ is a graded prime ideal of R by Corollary 6. Thus, Q is a graded semi-primary ideal of T by Proposition 4. On the other hand, it is well known that graded primary ideals of T are $I \times R$ and $R \times I$ with I is a graded primary ideal of R. Hence, Q is not a graded primary ideal of T. In fact, we could not give an example showing that arrow (4) is irreversible. However, if R is a principal ideal domain, then the three concepts will be equivalent as one can see in the next proposition that should be compared with ([7], Proposition 2).

Proposition 7. *Let R be a principal ideal domain. If R is graded and P is a graded ideal of R, then the following statements are equivalent:*

1. P is a graded primary ideal of R.

2. P is a graded 1-absorbing primary ideal of R.

3. P is a graded semi-primary ideal of R.

Proof: $(1) \Rightarrow (2)$: Clear.

$(2) \Rightarrow (3)$: Proposition 5.

$(3) \Rightarrow (1)$: Suppose that P is a graded semi-primary ideal of R. Then by Proposition 4, $Grad(P)$ is a graded prime ideal, and then as R is a principal ideal domain, $Grad(P)$ is a graded maximal ideal of R, which implies that P is a graded primary ideal of R by ([1], Proposition 1.11). $\qquad\square$

In [12], a graded ring R is said to be a graded domain if it has no homogeneous zero divisors. Clearly, every domain is a graded domain, but the converse is not necessarily true, see ([12], Example 3.6).

Proposition 8. *Let R be a graded domain. If every nonzero graded prime ideal of R is graded maximal, then every graded semi-primary ideal of R is graded primary.*

Proof: Let P be a graded semi-primary ideal of R. If $P = \{0\}$, then P is a graded prime ideal of R, and then P is a graded primary ideal of R. If $P \neq \{0\}$, then $Grad(P)$ is a nonzero graded prime ideal of R by Proposition 4, and then $Grad(P)$ is a graded maximal ideal of R by assumption, and hence P is a graded primary ideal of R by ([1], Proposition 1.11). $\qquad\square$

Corollary 9. *Let R be a graded domain. If every nonzero graded prime ideal of R is graded maximal, then every graded semi-primary ideal of R is graded 1-absorbing primary.*

Clearly, if every graded semi-primary ideal is graded primary, then every graded semi-primary ideal is graded 1-absorbing primary. However, we give an example that introduces a case where every graded semi-primary ideal is graded 1-absorbing primary, but there is a graded semi-primary ideal which is not graded primary:

Example 10. *Consider $T = K[X, Y]$, where K is a field, and $G = \mathbb{Z}$. Then T is G-graded by $T_n = \bigoplus_{i+j=n, i,j \geq 0} KX^i Y^j$ for all $n \in \mathbb{Z}$. Consider the graded ideal $J = \langle X^2, XY \rangle$ of T. Then $R = T/J$ is a graded ring and $P = \langle X, Y \rangle / J$ is a graded prime ideal of R. So, $S = h(R) - P$ is a multiplicative subset of $h(R)$, and then $S^{-1}R$ is a graded local ring with graded maximal ideal $M = \langle \frac{\overline{X}}{1}, \frac{\overline{Y}}{1} \rangle$. Note that, $P = Grad(\{0_{S^{-1}R}\}) = \langle \frac{\overline{X}}{1} \rangle$ is a graded prime ideal of $S^{-1}R$ which is not graded maximal, and $PM = \{0_{S^{-1}R}\}$. Let I be a graded semi-primary ideal of $S^{-1}R$. Then $Grad(I)$ is a graded prime ideal of $S^{-1}R$ by Proposition 4, and then either $Grad(I) = M$ or $Grad(I) = Grad(\{0_{S^{-1}R}\})$. If $Grad(I) = M$, then I is graded primary by ([1], Proposition 1.11), and so I is graded 1-absorbing primary. Suppose that $Grad(I) = Grad(\{0_{S^{-1}R}\})$. Assume that $a, b, c \in h(S^{-1}R)$ be nonunit elements such that $abc \in I$ and $c \notin Grad(\{0_{S^{-1}R}\})$. Then either $a \in Grad(\{0_{S^{-1}R}\})$ or $b \in Grad(\{0_{S^{-1}R}\})$, which*

implies in both cases that $ab = 0_{S^{-1}R} \in I$. Hence, I is a graded 1-absorbing primary ideal of $S^{-1}R$. Consequently, every graded semi-primary ideal of $S^{-1}R$ is graded 1-absorbing primary. On the other hand, $Grad(\{0_{S^{-1}R}\})$ is a graded prime ideal of $S^{-1}R$, so $\{0_{S^{-1}R}\}$ is a graded semi-primary ideal of $S^{-1}R$ by Proposition 4, but $\{0_{S^{-1}R}\}$ is not a graded primary ideal of $S^{-1}R$ since $\frac{\overline{X}}{1}, \frac{\overline{Y}}{1} \in h(S^{-1}R)$ with $\frac{\overline{X}}{1} \cdot \frac{\overline{Y}}{1} = 0_{S^{-1}R}, \frac{\overline{X}}{1} \neq 0_{S^{-1}R}$ and $\frac{\overline{Y}}{1} \notin Grad(\{0_{S^{-1}R}\})$.

Let R and S be two G-graded rings. In [9], a ring homomorphism $f : R \to S$ is said to be graded homomorphism if $f(R_g) \subseteq S_g$ for all $g \in G$.

Proposition 11. *[5] Let R and S be G-graded rings and $f : R \to S$ be a graded homomorphism such that $f(1_R) = 1_S$. Then the following hold:*

1. *If K is a graded 1-absorbing primary ideal of S and $f(x)$ is a nonunit element of S for every nonunit element x of R, then $f^{-1}(K)$ is a graded 1-absorbing primary ideal of R.*

2. *If P is a graded 1-absorbing primary ideal of R and f is surjective with $Ker(f) \subseteq P$, then $f(P)$ is a graded 1-absorbing primary ideal of S.*

Proof:

1. By ([13], Lemma 3.11 (1)), $f^{-1}(K)$ is a graded ideal of R. Let $x, y, z \in h(R)$ be nonunit elements such that $xyz \in f^{-1}(K)$. Then $f(x), f(y), f(z) \in h(S)$ are nonunit elements such that $f(x)f(y)f(z) = f(xyz) \in K$. Since K is a graded 1-absorbing primary ideal of S, we have that $f(xy) = f(x)f(y) \in K$ or $f(z) \in Grad(K)$, which implies that $xy \in f^{-1}(K)$ or $z \in f^{-1}(Grad(K)) = Grad(f^{-1}(K))$. Thus, $f^{-1}(K)$ is a graded 1-absorbing primary ideal of R.

2. By ([13], Lemma 3.11 (2)), $f(P)$ is a graded ideal of S. Let $a, b, c \in h(S)$ be nonunit elements such that $abc \in f(P)$. Then since f is surjective, there exist nonunit elements $x, y, z \in h(R)$ such that $f(x) = a, f(y) = b$ and $f(z) = c$. Now, $f(xyz) = f(x)f(y)f(z) = abc \in f(P)$. Since $Ker(f) \subseteq P$, we have that $xyz \in P$. Since P is a graded 1-absorbing primary ideal of R, we have that $xy \in P$ or $z \in Grad(P)$, which implies that $ab = f(x)f(y) = f(xy) \in f(P)$ or $c = f(z) \in f(Grad(P)) = Grad(f(P))$ as f is surjective and $Ker(f) \subseteq P$. Hence, $f(P)$ is a graded 1-absorbing primary ideal of S.

□

Corollary 12. *[5] Let P and K be proper graded ideals of a graded ring R with $K \subseteq P$. If $U(R/K) = \{a + K : a \in U(R)\}$, then P is a graded 1-absorbing primary ideal of R if and only if P/K is a graded 1-absorbing primary ideal of R/K.*

Proof: Define $f : R \to R/K$ by $f(x) = x + K$. Then f is surjective graded homomorphism and $f(1_R) = 1_{R/K}$. Suppose that P is a graded 1-absorbing primary ideal of R. Since f is surjective and $Ker(f) = K \subseteq P$, by Proposition 11 (2), we have that $f(P) = P/K$ is a graded 1-absorbing primary ideal of R/K. Conversely, $f^{-1}(P/K) = P$ is a graded 1-absorbing primary ideal of R by Proposition 11 (1). □

Proposition 13. *Let R be a graded ring and K be a graded ideal of R such that $U(R/K) = \{a + K : a \in U(R)\}$. If every graded semi-primary ideal of R is graded 1-absorbing primary, then every graded semi-primary ideal of R/K is graded 1-absorbing primary.*

Proof: Let P/K be a graded semi-primary ideal of R/K. Then $Grad(P/K) = Grad(P)/K$ is a graded prime ideal of R/K by Proposition 4, and then $Grad(P)$ is a graded prime ideal of R, so P is a graded semi-primary ideal of R by Proposition 4. Hence, P is a graded 1-absorbing primary ideal of R by assumption. Therefore, P/K is a graded 1-absorbing primary ideal of R/K by Corollary 12. □

Lemma 14. *Let R be a graded ring such that every homogeneous element is either unit or nilpotent. Then R has exactly one graded prime ideal, and hence R is a graded local ring with graded maximal ideal $Grad(\{0\})$.*

Proof: Let P be a graded prime ideal of R. Then $Grad(\{0\}) \subseteq P$. Assume that $x \in P$. Then $x_g \in P$ for all $g \in G$ as P is a graded ideal, and then x_g is nilpotent for all $g \in G$, which implies that $x_g \in Grad(\{0\})$ for all $g \in G$, and so $x \in Grad(\{0\})$. Therefore, $P \subseteq Grad(\{0\})$, and hence $P = Grad(\{0\})$. So, $Grad(\{0\})$ is the only graded prime ideal of R. Since every graded maximal ideal is graded prime, $Grad(\{0\})$ is the only graded maximal ideal of R. Hence, R is a graded local ring with graded maximal ideal $Grad(\{0\})$. □

Theorem 15. *Let R be a graded ring such that every homogeneous element is either unit or nilpotent. Then every graded semi-primary ideal of R is graded primary.*

Proof: By Lemma 14, $Grad(\{0\})$ is the only graded prime ideal of R. Let P be a graded semi-primary

ideal of R. Then $Grad(P)$ is a graded prime ideal of R by Proposition 4, and then $Grad(P) = Grad(\{0\})$ which is a graded maximal ideal by Lemma 14, and hence P is a graded primary ideal of R by ([1], Proposition 1.11). $\square$

Corollary 16. *Let R be a graded ring such that every homogeneous element is either unit or nilpotent. Then every graded semi-primary ideal of R is graded 1-absorbing primary.*

Theorem 17. *Let R be a graded local ring with a graded maximal ideal X. If $Grad(\{0\})$ and X are the only graded prime ideals of R, then every graded semi-primary ideal of R is graded 1-absorbing primary.*

Proof: Let P be a graded semi-primary ideal of R. Then $Grad(P)$ is a graded prime ideal of R by Proposition 4. If $Grad(P) = X$, then P is a graded primary ideal of R by ([1], Proposition 1.11), and hence P is graded 1-absorbing primary. Suppose that $Grad(P) = Grad(\{0\})$. Let $x, y, z \in h(R)$ be nonunit elements such that $xyz \in P$ and $z \notin Grad(\{0\})$. Then $x \in Grad(\{0\})$ or $y \in Grad(\{0\})$, and then $xy = 0 \in P$ in both cases. Therefore, P is a graded 1-absorbing primary ideal of R. $\square$

Theorem 18. *Let R be a graded ring such that every homogeneous element is either unit or nilpotent. Then every proper graded ideal of R is graded 1-absorbing primary.*

Proof: Let P be a proper graded ideal of R and $x \in P$. Then $x_g \in P$ for all $g \in G$, and then x_g is nilpotent for all $g \in G$, which yields that $x_g \in Grad(\{0\})$ for all $g \in G$, so $x \in Grad(\{0\})$). So, $P \subseteq Grad(\{0\})$, and then $Grad(\{0\}) \subseteq Grad(P) \subseteq Grad(\{0\})$, that is $Grad(P) = Grad(\{0\})$ which is graded prime by Lemma 14, and hence P is a graded semi-primary ideal of R by Proposition 4. Therefore, P is a graded 1-absorbing primary ideal of R by Corollary 16. $\square$

The intersection of two graded prime ideals is not necessarily to be a graded prime ideal, see ([7], Example 5). However, if the intersection is also graded prime, then the two graded prime ideals should be comparable as one can see in the following lemma:

Lemma 19. *Let R be a graded ring and P, K be two graded prime ideals of R. Then $P \bigcap K$ is a graded prime ideal of R if and only if $P \subseteq K$ or $K \subseteq P$.*

Proof: Suppose that $P \bigcap K$ is a graded prime ideal of R. Assume that $P \nsubseteq K$ and $K \nsubseteq P$. Then there exist $x \in P$ and $y \in K$ such that $x \notin K$ and $y \notin P$, and

then $x_g \notin K$ and $y_h \notin P$ for some $g, h \in G$. Note that $x_g \in P$ and $y_h \in K$ as P and K are graded ideals. So, $x_g y_h \in P \bigcap K$, and then either $x_g \in P \bigcap K$ or $y_h \in P \bigcap K$, which is a contradiction. Therefore, $P \subseteq K$ or $K \subseteq P$. The converse is clear. $\square$

Theorem 20. *Let R be a graded ring. If every graded ideal of R is graded 1-absorbing primary, then graded prime ideals of R are comparable.*

Proof: Let P and K be two graded prime ideals of R. Then $P \bigcap K$ is a graded ideal of R by Proposition 3, and then $P \bigcap K$ is a graded 1-absorbing primary ideal of R by assumption, which implies that $Grad(P \bigcap K) = P \bigcap K$ is a graded prime ideal of R by Corollary 6. Hence, $P \subseteq K$ or $K \subseteq P$ by Lemma 19. $\square$

Corollary 21. *Let R be a graded ring. If every graded ideal of R is graded 1-absorbing primary, then R is a graded local ring.*

Definition 22. *[5] Let R be a graded ring.*

1. *For $a, b \in h(R)$, we say that a divides b (we write $a|b$) if $b = ax$ for some $x \in h(R)$.*

2. *R is said to be a graded divided ring if for every graded prime ideal P of R and for every $a \in h(R) - P$, we have $a|p$ for every $p \in P$.*

Proposition 23. *[5] Let R be a graded divided ring. Then every graded 1-absorbing primary ideal of R is graded primary.*

Proof: Suppose that P is a graded 1-absorbing primary ideal of R. Let $a, b \in h(R)$ such that $ab \in P$ and $b \notin Grad(P)$. We may assume that a, b are nonunit elements of R. Since $Grad(P)$ is a graded prime ideal of R by Corollary 6 and $b \notin Grad(P)$, we have that $a \in Grad(P)$. Since R is a graded divided ring, we have that $b|a$, which means that $a = bw$ for some $w \in h(R)$. Since $b \notin Grad(P)$ and $a \in Grad(P)$, we achieve that w is a nonunit element of R. Since $ab = bwb \in P$ and P is a graded 1-absorbing primary ideal of R and $b \notin Grad(P)$, we have that $a = bw \in P$. Thus, P is a graded primary ideal of R. $\square$

The next result gives another case in which every graded 1-absorbing primary ideal of R is graded primary:

Proposition 24. *Let R be a graded local ring with a graded maximal ideal generated by a homogeneous element. Then every graded 1-absorbing primary ideal of R is graded primary.*

Proof: Let $M = Ra$ be the graded maximal ideal of R, where $a \in h(R)$. Suppose that P is a graded 1-absorbing primary ideal of R. Then $Grad(P)$ is a graded prime ideal of R by Corollary 6. Assume that $x, y \in h(R)$ such that $xy \in P$ and $y \notin Grad(P)$. Then $x \in Grad(P)$. If y is unit, then $x \in P$ and we are done. If x is unit, then $y \in P$, and then $y \in Grad(P)$, which is a contradiction. So, we assume that x and y are nonunit elements. Then $x \in M$, and then $x = ra$ for some $r \in R$. Clearly, $r \in h(R)$. If r is unit, then $M = Ra \subseteq Rx \subseteq Grad(P)$, and hence $Grad(P) = M$, which gives that P is a graded primary ideal of R by ([1], Proposition 1.11). Suppose that r is a nonunit element. Then $ary = xy \in P$, and then $ar \in P$ since P is graded 1-absorbing primary and $y \notin Grad(P)$, which gives that $x \in P$. Therefore, P is a graded primary ideal of R. □

Let R be a G-graded ring and P be a graded ideal of R. Assume that $g \in G$ such that $P_g \neq R_g$. In [5], P is said to be a g-1-absorbing primary ideal of R if whenever nonunit elements $x, y, z \in R_g$ such that $xyz \in P$, then $xy \in P$ or $z \in Grad(P)$. Also, P is said to be a g-primary ideal of R if whenever $x, y \in R_g$ such that $xy \in P$, then either $x \in P$ or $y \in Grad(P)$.

Proposition 25. *[5] Let R be a G-graded ring and $g \in G$. If R has a g-1-absorbing primary ideal that is not a g-primary ideal, then the sum of every nonunit element of R_g and every unit element of R_g is a unit element of R_g.*

Proof: Suppose that P is a g-1-absorbing primary ideal of R that is not a g-primary ideal of R. Hence, there exist nonunit elements $a, b \in R_g$ such that neither $a \in P$ nor $a \in Grad(P)$. Let w be a nonunit element of R_g. Since $wab \in P$ and P is a g-1-absorbing primary ideal of R and $b \notin Grad(P)$, we conclude that $wa \in P$. Let u be a unit element of R_g. Suppose that $w + u$ is a nonunit element of R_g. Since $(w + u)ab \in P$ and P is a g-1-absorbing primary ideal of R_g and $b \notin Grad(P)$, we conclude that $(w + u)a = wa + ua \in P$. Since $wa \in P$, we conclude that $a \in P$, which is a contradiction. Thus, $w + u$ is a unit element of R_g. □

In view of Proposition 25, we have the following conclusion:

Corollary 26. *[5] Let R be a G-graded ring and $g \in G$. If R_g has a nonunit element and a unit element whose sum is nonunit element in R_g, then a graded ideal P of R is a g-1-absorbing primary ideal of R if and only if P is a g-primary ideal of R.*

Corollary 27. *[5] Let R be a G-graded ring. If R has an e-1-absorbing primary ideal that is not an e-primary ideal, then R_e is a local ring.*

Proof: By Proposition 25, the sum of every nonunit element of R_e and every unit element of R_e is a unit element of R_e, and then by ([14], Lemma 1), R_e is a local ring. □

In view of Corollary 27, we have the following conclusion:

Corollary 28. *Let R be a graded ring such that R_e is non-local ring and P be a graded ideal of R. Then P is an e-primary ideal of R if and only if P is an e-1-absorbing primary ideal of R.*

Let R be a G-graded ring and P be a graded ideal of R. Assume that $g \in G$ such that $P_g \neq R_g$. In [15], P is said to be a g-prime ideal of R if whenever $x, y \in R_g$ such that $xy \in P$, then $x \in P$ or $y \in P$. We state the following:

Definition 29. *Let R be a G-graded ring and P be a graded ideal of R. Assume that $g \in G$ such that $P_g \neq R_g$. Then P is said to be a g-semi-primary ideal of R if whenever $x, y \in R_g$ such that $xy \in P$, then $x \in Grad(P)$ or $y \in Grad(P)$.*

Using the same technique that is used in the proof of Proposition 4, one can prove the following result:

Proposition 30. *Let R be a graded ring and P be a graded ideal of R. Then P is a e-semi-primary ideal of R if and only if $Grad(P)$ is an e-prime ideal of R.*

Proposition 31. *Let R be a graded ring over which every e-semi-primary ideal of R is e-1-absorbing primary and P be a graded ideal of R such that P is an e-prime ideal of R which is not a graded maximal ideal of R. Suppose that X is a graded maximal ideal of R with $P \subsetneq X_e$ and J is a proper graded ideal of R with $Grad(J) = P$. Then $P_e^2 \subseteq J$. Moreover, $P_e(X_e - P) \subseteq J$. Furthermore, if $P_e = P$, then $P_e X_e \subseteq J$.*

Proof: Let $m \in X_e - P$ and $p, q \in P_e$. Then $Grad(J + R_e pqm) = P$, and then $J + R_e pqm$ is an e-semi-primary ideal of R by Proposition 30, and so $J + R_e pqm$ is an e-1-absorbing primary ideal of R by assumption. Since $pqm \in J + R_e pqm$ and $m \notin P = Grad(J + R_e pqm)$, $pq \in J + R_e pqm$, and then there exists $r \in R_e$ such that $pq(1 - rm) \in J$. Since J is an e-1-absorbing primary ideal of R, either $pq \in J$ or $1 - rm \in J$. If $1 - rm \in J \subseteq Grad(J) = P \subset X_e \subseteq X$, then $1 \in X$ as $rm \in X$, which is a contradiction. So, $pq \in J$ and hence $P_e^2 \subseteq J$. Similarly, $Grad(J + R_e pm^2) = P$, which gives that $J + R_e pm^2$ is an e-1-absorbing primary ideal of R. So, we will have $pm \in J$. Therefore, $P_e(X_e - P) \subseteq J$. Furthermore, let $x \in X_e$. If $x \notin P$, then $P_e x \subseteq P_e(X_e - P) \subseteq J$. If $x \in P = P_e$, then $P_e x \subseteq P_e^2 \subseteq J$. Consequently, $P_e X_e \subseteq J$. □

3   Conclusion

In this article, we characterized graded commutative rings over which every graded semi-primary ideal is graded 1-absorbing primary and graded rings over which every graded 1-absorbing primary ideal is graded primary. As a proposal for future work, we will try to study graded primary, graded semi-primary and graded 1-absorbing primary ideals over non-commutative graded rings.

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