

Reduction of the Self-dual Yang-Mills Equations to Sinh-Poisson Equation and Exact Solutions

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Abstract: - The geometric properties of differential systems are used to demonstrate how the sinh-poisson equation describes a surface with a constant negative curvature in this paper. The canonical reduction of 4-dimensional self dual Yang Mills theorem is the sinh-poisson equation, which explains pseudo spherical surfaces. We derive the Bäcklund transformations and the travelling wave solution for the sinh-poisson equation in specific. As a result, we discover exact solutions to the self-dual Yang-Mills equations.

KeyWords :- Bäcklund transformations; Nonlinear evolution equations; Yang-Mills theory; Pseudo-spherical surfaces.

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1 Introduction

The concept of integrable structures has been a contentious issue in mathematics for the last three decades. The focus includes but not limited into mechanics, dynamics, mathematics, algebra, physics, evaluation, and geometry.

Many partial differential equations, which are still studied due to their significance in physics and mathematics, have relations with the geometry of surfaces embedded in 3D space [1]. This has long been known that there is a relationship between surfaces in Euclidean three-space with a constant negative Gaussian curvature, the SineGordon Formula, and Bäcklund transformations that are applicable to the equation given [2]. Furthermore, including pseudo spherical surfaces, the main Bäcklund transformation for something like the SineGordon Formula appears to have been a normal linear structure (pss) [3-5].

The Yang-Mills (YM) behavior is one of the key components of the standard model, which has been phenomenological very popular up to this point. A Yang-Mills field adds to the curvature of space-time in the same way as every other field does, according to General Relativity. There are some physically important circumstances in which gravitational fields are exceptionally strong, and the impact of curvature on the propagation of matter fields, as well as the Yang-Mills fields' back-reaction, cannot be overlooked [6].

Once Yang-Mills concept was proposed in the mid-twentieth century, it was understood that the quantum model of the Maxwell system, known as

Quantum Electrodynamics (QED), provided a detailed explanation of the quantum structure of electro_magnetic fields and forces. The issue of whether or not the nonabelian analog was needed to represent other natural forces, especially the weak force, which is concerned with certain types of radioactivity, and the powerful or nuclear force, which is concerned with the mixing of protons and neutrons around nuclei, among other things, arose.

Yang-Mills theories, which describe fundamental laws of interactions, are central to primary particle physics. Topological solitons such as instantons, monopoles, vortices, calorons, and merons played important roles in the study of non-perturbative properties, duality structures, and quark confinements, among other aspects, in these theories.

Since the weak and nuclear forces are unrelated to long-range fields or particles of low mass [7-9]. The masslessness of classical Yang-Mills waves was a major impediment to applying Yang-Mills theory to them. In the 1960s and 1970s, these obstacles to physical applications of non-abelian gauge theory were overcome.

The self-dual Yang-Mills equations (SDYM) are well-known for having a large number of integrable systems. The Painleve equations and classical soliton equations in 1+1 dimensions are finite-dimensional Lie algebra reductions of the SDYM equations (LG). It is important to use infinite dimensional algebras to minimize the SDYM equations [10-12].

Anti-self-dual Yang-Mills(ASDYM) equation, on the other side, has a close relationship with lower-

dimensional integrable equations such as KdV, Toda, and Painlevé equations, and so on [18, 21]. Any soliton solution towards these equations has energy densities that are local on hyper-plane over all three dimensions, allowing them to be viewed as space-time domain walls. Even so, the integrability of those kind of solitons systems is dependent on the presence of infinite quantities and undefined infinite transformations.

The objective of this research is to propose and exact solutions to the canonical reduction of 4D SDYM theory to 2D sinh-poisson (SP) equation. The section is structured as follows. Section 2 establishes the relationship in between SP equation and pss. Sections 3 and 4 look at the canonical reduction of 4D SDYM to the above equation. In addition, the travelling wave solution for the SP equation and the gauge potential A_μ are obtained. Section 5 concludes with certain assumptions.

2 Description of Pseudo-Spherical Surface by Sinh-Poisson Equation

The definition of pss first appeared in geometry in the far past. That was a watershed point in the history of mathematics. A pss is the final element in the visual representation of non-Euclidean hyperbolic geometry, according to Klingenberg [13]. Pss was realized to have a close relationship with many theorems, such as theory of solitons, nets, attractors, a lot of nonlinear evolution equations (NLEEs) of mechanics, BTs, and other advanced approaches in mathematics [14-16]. For a long time, mathematical physics has studied the relationship between geometry and nonlinear partial differential equations (NLPDEs). For example, the classic Liouville equation indicates minimum surfaces in space E^3 , while the sine Gordon equation is identical to the geometry of pss, or surfaces with a negative Gaussian curvature [17,18].

A scalar differential equation (DE)

$$f(x, t, G, G_x, \dots, G_{x^n t^m}) = 0, \quad G_{x^n t^m} = \frac{\partial^{n+m} u}{\partial x^n \partial t^m}, \quad (1)$$

Consequently, if $F(x, t, G, G_x, \dots, G_{x^n t^m}) = 0$ describe pss with associated one-forms y_i (2). A solution $G(x, t)$ of $F(x, t, G, G_x, \dots, G_{x^n t^m}) = 0$ is

I-generic if $(y_3 \wedge y_2)(G(y, t)) \neq 0$; II-generic if $(y_1 \wedge y_3)(G(y, t)) \neq 0$; and III-generic if $(y_1 \wedge y_2)(G(x, t)) \neq 0$.

The smooth functions f_{ij} , $1 \leq i \leq 3$, $1 \leq j \leq 2$, depending on $G(x, t)$ and its derivatives, one-forms [5]

$$y_i = f_{i1} dx + f_{i2} dt, \quad 1 \leq i \leq 3, \quad (2)$$

The constant Gaussian curvature = -1, that is

$$dy_1 = y_3 \wedge y_2, \quad dy_2 = y_1 \wedge y_3, \quad dy_3 = y_1 \wedge y_2. \quad (3)$$

It is equivalent to say that the NLEE for $G(x, t)$ is necessary and sufficient for the integrability of the linear system [18]

$$d\varphi = \alpha \varphi, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad \alpha = \frac{1}{2} \begin{pmatrix} y_2 & y_1 - y_3 \\ y_1 + y_3 & -y_2 \end{pmatrix} \quad (4)$$

Where d denotes exterior differentiation, then

$$\alpha = \begin{pmatrix} \frac{f_{21}}{2} dx + \frac{f_{22}}{2} dt & \frac{(f_{11}-f_{31})}{2} dx + \frac{(f_{12}-f_{32})}{2} dt \\ \frac{(f_{11}-f_{31})}{2} dx + \frac{(f_{12}-f_{32})}{2} dt & -\frac{f_{21}}{2} dx + \frac{f_{22}}{2} dt \end{pmatrix} = H dx + L dt, \quad (5)$$

Eqs. (4), (5), we comes

$$\varphi_x = H\varphi, \quad \varphi_t = L\varphi,$$

where H and L are two 2×2 null-trace matrices

$$H = \begin{pmatrix} \frac{f_{21}}{2} & \frac{f_{11}-f_{31}}{2} \\ \frac{f_{11}+f_{31}}{2} & -\frac{f_{21}}{2} \end{pmatrix}, \quad (7)$$

$$L = \begin{pmatrix} \frac{f_{22}}{2} & \frac{f_{12}-f_{32}}{2} \\ \frac{f_{12}+f_{32}}{2} & -\frac{f_{22}}{2} \end{pmatrix}. \quad (8)$$

Now

$$d^2\varphi = d\alpha\varphi - \alpha \wedge d\varphi = (d\alpha - \alpha \wedge \alpha)\varphi,$$

which

$$\Theta \equiv d\alpha - \alpha \wedge \alpha = 0, \quad (9)$$

(9)

or in component form

$$f_{12,x} - f_{11,t} = f_{31}f_{22} - f_{21}f_{32}, \quad (10)$$

$$f_{22,x} - f_{21,t} = f_{11}f_{32} - f_{12}f_{31}, \quad (11)$$

$$f_{32,x} - f_{31,t} = f_{11}f_{22} - f_{21}f_{12}, \quad (12)$$

The SP equation describe pss as well. The equation has many physical applications [2,4].

Example: Let a differentiable surface M^2 ,

The SP equation

$$\begin{aligned} \gamma_1 &= G_x dx, \\ \gamma_2 &= \eta dx + \frac{1}{\eta} \cosh G dt, \end{aligned}$$

(13)

$$\gamma_3 = \frac{1}{\eta} \sinh G dt,$$

Hence M^2 is a pss iff G satisfies the SP equation

$$G_{xt} = \sinh G.$$

(14)

3 Reducing "SDYM" Equations to "SP" Equation

The SDYM equations are particularly significant in their own right, with wide range of applicable issues in science of math besides physics. SDYM equations is being used in gauge principle and classical quantum theory, and they're used to efficiently analyze four-manifolds [19, 20]. The SDYM equations are important in analyzing of integrable systems, as well as in plenty other aspects of math and physics. The master integrable scheme is another name for SDYM equations.

SDYM include a feature for developing and categorizing a large number of integrable structures, as well as a geometrical framework for analyzing them. Furthermore, an integrable equation considers comprehensive solution achieved by resolving related linear problems using the inversed scattering transformation [21-34].

The equations of SDYM are a set of 4D paired with 2nd order partial differential equations (PDEs) for Lie algebras defined gauge potential functions

$A_\mu^I s$ ($\mu = 0, 1, 2, 3$), which are notoriously difficult to solve in general.

$$*F = \lambda F, \tag{15}$$

for some constant λ , $*F$ is the dual two-form of F . [1-3], the SDYM equation

$$F_{01} = F_{23}, F_{02} = F_{31}, F_{03} = F_{12},$$

(16)

where $F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu$. Introduce the null coordinates y, \bar{y}, z, \bar{z} Eq.(16) is cast into

$$F_{yz} = 0, F_{\bar{y}\bar{z}} = 0, F_{y\bar{y}} + F_{z\bar{z}} = 0,$$

(17)

where

$$\begin{aligned} y &= \frac{1}{\sqrt{2}} (x_1 + ix_2), \bar{y} = \frac{1}{\sqrt{2}} (x_1 - ix_2), \\ z &= \frac{1}{\sqrt{2}} (x_0 + ix_3), \bar{z} = \frac{1}{\sqrt{2}} (x_0 - ix_3). \end{aligned}$$

(18)

It is verified that Eq. (16) is the compatibility of the following isospectral problem:

$$\begin{aligned} (\partial_y + \xi \partial_{\bar{z}})v &= (E_y + \xi E_{\bar{z}})v, (\partial_z + \xi \partial_{\bar{y}}) \\ v &= (E_z + \xi E_{\bar{y}})v, \end{aligned}$$

(19)

where

$$E = E_\mu dx^\mu = E_y dy + E_{\bar{y}} d\bar{y} + E_z dz + E_{\bar{z}} d\bar{z} \xi.$$

We see that

$$\begin{aligned} E_{yz} - E_{zy} - [E_y, E_z] &= 0, \\ E_{y\bar{y}} + E_{z\bar{z}} - E_{\bar{z}z} - E_{\bar{y}y} - [E_y, E_{\bar{y}}] - \\ [E_z, E_{\bar{z}}] &= 0, \end{aligned}$$

(20)

$$E_{\bar{y}\bar{z}} - E_{\bar{z}\bar{y}} - [E_{\bar{y}}, E_{\bar{z}}] = 0,$$

where

$$E_{\mu\nu} = \partial_\mu E_\nu, F_{\mu\nu} = \partial_\mu E_\nu - \partial_\nu E_\mu - [E_\mu, E_\nu]$$

and

$$\begin{aligned} E_0 &= \frac{1}{\sqrt{2}} (E_z + E_{\bar{z}}), E_1 = \frac{1}{\sqrt{2}} (E_y + E_{\bar{y}}), \\ E_2 &= \frac{i}{\sqrt{2}} (E_y - E_{\bar{y}}), E_3 = \frac{-i}{\sqrt{2}} (E_z - E_{\bar{z}}) \end{aligned}$$

(21)

In the end of this section, we discuss reduction of the SDYM equations to integrable PDE, many integrable equations are reductions of the SDYM, As this application, a good NLEE could be obtained in the following example:

3.1 The SP Equation

Suppose that $E'_\mu s$ which $E_{\bar{y}} = 0$, then the equation (19) becomes

$$\partial_x v = (E_y + \xi E_{\bar{z}})v, \partial_t v = -\frac{1}{\xi} E_z v$$

(22)

The $E'_\mu s$ are skew-Hermitian. Then the parameterization

$$E_y = \begin{pmatrix} \frac{-ik}{2} & 0 \\ 0 & ik \end{pmatrix}, E_z = \begin{pmatrix} 0 & \frac{-m+m}{2} \\ \frac{-m+m}{2} & 0 \end{pmatrix},$$

$$E_{\bar{z}} = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix},$$

(23)

where k, m, n are real functions. And SDYM equivalent to

$$\frac{\partial n}{\partial x} = mk, \frac{\partial m}{\partial x} = nk, \frac{\partial k}{\partial t} = m.$$

(24)

It follows from the fi two equations of (20) that $n^2 - m^2$ is independent of x . We choose $n^2 - m^2 = 1$ and introduce the paramertization $m = \sinh G, n = \cosh G$. this gives $k = \frac{\partial u}{\partial x}$. The third equation in (20) then becomes the SP equation (14).

4 Exact Solutions of SP Equation and the SDYM Equations

The problem [3], with Gaussian curvature = -1 on M^2 take

$$\alpha = \begin{pmatrix} \frac{\eta}{2} dx + E dt & q dx + B dt \\ r dx + C dt & -\frac{\eta}{2} dx - E dt \end{pmatrix} = H dx + L dt,$$

(25)

where H and L are two 2×2 null-trace matrices

$$H = \begin{pmatrix} \frac{\eta}{2} & q \\ r & -\frac{\eta}{2} \end{pmatrix},$$

(26)

$$L = \begin{pmatrix} E & B \\ C & -E \end{pmatrix},$$

(27)

Here η is a parameter, independent of x and t , q and r are functions of x and t .

The equations (10) - (12) become,

$$E_x = qC - rB,$$

$$q_t = -2Eq - B_x + \eta B = 0,$$

(28)

$$C_x = r_t + 2Er - \eta C,$$

and Tenenblat [5] obtained equations (28) directly from the structure equations (25). By suitably choosing r, E, B and C in (28), we shall obtain various NLEEs which q must satisfy. From equations (4)-(6), we obtain the following scattering problem

$$\varphi_{1x} = \frac{\eta}{2} \varphi_1 + q \varphi_2,$$

$$\varphi_{2x} = r \varphi_1 + \frac{\eta}{2} \varphi_2,$$

(29)

$$\varphi_{1t} = E \varphi_1 + B \varphi_2,$$

$$\varphi_{2t} = C \varphi_1 + E \varphi_2,$$

(30)

Konno and Wadati [14] introduced the function

$$\Gamma = \frac{\varphi_1}{\varphi_2},$$

(31)

where the function Γ determines geodesic coordinates on the pss determined by $G_t = F(G, G_x, \dots, G_{x^k})$ ($G_{x^k} = \partial^k G / \partial x^k$) [15]. Then equations (29), (30) are reduced to the Riccati equations:

$$\frac{\partial \Gamma}{\partial x} = \eta \Gamma - r \Gamma^2 + q,$$

(32)

$$\frac{\partial \Gamma}{\partial t} = 2E \Gamma - C \Gamma^2 + B,$$

(33)

Our procedure in the following is that we construct a transformation Γ' satisfying the same equation as (32) with a potential u' where

$$u' = u + f(\Gamma, \eta),$$

(34)

For any solution $G(x, t)$ of the SP equation (14), the matrices H and L are

$$H = \begin{pmatrix} \frac{\eta}{2} & \frac{u_x}{2} \\ \frac{u_x}{2} & -\frac{\eta}{2} \end{pmatrix}$$

(35)

$$L = \begin{pmatrix} \frac{1}{2} \left(\frac{\cosh u}{\eta} \right) & -\frac{1}{2} \left[\frac{\sinh u}{\eta} \right] \\ \frac{1}{2} \left[\frac{\sinh u}{\eta} \right] & -\frac{1}{2} \left(\frac{\cosh u}{\eta} \right) \end{pmatrix}$$

(36)

the above matrices H, L satisfy the equation (28). Then equation (32) becomes

$$\frac{\partial \Gamma}{\partial x} = \eta \Gamma + \frac{1}{2} u_x (1 - \Gamma^2).$$

(37)

If we choose Γ' and u' as

$$\Gamma' = \frac{1}{\Gamma},$$

(38) $u' =$

$$-u + 4 \tanh^{-1} \Gamma,$$

(39)

then Γ' and u' satisfies equation (37).

Substitute $G = G_0$ into the matrices H and L in (35) and (36), then by (6) we have

$$d\varphi = \varphi_x dx + \varphi_t dt = H\varphi dp, \quad (40)$$

Where

$$H = \begin{pmatrix} \frac{\eta}{2} & 0 \\ 0 & -\frac{\eta}{2} \end{pmatrix}, \quad (41)$$

$$p = x + kt, k = \frac{-\cosh u_0}{\eta^2}, \quad (42)$$

The solution of equation (40) is

$$\varphi = e^{\rho P} \varphi_0 = \left(I + \rho P + \frac{\rho^2 p^2}{2!} + \frac{\rho^3 p^3}{3!} + \dots \right) \varphi_0, \quad (43)$$

where φ_0 is a constant column vector. The solution of equation (43) is

$$\varphi \begin{pmatrix} \cosh \frac{\eta}{2} \rho + \sinh \frac{\eta}{2} \rho & 0 \\ 0 & \cosh \frac{\eta}{2} \rho - \sinh \frac{\eta}{2} \rho \end{pmatrix}$$

$$\varphi_0, \quad (44)$$

Now, we choose $\varphi_0 = (1,1)^T$ in (44), then we have

$$\varphi = \begin{pmatrix} \frac{\eta \rho}{e^2} \\ \frac{\eta \rho}{e^2} \end{pmatrix}. \quad (45)$$

Substitute (45) into (39), then by (31), we obtain the new solutions of the SP equation (14)

$$u' = u_0 + 4 \tanh^{-1}(e^{\eta \rho}) \rho = x - \frac{\cosh u_0}{\eta^2} t \quad (46)$$

Consequently we obtain the gauge potential E_μ as follows:

$$E_{\bar{y}} = 0,$$

$$E_y = \begin{pmatrix} \eta i \operatorname{csch}(\eta \rho) & 0 \\ 0 & -\eta i \operatorname{csch}(\eta \rho) \end{pmatrix},$$

$$E_z = \begin{pmatrix} 0 & \frac{a}{2} \\ \frac{a}{2} & 0 \end{pmatrix}, \quad E_{\bar{z}} = \begin{pmatrix} 0 & \frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}, \quad (47)$$

Where

$$a = -\sinh[-u_0 + 4 \tanh^{-1}(e^{\eta \rho})] + i \cosh[-u_0 + 4 \tanh^{-1}(e^{\eta \rho})]$$

5 Conclusions

Since the fundamental laws of quantum mechanics, gravity, electromagnetism, and

elementary particle interactions are all clarified in the manner of partial differential equations (PDEs), science has produced a great success in physics in which partial differential equations contribute significantly. In exchange, advances in physics include immense opportunities, motivation, and obstacles, all of which have fueled research in partial differential equations. We will hope to discover a relationship between the SP equation and the pss in this paper. The SDYM equations are fundamental in many fields of mathematics and physics, as well as in the field of integrable systems. The reality that indeed SDYM equations are the compatible scenario of a relevant linear problem, which accepts enormous flexibility if the involved gauge algebra(LG) is random, implies that they are a valuable source of integrable models. The SDYM equations are finite-dimensional gauge algebra reductions of the classic soliton equations in 1+1 dimensions. Our study shows how the SDTM equations can be reduced to SP equations that explain pss and precise solutions.

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