On approximation of the panjer distribution by the poisson and binomial distributions

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Abstract: The aim of this paper is to approximate the panjer distribution by the poisson and binomial distributions, where each bound is obtained by using the z-function and the Stein-Chen identity. For these bounds, it is indicated that a result of each of the Poisson and Binomial approximations yields a good approximation if both α and λ are small.

Key-Words: Binomial distribution, Panjer distribution, Poisson Approximation, Stein-Chen identity, z-function.

Received: February 2, 2021. Revised: September 24, 2021. Accepted:October 1,2021. Published: October 13, 2021.

1 Introduction

Let X be the Panjer random variable with parameters λ and α, and its probability distribution is defined by

$$\mathcal{P}_{(\lambda,\alpha)}(x) = (1+\frac{\lambda}{\alpha})^{\alpha}\frac{\lambda^x}{x!}\prod_{i=0}^{x-1}\frac{\alpha+i}{\alpha+\lambda}$$
$$= (1+\frac{\lambda}{\alpha})^{-\alpha}\frac{\lambda^x}{\alpha+\lambda}\binom{\alpha+x-1}{x}, \quad (1)$$

where $x = 0, 1, 2, \ldots$ and mean and variance are λ and $\lambda(1 + \frac{\lambda}{\alpha})$, respectively. This distribution was found the existing in this class of distribution of the Binomial, the Poisson, and the Negative Binomial distributions by Sundt and Jewell [2] and Aktuar [8]. In view of the interest in this the Panjer distribution, it have been studied extensively by several researchers. For more detailed information, please refer to, for example, [5, 6, 9, 10] and the reference therein.Moreover, there were many researchers who applied the Panjer distribution in other fields such as [4, 12]

The more common parameters of these distributions are determined by both a and b. The properties of Panjer, Poisson and Binomial distributions related to the present class of distributions are summarized in the following table 1.

Table 1: The properties of Panjer, Poisson and Binomial distributions related to the present class of distributions

Distribution	P(N=k)	a	b
Binomial	$\binom{n}{k}p^k(1-p)^k$	$\frac{-p}{1-p}$	$\frac{p(n+1)}{1-p}$
Poisson	$e^{-\lambda}\frac{\lambda^k}{k!}$	0	λ
Panjer	$(1+\frac{\lambda}{\alpha})^{-\alpha}\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{\alpha+i}{\alpha+\lambda}$	$\frac{\lambda}{\alpha+\lambda}$	$\frac{(\alpha-1)\lambda}{\alpha+\lambda}$

As presented in Table 1, it showed that the Panjer distribution can be reduced to be the Poisson or Binomial distribution. So, the Panjer distribution can be approximated by the Poisson and Binomial distributions providing that certain conditions on their parameters are satisfied. Thus, if the parameters of the Poisson and Binomial distributions are set to correspond the parameter of the Panjer distribution, the later can also be approximated by the Poisson and binomial ones. For both approximations, the accuracy of each approximation can be measured in terms of the total variation distance between two distributions. The total variation distance between the Panjer and Poisson distributions is defined by

$$d_{TV}(\mathcal{P}(\lambda,\alpha), \mathcal{P}o(\lambda))$$
$$= \sup_A |\mathcal{P}_{(\lambda,\alpha)}\{A\} - \mathcal{P}o_{(\lambda)}\{A\}| \quad (2)$$

and the total variation distance between the Panjer and Binomial distribution is

$$d_{TV}(\mathcal{P}(\lambda,\alpha),\mathcal{B}(n,p)) = \sup_A |\mathcal{P}_{(\lambda,\alpha)}\{A\} - \mathcal{B}_{(n,p)}\{A\}| \qquad (3)$$

where A is a subset of $\mathbb{N}\cup\{0\}$ and $\mathcal{P}_{(\lambda,\alpha)}$, $\mathcal{P}o_{(\lambda)}$ and $\mathcal{B}_{(n,p)}$ are generalized panjer, poisson and binomial distributions, respectively.

The main tools for determining an upper bound for each total variation distance are the z-function associated with the Panjer random variable X random variable and the Stein identities for binomial and Poisson distributions. Sudheesh Kumar [11] adapted the relation of z-function associated with a non-negative integer-valuedrandom variable X, which are mentioned as follows:

$$z(x) = \frac{p_X(x-1)}{p_X(x)} z(x-1) + \mu - x, \qquad (4)$$

where $x \in \mathcal{S}(x)\backslash\{0\}$, $z(0) = \mu$, $S(x)$ is support of X, $p_X(x) > 0$ for all $x \in S(x)$, μ and $\sigma^2 \in (0,\infty)$.

In this paper, we give two results of the Poisson and Binomial approximations to the Panjer distribution, where a result of each of the approximations has been presented in terms of the total variation distance (2) or (3) and we use the z-function associated with the random variable X together with the Stein-Chen identity to derive bounds which is introduced in section 2.

2 Main results

Let X be the generalized binomial random variable with probability distribution defined as in [10]. Its mean and variance are λ and $\lambda(1+\frac{\lambda}{\alpha})$, respectively, and its associated z-function is given as the following proposition.

Proposition 2.1. *Let $z(x)$ be the z-function associated with the Panjer random variable X, then*

$$z(k) = \frac{k\lambda + \alpha\lambda}{\alpha}, k \in \{1, 2, \ldots, S\} \qquad (5)$$

where $\mu = \lambda$ and $\sigma^2 = \lambda(1+\frac{\lambda}{\alpha})$

The next step, we will use the z-function (5) and the Stein-Chen indentity to derive bounds for Poisson and Binomial approximations.

2.1 A result for the Poisson approximation

The Stein identity was introduced by Stein [3] and Chen [7]. The Stein-Chen identity or the Stein identity for the Poisson distribution with parameter $\lambda > 0$, for any subset A of $\mathbb{N}\cup\{0\}$ and the bounded real valued function $f : f_A : \mathbb{N}\cup\{0\} \to \mathbb{R}$, Barbour et al [1], proved that

$$\begin{aligned}
&\sup_{x,A} |\Delta f(x)| \\
&= \sup_{x,A} |\Delta f(x+1) - f(x)| \\
&\le \frac{1}{\lambda}(1 - \frac{1}{e^\lambda}), \qquad (6)
\end{aligned}$$

and defined that

$$\begin{aligned}
&\mathcal{P}_{(\lambda,\alpha)}\{A\} - \mathcal{P}o_{(\lambda)}\{A\} \\
&= \mathbb{E}[\lambda f(X+1) - X f(X)] \qquad (7)
\end{aligned}$$

The theorem below gives bound for the total variation distance between the Panjer and the Poisson distribution.

Theorem 2.2. *For $A \subseteq \mathbb{N}\cup\{0\}$, then we have*

$$\begin{aligned}
&\lambda\{\frac{1}{\lambda^2} - \frac{1}{\lambda^3} - (\frac{\alpha}{\alpha+\lambda})^{\alpha+1}\} \\
&\le d_{TV}(\mathcal{P}(\lambda,\alpha), \mathcal{P}o(\lambda)) \le \frac{\lambda}{\alpha} \qquad (8)
\end{aligned}$$

Proof. From (14), (7) and the view of proposition 2.1, it follows that

$$\begin{aligned}
&|\mathcal{P}_{(\lambda,\alpha)}\{A\}, \mathcal{P}o_{(\lambda)}\{A\}| \\
&= \mathbb{E}|\lambda f(X+1) - X f(X)| \\
&= |\lambda\mathbb{E}[f(X+1)] - \mathbb{E}[X f(X)]| \\
&= |\lambda\mathbb{E}[f(X+1)] - \mathbb{C}ov(X, f(X)) \\
&\quad - \mu\mathbb{E}[f(X)]| \\
&= |\lambda\mathbb{E}[f(X+1)] - \mathbb{C}ov(X, f(X)) \\
&\quad - \lambda\mathbb{E}[f(X)]| \\
&= |\lambda\mathbb{E}[\Delta f_x(X)] - \mathbb{C}ov(X, f(X))| \\
&= |\lambda\mathbb{E}[\Delta f_x(X)] - \mathbb{E}[z(x)\Delta f(X)]| \\
&\le \mathbb{E}|[\lambda - z(x)]\Delta f(X)| \\
&= \sum_{k=0}^{\infty} |[\lambda - z(k)]||\Delta f(k)|p(k) \\
&\le \sum_{k=1}^{\infty} |[\lambda - z(k)]|\frac{1}{k}p(k) \\
&= \sum_{k=1}^{\infty} \{\frac{\lambda}{\alpha}\}p(k) \\
&= \frac{\lambda}{\alpha}(1 - p(0)) \\
&= \frac{\lambda}{\alpha}\{1 - (\frac{\alpha}{\alpha+\lambda})^{\alpha}\} \\
&\le \frac{\lambda}{\alpha}, \qquad (9)
\end{aligned}$$

and, by using (2), we get

$$d_{TV}(\mathcal{P}_{(\lambda,\alpha)}\{A\}, \mathcal{P}o_{(\lambda)}\{A\}) \leq \frac{\lambda}{\alpha} \tag{10}$$

Since

$$|\mathcal{P}_{(\lambda,\alpha)}\{1\} - \mathcal{P}o_{(\lambda)}\{1\}| \leq d_{TV}(\mathcal{P}(\lambda,\alpha), \mathcal{P}o(\lambda)), \tag{11}$$

and

$$\begin{aligned}
&|\mathcal{P}_{(\lambda,\alpha)}\{1\} - \mathcal{P}o_{(\lambda)}\{1\}| \\
&= \Big|(1+\frac{\lambda}{\alpha})^{-\alpha}\frac{\lambda}{\alpha+\lambda}\alpha - \lambda e^{-\lambda}\Big| \\
&= \Big|(\frac{\alpha}{\alpha+\lambda})^{\alpha+1}\lambda - \lambda(\frac{1}{e})^{\lambda}\Big| \\
&= \{(\frac{1}{e})^{\lambda}\lambda - (\frac{\alpha}{\alpha+\lambda})^{\alpha+1}\lambda\} \\
&\geq \{\frac{\lambda-1}{\lambda^2} - (\frac{\alpha}{\alpha+\lambda})^{\alpha+1}\lambda\} \\
&= \lambda\{\frac{1}{\lambda^2} - \frac{1}{\lambda^3} - (\frac{\alpha}{\alpha+\lambda})^{\alpha+1}\},
\end{aligned}$$

we obtain

$$d_{TV}(\mathcal{P}(\lambda,\alpha), \mathcal{P}o(\lambda)) \geq \lambda\{\frac{1}{\lambda^2} - \frac{1}{\lambda^3} - (\frac{\alpha}{\alpha+\lambda})^{\alpha+1}\}. \tag{12}$$

Therefore, by (10) and (12), we obtain (8) □

2.2 A result for the Binomial approximation

The Stein identity for the Binomial distribution is the special case, $n \geq 1$ and $p = (1-q) \in (0,1)$, every subset A of $\{0, 1, \dots, n\}$ and the bounded real valued function $g = g_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ (defined as in [1])

$$\begin{aligned}
&\mathcal{P}_{(\lambda,\alpha)}\{A\} - \mathcal{B}_{n,p}\{A\} \\
&= \mathbb{E}[(n-X)pg(X+1) - qXg(X)]
\end{aligned} \tag{13}$$

For any subset A of $\{0, ..., n\}$, Ehm [13] showed that

$$\begin{aligned}
\sup_{x,A}|\Delta g(x)| &= \sup_{x,A}|\Delta g(x+1) - g(x)| \\
&\leq \frac{1 - p^{n+1} - q^{n+1}}{(n+1)pq},
\end{aligned} \tag{14}$$

The theorem below gives bound on the Panjer to the Binomial distribution.

Theorem 2.3. *For $A \subseteq \mathbb{N} \cup \{0\}$, $p = \frac{\lambda}{n}$, then we have*

$$\begin{aligned}
&\lambda\Big\{(\frac{n-\lambda}{n})^{n} - (\frac{\alpha}{\alpha+\lambda})^{\alpha}\Big\} \\
&\leq d_{TV}(\mathcal{P}(\lambda,\alpha), \mathcal{B}(n,p)) \leq \frac{\lambda(\alpha+n)}{\alpha n}
\end{aligned} \tag{15}$$

Proof. From (13) and proposition 2.1, we have

$$\begin{aligned}
&|\mathcal{P}_{(\lambda,\alpha)}\{A\}, \mathcal{B}_{(n,p)}\{A\}| \\
&= \mathbb{E}|(n-X)pg(X+1) - qXg(X)| \\
&= \mathbb{E}|[npg(X+1) - pX\Delta g(X) - Xg(X)]| \\
&= |\mu\mathbb{E}[g(X+1)] - p\mathbb{E}[X\Delta g(X)] - \mathbb{E}[Xg(X)]| \\
&= |\mu\mathbb{E}[g(X+1)] - p\mathbb{E}[X\Delta g(X)] - \mathbb{C}ov(X, g(X)) \\
&\quad - \mathbb{E}[\mu g(X)]| \\
&= |\mu\mathbb{E}[\Delta g(X)] - p\mathbb{E}[X\Delta g(X)] - \mathbb{E}[z(X)\Delta g(X)]| \\
&\leq \mathbb{E}|[\mu - pX - z(X)]\Delta g(X)| \\
&= \sum_{k=0}^{\infty}|\mu - pk - z(k)|\frac{1}{k}p(k) \\
&= \sum_{k=1}^{\infty}|\frac{-k\lambda(\alpha+n)}{n\alpha}|\frac{1}{k}p(k) \\
&= \frac{\lambda(\alpha+n)}{n\alpha}(1 - p(0)) \\
&= \frac{\lambda(\alpha+n)}{n\alpha}(1 - (\frac{\alpha}{\alpha+\lambda})^{\alpha}) \\
&\leq \frac{\lambda(\alpha+n)}{n\alpha},
\end{aligned}$$

and by using (3), we get

$$d_{TV}(\mathcal{P}(\lambda,\alpha), \mathcal{B}(n,p)) \leq \frac{\lambda(\alpha+n)}{n\alpha}. \tag{16}$$

Since

$$|\mathcal{P}(\lambda,\alpha)\{1\}, \mathcal{B}(n,p)\{1\}| \leq d_{TV}(\mathcal{P}(\lambda,\alpha), \mathcal{B}(n,p)),$$

and

$$\begin{aligned}
&|\mathcal{P}(\lambda,\alpha)\{1\}, \mathcal{B}(n,p)\{1\}| \\
&= \Big|(\frac{\alpha}{\alpha+\lambda})^{\alpha+1}\lambda - (\frac{n-\lambda}{n})^{n-1}\lambda\Big| \\
&= \lambda\Big|(\frac{\alpha}{\alpha+\lambda})^{\alpha+1} - (\frac{n-\lambda}{n})^{n-1}\Big| \\
&= \lambda\Big\{(\frac{n-\lambda}{n})^{n-1} - (\frac{\alpha}{\alpha+\lambda})^{\alpha+1}\Big\} \\
&\geq \lambda\Big\{(\frac{n-\lambda}{n})^{n} - (\frac{\alpha}{\alpha+\lambda})^{\alpha}\Big\},
\end{aligned}$$

we obtain

$$\begin{aligned}
&d_{TV}(\mathcal{P}(\lambda,\alpha), \mathcal{B}(n,p)) \\
&\geq \lambda\Big\{(\frac{n-\lambda}{n})^{n} - (\frac{\alpha}{\alpha+\lambda})^{\alpha}\Big\}.
\end{aligned} \tag{17}$$

Therefore, by (16) and (17), (15) is obtained. □

3 Conclusions

This study focusing on Stein's method and z-functions was attempted to give the bounds for the total variation distance between the panjer distribution and poisson one, and for the total variation distance between the panjer distribution and binomial one respectively. Furthermore, each of upper bound in (8) and (15), was also specified as a criterion for measuring the accuracy of the corresponding approximation via this conditions: if the obtained bound is small, then the good poisson or binomial approximation to the panjer distribution is obtained, whereas, if such the bound is large, then the poisson or binomial distribution is not appropriate to approximate the panjer distribution. Regarding the results of (8) and (15), it was found that the bounds of the poisson approximation was found to be small when both α and λ were small, and the upper bound of the binomail was small as both α and λ were small and n was large.

Acknowledgment. The authors would like to thanks the Kalasin University for the supports of infrastructures. We would also like to thanks the anonymous reviewers for their remarks which improved the exposition. Finally, we thank you Mr. Jonathan Wary for editing the language usage in the manuscript. .

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